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Second Thoughts on Bubbles and Global Games¹

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Abstract

In models of asset-price bubbles, we show that introducing second thoughts (reconsideration) makes the bubbly equilibrium a unique equilibrium. We reformulate the model of bubbles as a simultaneous game, in which we introduce the second thought stage, where some players can change their strategies after observing the strategies chosen by other players. With a second thought stage, only the bubbly equilibrium survives the backward elimination of weakly dominated strategies (BEWDS). We can also show that when one player reconsiders and the other player then also reconsiders, the bubbly equilibrium becomes a unique subgame-perfect equilibrium. The uniqueness result can also be replicated in a macroeconomic binary action game with a continuum of players, which is typically analyzed in the global game literature, and is modified so that individual agents can interact bilaterally and have second thought stages. Our result implies that the second thought mechanism could be an effective method of equilibrium selection, which could be a promising alternative to the global game.

Keywords: Bubble, second thought stage, uniqueness.

JEL classification: E44, G01, G12.

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1 Introduction

As emphasized in Abreu and Brunnermeier (2003, AB hereafter), most of the episodes of bubbles and crashes are associated with investors' riding bubbles. Investors continue to purchase and hold the bubbly asset, even though they know the price of the asset is overvalued. AB argue that "rational arbitrageurs understand that the market will eventually collapse but meanwhile would like to ride the bubble as it continues to grow and generate high returns. Ideally, they would like to exit the market just prior to the crash ... However, market timing is a difficult task. ... [The] dispersion of exit strategies and the consequent lack of synchronization are precisely what permit the bubble to grow."

AB proposed a model which has a unique equilibrium in which the bubble grows because investors ride the bubble. Their model is based on the mechanism of higher order inference just like global games (see, e.g., Morris and Shin (1998, 2006)), and the investors in their model do not observe or respond to other investors' actions directly.

In this paper, we propose a simple model of bubbles that has a unique equilibrium, on the premise that investors observe and respond to the choice of other investors. In an environment where strategic interactions among agents may arise, an agent will fear that his selling may trigger selling by others, leading to his own difficulty in selling at a high price. This fear could cause agents to stop selling and the bubble to continue.

To demonstrate these features, we introduce the second thought stage in a simplified version of Jiang, Norman, Puzzello, Sultanum and Wright (2024, JNPSW hereafter). We modify the 2-period model of JNPSW to a simultaneous game and introduce the following second thought stage in the game. Investors choose their strategies (to buy or not to buy the asset) simultaneously in the first stage. Then, they observe the opponents' strategies and have the opportunity to change their own strategy. This is the second thought stage. Introducing the second thought stage in a simultaneous game is proposed by Saijo (2019). He shows that the introduction of the second thought stage in the prisoners' dilemma game produces the equilibrium in which both prisoners cooperate. What we show in this paper is that the bubbly equilibrium becomes the unique equilibrium with the second thoughts by investors. Our result indicates that the second thought mechanism that makes the equilibrium unique could be an alternative mechanism to global games, in which higher order beliefs under noisy signals make a unique equilibrium in the models that otherwise generate multiple equilibria.

Actually, in the latter part of this paper, we generalize our basic model, which is a game between two individual agents, to a model with a continuum of agents, in which individual agents are atomistic but still have the opportunity to interact with each other strategically. We also analyze the second thoughts in a binary action continuum player game that is typically analyzed in the global game literature (Morris and Shin 2006) and

is modified such that players can also interact bilaterally. In these macroeconomic models, we show that the equilibrium is unique and that the agents' expectations are coordinated to achieve an efficient outcome. This result implies that the second thought mechanism may be a promising alternative to the global games in making the equilibrium unique.

The organization of the paper is as follows. In the next section, we present a modified version of the JNPSW model, which is originally a two-agent-two-period model. We modify it to a simultaneous game of two agents. It is easily shown that the game has two equilibria: one with bubbles and the other without bubbles. In Section 3, we introduce a second thought (ST) stage into the model and show that the bubbly equilibrium is the unique equilibrium under a reasonable equilibrium-selection mechanism, which we call the backward elimination of weakly dominated strategies (BEWDS). In Section 4, we further modify the model by introducing two ST stages and show that the bubbly equilibrium becomes the unique subgame perfect equilibrium. Section 5 generalizes the two-agent game to a macroeconomic model and shows that the introduction of a second thought stage makes equilibrium unique under BEWDS. Section 6 formulates a strategic version of a binary action continuum player game and shows that second thought stages generate a unique subgame perfect equilibrium. Section 7 concludes.

2 Modifying the JNPSW model to a simultaneous game

Our model is built on a simple 2-period model proposed in Jiang et al. (2024), which we call JNPSW model. The prototype of the JNPSW model is as follows.

Prototype model: The model is described in Jiang et al. (2024). The economy continues for two periods: period 1 and period 2. There are three agents: agent 0, agent A, and agent B. Agent 0 has been waiting in the market at the beginning of period 1 with one unit of intrinsically useless asset, which may be traded with one unit of consumer goods. Agents A and B enter the market sequentially, either in period 1 or in period 2. We denote the entrant in period i by entrant i for $i = 1, 2$. Nature chooses entrant 1 randomly from agents A and B. The other agent becomes entrant 2. Jiang et al. (2024) assume the following incomplete information about the timing of entry: agents A and B do not know that they are either entrant 1 or 2, even when they have entered the market. For example, agent A does not know whether the current period is period 1 or period 2, even when she decides whether to buy the asset or not upon entry, which we describe next. When entering the market, the entrant can choose whether to buy or not to buy the asset. To buy the asset, the entrant must produce 1 unit of consumer good by spending utility cost c . The seller of the asset receives the consumer good and obtains the utility u

by consuming the good. The values of c and u are measured in the utility units, and

$$u > 2c.$$

The timeline is the following: In period 1, entrant 1 (either agent A or B) enters the market and if she decides to buy the asset, she produces one unit of consumer good by expending utility cost c , and receives the asset from agent 0 in exchange for the good. Agent 0 obtains utility u by consuming the good. In period 2, entrant 2 enters the market and if she decides to buy the asset, she produces one unit of consumer good and receives the asset from entrant 1 in exchange for the consumer good. Entrant 1 obtains u units of utility by consuming the good, whereas entrant 2 receives nothing in exchange for the asset because the economy ends in period 2. Here, we assume that agents A and B do not know which is entrant 1 (the winner) and entrant 2 (the loser) even when they decide whether to buy the asset or not upon entry. We necessarily assume also that entrants cannot identify the sellers of the asset when they enter the market, otherwise they can infer their entry periods from the sellers. If agents know beforehand who is entrant 1 and entrant 2, then the usual backward induction obviously implies that the asset is never traded in equilibrium. Thus, if we assume the complete information about the timing of entry, the unique equilibrium is the non-bubble equilibrium. However, Jiang et al. (2024) show that, given the incomplete information about the timing of entry, there exists a bubbly equilibrium in which the asset, which is intrinsically useless, is traded. The proof is straightforward. Each of agents A and B believes that the current period in which the agent enters the market is period 1 with probability 0.5 and period 2 with probability 0.5. Given this belief, the agent optimally chooses to buy the asset because the expected payoff of buying the asset is larger than that of not buying the asset, as $\frac{1}{2}(u - c) + \frac{1}{2}(0 - c) = \frac{1}{2}(u - 2c) > 0$, given the expectation that the other agent buys the asset when she enters the market. It has been shown that there exists a bubbly equilibrium in which both agents buy the asset in exchange for one unit of consumer good. Note that there is also the non-bubbly equilibrium in the JNPSW model, because the expected payoff of buying the asset is $-c < 0$, which is smaller than the payoff of not buying the asset, given the expectation that the other agent does not buy the asset. Thus, the JNPSW model has multiple equilibria.

Simultaneous Game: We modify the above JNPSW model as a simultaneous game between agent A and agent B. We assume that there exists period 0 before periods 1 and 2 arrive. In period 0, agents A and B simultaneously submit the order form in which they write their order: to buy or not to buy the asset. The order forms are submitted to the Central Stock Exchange, which fulfills the trading orders of agents A and B in random order in period 1 or period 2. Due to incomplete information, agents A and B do not know which period their orders are fulfilled. Introducing the decisions in period 0, we

can make the JNPSW model a simultaneous game between agents A and B in period 0. Strategies that the agents can choose in period 0 are P (to purchase the asset) and N (not to purchase the asset). If both agents A and B choose P , the expected payoff for each agent is $\frac{1}{2}(u - c) + \frac{1}{2}(0 - c) = \frac{1}{2}u - c$. This is because both agents enter the market in periods 1 and 2 to implement their strategies chosen in period 0. If agent A chooses P and agent B chooses N , the expected payoff of agent A is $\frac{1}{2}(0 - c) + \frac{1}{2}(0 - 0) = -\frac{1}{2}c$, and the expected payoff of agent B is 0. If both agents choose N , their expected payoff is 0. Thus, the expected payoffs are summarized in the following table.

Table 1: Expected payoffs of agents A and B

Agent A / Agent B	P	N
P (Purchase)	$(\frac{1}{2}u - c, \frac{1}{2}u - c)$	$(-\frac{1}{2}c, 0)$
N (Not purchase)	$(0, -\frac{1}{2}c)$	$(0, 0)$

Multiple equilibria: We denote the state where agent A chooses X and agent B chooses Y by (X, Y) , where $X \in \{P, N\}$ and $Y \in \{P, N\}$. It is obvious that both (P, P) and (N, N) are the Nash equilibria. The state (P, P) is the bubbly equilibrium and (N, N) is the non-bubbly equilibrium. This result is the same as the original JNPSW model.

3 A second thought makes the BEWDS equilibrium unique

In period 0, agents A and B simultaneously offer their strategies, P or N . It is the first stage. In reality, investors in asset markets observe the decisions and actions of other investors and then change their own decisions with second thoughts. To model these features of second thoughts after observing the other agent's action in period 0, we modify the simultaneous game to a two-stage game as follows:

- **First Stage:** In period 0, agents A and B simultaneously offer their strategies, P or N . Their strategies are observable to each other.
- **Second Thought (ST) Stage:** If one agent chooses P and the other agent chooses N in the First Stage, the agent who chose P has the opportunity to change her strategy to N . Otherwise, the game ends in the First Stage.

Why do only those who chose P have the opportunity to change their strategy? Saijo (2019) pick the example of the prisoner's dilemma: The two prisoners can choose cooperation (to keep silent) or non-cooperation (to confess their crime). Once a prisoner confessed their crime, he cannot cancel his action, while if he kept silent, he can confess when the second chance is given. Although this example may not be directly applied to the decision

in asset purchase, we simply follow Saijo (2019) in this section to clearly show the simple result. We generalize the second thought stage further in the next section.

Definition of equilibrium: We define BEWDS equilibrium as the state (X, Y) that survives the backward elimination of weakly dominated strategies (BEWDS), where $X \in \{P, N\}$ and $Y \in \{P, N\}$. The BEWDS equilibrium requires two properties: (i) subgame perfection and (ii) that agents do not choose weakly dominated strategies in each subgame and in the reduce normal form game. The BEWDS is adopted, for example, in Saijo (2019) and Kalai (1981).

Uniqueness of equilibrium: Only if the state (P, N) or (N, P) is chosen in the First Stage, the Second Thought (ST) Stage opens. Suppose agent A chose P and agent B chose N in the First Stage. Then, agent A faces the choice of P or N in the ST Stage. If agent A chooses P in the ST Stage, her expected payoff becomes $-\frac{1}{2}c$, whereas it becomes 0 if she chooses N . Thus, strategy P is strictly dominated, and agent A chooses N in the ST Stage, causing the expected payoff to be zero for both agents. Similarly, if (N, P) is chosen in the First Stage, agent B chooses N in the ST Stage, and the expected payoff becomes zero for both agents. Given these subgame outcomes, the reduced normal form game is written as follows in Table 2.

Table 2: Expected payoffs with a second thought

Agent A / Agent B	P	N
P (Purchase)	$(\frac{1}{2}u - c, \frac{1}{2}u - c)$	$(0, 0)$
N (Not purchase)	$(0, 0)$	$(0, 0)$

Given this payoff matrix of the reduced normal form game, agents A and B eliminate the weakly dominated strategy N and both of them choose P in the First Stage. Thus, the equilibrium, which is the state that survives the BEWDS, is only (P, P) . We have shown the uniqueness of the BEWDS equilibrium. Note that the states (P, P) and (N, N) are both subgame perfect in Table 2. So if we adopt the subgame perfect equilibrium as the equilibrium concept, we still have multiple equilibria. In the next section, we will make the subgame perfect equilibrium unique by introducing two stages of the second thoughts.

4 Second thoughts make the subgame perfect equilibrium unique

We generalize the second thought stages so that both of agents A and B have the opportunity to change their strategies after observing the other agent's choice. There are three stages: First Stage, Second Thought Stage 1, and Second Thought Stage 2.

- **First Stage:** In period 0, agents A and B simultaneously offer their strategies, P or N . Their strategies are observable to each other.
- **Second Thought Stage 1 (ST-1):** If one agent chooses P and the other agent chooses N in the First Stage, the agent who chose N is given a chance to change her strategy to P . If both agents chose P or both chose N in the First Stage, then game is over (no second thought stages).
- **Second Thought Stage 2 (ST-2):** If an agent was given a chance to change her strategy in ST-1, the other agent in the pair (who chose P in the First Stage) is now given a chance to change her strategy.

The game ends after both agents A and B had second thoughts in ST-1 and ST-2. After their strategies are determined in the game in period 0, they enter the market sequentially and randomly in periods 1 and 2, and implement their strategies chosen in ST-2 of the game. Agents A and B believe that the current period in which they have entered the market is period 1 with probability $\frac{1}{2}$ and period 2 with probability $\frac{1}{2}$.

We can show the following proposition:

Proposition 1. *In the game with the second thought stages, ST-1 and ST-2, the unique subgame perfect equilibrium is (P, P) .*

Proof. First, we consider the case where the state (P, N) is chosen in the First Stage. We can show that agent B chooses P in ST-1 because of the following:

- When the state is (P, P) in ST-1, agent A obtains $\frac{1}{2}u - c > 0$ by choosing P in ST-2 and 0 by choosing N in ST-2. Thus, agent A definitely chooses P in ST-2. In this case, agent B obtains $\frac{1}{2}u - c > 0$.
- When the state is (P, N) in ST-1, agent A obtains 0 by choosing N in ST-2, and $-\frac{1}{2}c$ by choosing P in ST-2. Thus, agent A definitely chooses N in ST-2. In this case, agent B obtains 0.
- Comparing the above two cases, agent B obtains $\frac{1}{2}u - c > 0$ when she chooses P in ST-1 and 0 when she chooses N in ST-1. Therefore, agent B chooses P in ST-1, and the final outcome in ST-2 is (P, P) .

Second, we consider the case where the state in the First Stage is (N, P) . Similar arguments imply that agent A chooses P in ST-1 and agent B chooses P in ST-2. Thus, the final outcome in ST-2 is also (P, P) .

In summary, the payoff matrix for the First Stage is written as follows, implying that the final choice (after ST-2) is uniquely (P, P) in the subgame perfect equilibrium.

This completes the proof. □

Table 3: Expected payoffs with two second thoughts

Agent A / Agent B	P	N
P (Purchase)	$(\frac{1}{2}u - c, \frac{1}{2}u - c)$	$(\frac{1}{2}u - c, \frac{1}{2}u - c)$
N (Not purchase)	$(\frac{1}{2}u - c, \frac{1}{2}u - c)$	$(0, 0)$

We have shown that the bubbly equilibrium, where both agents A and B purchase the bubbly asset is the unique subgame perfect equilibrium with two stages of the second thoughts.

5 Macroeconomic version of the model of bubbles

So far, we have treated agents A and B as individuals and focused on their strategic interactions. Our result of the uniqueness of the equilibrium is based on the premise that the economy is a game between two individuals. In this section, we argue that the uniqueness result is sustained in the macroeconomic setting of the market where an individual agent is atomistic but there is a chance that her action can strategically affect another agent's choice, as we describe shortly. Our result indicates that the second thought mechanism may be a useful and practical model of selection of a unique equilibrium. In particular, the second thought mechanism may be considered an alternative mechanism to the global games, which make the equilibrium unique by eliminating dominated strategies in higher-order beliefs with noisy signals.

Setting: We generalize the simultaneous game in Section 2 from a game between two individuals to a game between two groups, in which each member of the groups individually chooses strategy P (purchasing the bubbly asset) or N (not purchasing the bubbly asset).¹ There are three groups of agents: Group 0, Group A and Group B. Each group is a continuum of agents with a unit mass. The economy continues for two periods: period 1 and period 2. Before these periods, there exists period 0, in which agents of Groups A and B play the simultaneous game to determine their strategies, P or N . Each agent of Group 0 is already in the market at the beginning of period 1 holding one unit of the bubbly asset, which is intrinsically useless. Nature determines two things.

- **Matching Stage:** In period 0 after agents decide their strategies P or N , Nature makes the random matching between an agent in Group A (we call “agent a”) and an agent in Group B (“agent b”). Each agent in Group A is matched with an agent in Group B, and vice versa. In each pair, the strategies of agents a and b are observable

¹Note that the strategy P entails selling of the bubbly asset in the next period. Strategy N entails no selling in the next period, as she has nothing to sell.

with each other. So, agents a and b know each other in period 0 that they will play the game of the JNPSW model in periods 1 and 2.

- **Entry:** Nature decides the timing of entry on a group basis. All agents in Group A enter the market together in either period 1 or 2. The same is true for Group B. Thus, if an agent in Group A enters the market in period 1, then all other agents in Group A enter in period 1 as well, and all agents in Group B enter in period 2.

These agents do not know which period they enter just like agents in the JNPSW model.

Note on the relevance of matching stage: Although there is a continuum of agents in this game, the choice of whether to buy the asset such as real estate (strategy P) or not buy (N) can only be realized within an individual, bilateral transaction. Thus, the matching stage, where agents are paired with trading partners from the other group, should be a very natural setting. Further discussion on the relevance of matching stage is provided in Section 6.1.

Macroeconomic variables: The macroeconomic state of the economy is described by the measure of agents who choose the strategy P . We define variables, θ_A , θ_B , and $\bar{\theta}$, as follows.

- θ_A is the measure of agents in Group A who choose the strategy P .
- θ_B is the measure of agents in Group B who choose the strategy P .
- We assume that the measure $1 - \bar{\theta}$ of agents in each group, i.e., Group A and Group B, has no choice other than to choose N for an unspecified fundamental reason. We call these agents who choose N unconditionally “the fundamental pessimists,” in what follows. The other $\bar{\theta}$ agents (in Group A and Group B) can choose P or N by their own discretion. The variable $\bar{\theta} \in [0, 1]$ represents the macroeconomic fundamentals of this economy.

In the simultaneous game in period 0, θ_A agents in Group A choose P and $1 - \theta_A$ agents choose N , while θ_B agents in Group B choose P and $1 - \theta_B$ agents choose N .

Values of agents according to their strategies: We calculate the expected payoff for an agent of Group A who chooses the strategy P , in what follows. We call her “agent ap.” First, with probability θ_B , agent ap is matched with an agent in Group B who chose P . In this pair, agent ap’s expected payoff is $\frac{1}{2}(u - c) + \frac{1}{2}(-c)$ as described in Section 2. Thus, the part of the expected payoff coming from matching with an agent who chose P is $\theta_B(\frac{1}{2}u - c)$. Second, with probability $1 - \theta_B$, agent ap is matched with an agent in Group B who chose N . In this pair, agent ap’s expected payoff is $\frac{1}{2}(-c)$ because if agent

ap enters in period 1, she buys the asset from an agent in Group 0. Thus, the part of the expected payoff coming from matching with an agent who chose N is $-(1 - \theta_B)\frac{1}{2}c$. Therefore, the expected payoff for agent ap is

$$V_{AP} = \theta_B(\frac{1}{2}u - c) - (1 - \theta_B)\frac{1}{2}c = \frac{1}{2}\{\theta_B u - (1 + \theta_B)c\}.$$

The expected payoff for an agent in Group B who chooses P is also

$$V_{BP} = \frac{1}{2}\{\theta_A u - (1 + \theta_A)c\}.$$

The expected payoff for an agent in Groups A and B who chooses N is

$$V_{AN} = V_{BN} = 0.$$

There exists the threshold θ_T , such that $V_{AP} > 0$ when $\theta_B > \theta_T$ and $V_{BP} > 0$ when $\theta_A > \theta_T$, where

$$\theta_T \equiv \frac{c}{u - c}.$$

As we assumed in Section 2 that $u > 2c$, we have $0 < \theta_T < 1$.

Definition of equilibrium: In this section, we use the BEWDS equilibrium as an equilibrium concept. The BEWDS procedure in this macroeconomic game is as follows. For an agent in Group A, the strategy X is weakly dominated by the strategy Y , where $X \in \{P, N\}$ and $Y \in \{P, N\}$, if $V_{AX} \leq V_{AY}$ for all $\theta_B \in [0, \bar{\theta}]$ and $\exists \theta_B \in [0, \bar{\theta}]$ such that $V_{AX} < V_{AY}$. In this case, all agents in Group A eliminate X and choose Y in equilibrium.

Multiple equilibria without the second thought stage: Note that $\bar{\theta}$ is a variable that represents the fundamentals of the economy, which is determined by Nature. We assume $\bar{\theta}$ is observable. First, consider the case where $\bar{\theta} < \theta_T$. Then all agents know that at least the fundamental pessimists, whose measure is $1 - \bar{\theta}$, in each Groups A and B choose the strategy N . Then, $V_{AP} < 0$ and $V_{BP} < 0$ so that all agents choose N . In this case, the equilibrium is unique and it is the non-bubbly equilibrium where $\theta_A = \theta_B = 0$. Second, consider the case where $\bar{\theta} > \theta_T$. In this case, if $\theta_A = \theta_B = \bar{\theta}$, then $V_{AP} > 0$ and $V_{BP} > 0$, and all agents who are not the fundamental pessimists will buy the bubbly asset in equilibrium. This is the bubbly equilibrium. There also exists another equilibrium, which is a non-bubbly equilibrium, where $\theta_A = \theta_B = 0$, meaning that all agents choose N , no matter whether they are fundamental pessimists or not. This is because $V_{AP} < 0$ and $V_{BP} < 0$, given that $\theta_A = \theta_B = 0$. Therefore, in the case where $\bar{\theta} > \theta_T$, there exist both the bubbly and the non-bubbly equilibria in this simultaneous game in which each individual is atomistic.

A second thought stage makes the equilibrium unique: After all agents choose their strategies P or N , agents of Group A and agents of Group B form pairs in period 0. Then, they have the second thought stage in which they can change their strategies. Therefore, the stages of the game in period 0 are as follows.

- **First Stage:** All agents in Groups A and B choose their strategies, P or N simultaneously.
- **Matching Stage:** After the First Stage, the values of θ_A and θ_B are revealed. Each member of Group A and each member of Group B form a pair at random.
- **Second Thought Stage:** After a unit mass of pairs is formed by the random matching, the two agents in each pair observe their opponents' strategies with each other. Only agents who chose P and whose opponents chose N in the First Stage have the opportunity to change their strategy from P to N .

We can prove the following proposition.

Proposition 2. *The BEWDS equilibrium is unique and bubbly, i.e. $\theta_A = \theta_B = \bar{\theta}$, which means that all agents who are not fundamental pessimists choose P and only fundamental pessimists choose N .*

Proof. If an agent chooses N in the First Stage, she has no second chance to change her strategy, and the expected value of the payoff is zero:

$$V_{AN} = V_{BN} = 0.$$

Next, we consider the value of the payoff for agent ap , who is an agent in Group A who chooses P in the First Stage. We consider backwards. In the Second Thought Stage, agent ap has no chance to change her strategy if the opponent chose P in the First Stage, and agent ap 's expected payoff becomes $\frac{1}{2}u - c > 0$. Note that the opponent has no chance to change her strategy in this case, and both agents in the pair know this. If the opponent chose N in the First Stage, agent ap optimally chooses N in the Second Thought Stage and her payoff becomes zero. In the Matching Stage, agent ap is matched with an opponent who chose P with probability θ_B and N with probability $1 - \theta_B$. Thus, the expected payoff (before the matching) for agent ap is

$$V_{AP} = \theta_B \left(\frac{1}{2}u - c \right).$$

$V_{AP} > 0$ for $\theta_B > 0$ and $V_{AP} = 0$ for $\theta_B = 0$. Similar arguments imply $V_{BP} = \theta_A \left(\frac{1}{2}u - c \right)$. In the First Stage, agent ap compares V_{AP} and V_{AN} for all $\theta_B \in [0, 1]$ and chooses P because strategy P weakly dominates strategy N .

The BEWDS procedure implies that all agents who are not fundamental pessimists choose P in the First Stage and therefore it is realized in equilibrium that $\theta_A = \theta_B = \bar{\theta}$. Thus, the bubbly equilibrium is the only equilibrium in the macroeconomic model. $\square$

It would be easily shown that introduction of two second thought stages makes the bubbly equilibrium a unique subgame perfect equilibrium, just as in Section 6.3 below.

6 Second Thoughts on Global Games

We can demonstrate the uniqueness of equilibrium in a version of the stylized binary action continuum player game that is typically analyzed in the global game literature, e.g. in Morris and Shin (2006).

6.1 The baseline model

We consider the following binary action game, in which agents form pairs randomly after choosing their strategies and then interact bilaterally with their opponents in the pairs.

Setting: There exists a continuum of agents with a unit mass. Each agent chooses I (to invest) or N (not to invest). If an agent chooses N , the payoff becomes zero. If an agent chooses I , the agent produces one unit of good at no cost. She can obtain the utility $\theta - 1 + y$, where θ is a macroeconomic variable that represents the fundamentals of the economy, where $\theta \in \mathbb{R}$, and y is consumption of good. The value of θ is revealed at the beginning of the game. However, the agent cannot consume her own product, and she can consume only the goods that are produced by the other agents. The steps in this game are as follows.

- **First Stage:** The value of θ is revealed and all agents observe θ . Then, all agents decide simultaneously their strategies I or N . Their decisions are observable. We define ι as the measure of agents who chose I , where $0 \leq \iota \leq 1$. The value of ι is revealed in the First Stage.
- **Matching Stage:** After the First Stage when agents decide their strategies, they form pairs randomly, i.e., each agent is matched with another agent at random.
- **Consumption Stage:** After pairs are formed, a pair-wise trading of the goods may or may not occur. That is, if both agents of a pair chose I in the First Stage, they exchange their products with each other and consume them. Otherwise, trading does not occur, and there is no consumption.

Note on the relevance of matching stage: As we saw in Section 5, the matching stage is a natural setting for the asset market to analyze asset bubbles. In addition, macroeconomic regime changes such as bank runs, currency crises, and sovereign bond crashes, are also popular and intensively analyzed in the global game literature. We would like to argue below that the introduction of matching stage should be quite natural in modeling these phenomena as well.

- **Bank runs:** We can consider the case where a bank is relatively small and regional, and there are a few large depositors who can exert significant influences on the bank. In this case, a large depositor may worry about the reaction of another large depositor when she chooses whether to withdraw her deposit or not. We can model these large depositors as forming a pair in which they interact strategically with each other. This is formalized as the matching stage in our game.
- **Currency or bond crises:** In actual currency or sovereign bond markets, investors alter their strategies based on the actions of other market participants. In such an environment, a large institutional investor may worry that her selling of the asset may trigger a wave of selling of other investors. This consideration about the reactions of others to her own action can be simplified as strategic interaction within a pair of investors in our model.

Definition of equilibrium: In this section, we use two equilibrium concepts, the subgame perfect equilibrium and the BEWDS equilibrium. We denote the payoff of choosing $X \in \{I, N\}$ by V_X . X is weakly dominated by $Y \in \{I, N\}$ if $V_X \leq V_Y$ for all $\iota \in [0, 1]$ and $\exists \iota$ such that $V_X < V_Y$. The BEWDS equilibrium is the outcome that survives the BEWDS procedure, which is to eliminate the weakly dominated strategy X (and to choose the weakly dominant strategy Y) in each subgame. We adopt the subgame perfect equilibrium unless otherwise specified.

Multiple equilibria without second thoughts: Suppose that there exists no second thought stages. The payoff matrix for a pair of agents is given as follows.

Table 4: Payoffs without second thoughts

Agent 1 / Agent 2	I	N
I (Invest)	(θ, θ)	$(\theta - 1, 0)$
N (Not invest)	$(0, \theta - 1)$	$(0, 0)$

Given this payoff matrix, we consider in what follows the expected payoff for an agent in the First Stage. If an agent chooses N in the First Stage, the payoff for the agent, V_N , is zero: $V_N = 0$. If an agent chooses I in the First Stage, the expected payoff, V_I , is given

by

$$V_I = \theta - 1 + \iota.$$

Note that the expectation in V_I is taken before the agent forms a pair with another agent. The equilibrium is specified as follows:

- For $\theta > 1$, it is obvious that $V_I > V_N$ for all $\iota \in [0, 1]$. Thus, all agents choose I so that $\iota = 1$ and $V = \theta$ in equilibrium.
- For $\theta < 0$, it is obvious that $V_I < V_N$ for all $\iota \in [0, 1]$. Thus, all agents choose N so that $\iota = 0$ and $V = 0$ in equilibrium.
- For $0 \leq \theta \leq 1$, there are two equilibria. On one hand, that all agents choose I and $\iota = 1$ is an equilibrium because $V_I \geq V_N$ if $\iota = 1$. On the other hand, that all agents choose N and $\iota = 0$ is an equilibrium because $V_I \leq V_N$ if $\iota = 0$.

6.2 A second thought makes the BEWDS equilibrium unique

Now we introduce the second thought stage(s) after the Matching Stage in this game to make the equilibrium unique. In this subsection, we introduce one second thought stage and show that it makes the BEWDS equilibrium unique. In the next subsection, we show that introduction of two second thought stages makes the subgame perfect equilibrium unique. The steps in the game are now as follows:

- **First Stage:** All agents decide simultaneously their strategies I or N . The value of ι is revealed.
- **Matching Stage:** After agents decide their strategies, they form pairs with random matching.
- **Second Thought Stage:** After the pairs are formed, only the agents who chose I in the First Stage and whose opponents chose N in the First Stage are given the opportunity to change their strategy from I to N .
- **Consumption Stage:** After the Second Thought Stage, a pair-wise trading of the goods may or may not occur. That is, if both agents of a pair chose I in the Second Thought Stage, they exchange their products with each other and consume them. Otherwise, trading does not occur, and there is no consumption.

Note that only agents who chose I and whose opponents chose N in the First Stage are given the opportunity to change their strategy in the Second Thought Stage. We have the following proposition.

Proposition 3. *Consider the case where $0 \leq \theta \leq 1$. The Second Thought Stage leads to a unique BEWDS equilibrium, where all agents choose I and $\iota = 1$ in the First Stage.*

Proof. If an agent chooses N , the expected payoff is $V_N = 0$ for all $\iota \in [0, 1]$. If an agent chooses I , she will be matched with an agent who chose I with probability ι and, in this case, there is no second stage and the payoff is $\theta - 1 + 1 = \theta$. Thus, the part of the expected payoff coming from the matching with an agent who chose I is $\iota\theta$. The agent who chooses I will be matched with an agent who chose N with probability $1 - \iota$. In this case, she will get $\theta - 1 + 0 = \theta - 1 < 0$ if she does not change her strategy, and she will get 0 if she changes her strategy from I to N . So, the agent optimally changes her strategy from I to N , and the part of expected payoff coming from the matching with an agent who chose N is 0. Thus, the total expected payoff of choosing I is $V_I = \iota\theta$, which satisfies $V_I \geq 0$ for all ι and $\exists \iota$ such that $V_I > 0$. Therefore, strategy I weakly dominates strategy N . Thus, all agents choose I and $\iota = 1$ in the First Stage. For reference, the following Table 5 is the payoff matrix of a pair of agents with their choices in the First Stage:

Table 5: Payoffs with a second thought

Agent 1 / Agent 2	I	N
I (Invest)	(θ, θ)	$(0, 0)$
N (Not invest)	$(0, 0)$	$(0, 0)$

□

6.3 Two second thoughts make the subgame perfect equilibrium unique

In this subsection, we introduce two second thought stages to the baseline model. The steps in the game are now as follows:

- **First Stage:** All agents decide simultaneously their strategies I or N . The value of ι is revealed.
- **Matching Stage:** After agents decide their strategies, they form pairs with random matching.
- **Second Thought Stage 1 (ST-1):** After the Matching Stage, each pair moves into ST-1. In each pair of agents, if one agent chooses I and the other agent chooses N in the First Stage, the agent who chose N is given a chance to change her strategy to I . If both agents chose I or both chose N in the First Stage, then the game is over (no second thought stages).
- **Second Thought Stage 2 (ST-2):** If an agent was given a chance to change her strategy in ST-1, the other agent in the pair (who chose P in the First Stage) is now given a chance to change her strategy.

- **Consumption Stage:** After the second thought stages, a pair-wise trading of the goods may or may not occur. That is, if both agents of a pair have chosen I when ST-2 is over, they exchange their products and consume them. Otherwise, trading does not occur, and there is no consumption.

We have the following proposition.

Proposition 4. *We assume $\theta \neq 0$. With the two second thought stages, the subgame perfect equilibrium is unique and*

- *all agents choose I and $\iota = 1$ in equilibrium for $\theta > 0$,*
- *all agents choose N and $\iota = 0$ in equilibrium for $\theta < 0$.*

Proof. It is obvious that all choose N when $\theta < 0$, because the payoff of choosing I would be $\theta < 0$ even if all the other agents choose I . It is also obvious that all choose I when $\theta > 1$, because the payoff of choosing I would be $\theta - 1 > 0$ even if all the other agents choose N . So we focus on the case where $0 < \theta \leq 1$.² Consider a representative pair of agents, agent 1 and agent 2. We denote the state that agent 1 chooses X and agent 2 chooses Y by (X, Y) , where $X \in \{I, N\}$ and $Y \in \{I, N\}$. Suppose the state is (I, N) after the First Stage. Then, agent 2 is given a chance to change her strategy in ST-1.

- Now, suppose agent 2 chooses N in ST-1. Then, the state becomes (I, N) . In this case, agent 2's final payoff would be 0, because agent 1 chooses N in ST-2 and the final state would be (N, N) because
 - if agent 1 chooses N in ST-2, the state would be (N, N) , and agent 1 would get 0,
 - if agent 1 chooses I in ST-2, the state would be (I, N) , and agent 1 would get $\theta - 1 \leq 0$,
 - and comparing these cases, agent 1 optimally chooses N in ST-2 for $\theta < 1$. For $\theta = 1$, agent 1's choice is indeterminate, but her payoff and agent 2's payoff are both 0.
- Next, suppose agent 2 chooses I in ST-1. Then, the state becomes (I, I) . In this case, agent 2's final payoff would be $\theta > 0$, because agent 1 chooses I in ST-2 and the final state would be (I, I) because
 - if agent 1 chooses N in ST-2, the state would be (N, I) , and agent 1 would get 0,

²We assume away the knife edge case of $\theta = 0$.

- if agent 1 chooses I in ST-2, the state would be (I, I) , and agent 1 would get $\theta > 0$,
- and comparing these cases, agent 1 optimally chooses I in ST-2.
- Therefore, if the state after the First Stage is (I, N) , agent 2 optimally chooses I in ST-1 and the final state is (I, I) .

This argument implies that the payoff matrix of agents 1 and 2 is given as follows with respect to the choice in the First Stage:

Table 6: Payoffs with two second thoughts

Agent 1 / Agent 2	I	N
I (Invest)	(θ, θ)	(θ, θ)
N (Not invest)	(θ, θ)	$(0, 0)$

We have shown that, in any pair of agents, the final state after the second thoughts will be (I, I) unless (N, N) in the First Stage. Thus, the final choice (after ST-2) is uniquely (I, I) in the subgame perfect equilibrium. $\square$

The second thought makes the equilibrium unique, and it is an optimistic and efficient equilibrium where all agents choose I . This result contrasts with the global game in which some agents choose N inefficiently in the unique equilibrium (Morris and Shin 2006).

7 Conclusion

We have shown that second thoughts by agents make the bubbly equilibrium a unique equilibrium in the JNPSW model. The uniqueness holds with one second-thought stage if we adopt the BEWDS equilibrium as the equilibrium concept, whereas it holds with two second-thought stages if we adopt the standard subgame perfect equilibrium. In the first half of the paper, we focused on the bilateral game between two individual agents.

In the latter half, we showed that our uniqueness result holds in the macroeconomic version of the model of the bubble and in a version of a binary action continuum player game that is typically analyzed in the global game literature. In these models, an individual agent is atomistic when she chooses the strategy, while she forms a pair with another agent after choosing the strategy so that her second thoughts exert strategic interaction with the strategy of the opponent in the pair.

As a selection mechanism for a unique equilibrium, the second thought mechanism could be a simple alternative to the global game. The equilibrium becomes unique when agents can respond to the strategies of other agents in the second thought theory, while there are no strategic interactions among agents in the global games, which make the

equilibrium unique by eliminating dominated strategies in higher-order beliefs with noisy signals. We have shown that the unique equilibrium is efficient with second thoughts, while the equilibrium is inefficient in the global games. The second thought theory may explain why efficiency is achieved so often in reality.

8 References

Abreu, D., and M. Brunnermeier (2003) “Bubbles and Crashes.” *Econometrica*, 71(1): 173–204.

Jiang, Janet Hua, Peter Norman, Daniela Puzzello, Bruno Sultanum, and Randall Wright (2024) “Is money essential? An experimental approach.” *Journal of Political Economy*, 132(9):2972–98.

Kalai, E. (1981). “Preplay Negotiations and the Prisoner’s Dilemma” *Mathematical Social Sciences*, 1(4), 375–379.

Morris, Steven, and Hyun Song Shin (1998), “Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks,” *American Economic Review*, 88:587–597.

Morris, Steven, and Hyun Song Shin (2006). “Global Games: Theory and Applications.” Chapter 3 in Dewatripont, Mathias, Lars Peter Hansen and Stephen J. Turnovsky eds. *Advances in Economics and Econometrics: Theory and Applications*, Eighth World Congress, pp. 56 - 114.

Saijo, Tatsuyoshi (2019). “Second thoughts of social dilemma in mechanism design.” In W. Trockel (Ed.), *Social Design: Essays in Memory of Leonid Hurwicz* (Studies in Economic Design). Springer. <https://doi.org/10.1007/978-3-319-93809-7>