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R&D Subsidies and Multi-product Firms*

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Abstract

We analyze firm subsidies directed at the fixed costs of developing new products in a setting with international trade and multiproduct heterogeneous firms that can move between countries. Socially optimal unilateral subsidies balance the welfare gains from increased variety against taxes. Freer trade implies lower optimal unilateral subsidies as more of the benefits spill over to foreign consumers. For similar reasons, the simulated Nash subsidies will be lower with lower trade costs. This is consistent with the current situation in the world economy, where trade protection and higher subsidies seem to go hand in hand.

Keywords: multi-product firms, heterogeneous firms, trade liberalization, firm location
JEL classification: F12, F15

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1 Introduction

Subsidies are on the rise (Hoekman and Nelson 2020). For example, the Inflation Reduction Act (2022), along with the CHIPS & Science Act (2021), implies a massive U.S. industrial policy to spur R&D and commercialization of cutting-edge technologies, such as domestic production of semiconductors. The IRA also implies subsidies for the development and production of clean product varieties, such as energy-efficient homes. The policy has raised concerns among U.S. trading partners, who are considering similar policies in response. The stage is thus set for subsidy competition among the major trading blocs in the global economy. Positive subsidies may be optimal in the presence of externalities, but since several recent subsidy schemes explicitly benefit domestic firms, they raise concerns about unfair competition.

Here, we analyze firm subsidies directed at the fixed costs of developing new product varieties in a setting of international trade and heterogeneous multiproduct firms. We allow firms to move between countries in response to the subsidies. Unilateral subsidies will attract firms in the model, and since larger firms with a broader product range benefit most from subsidies, they will be the first to relocate. The optimal level of subsidies for a country balances the welfare gains from greater variety against taxes. Freer trade implies lower optimal unilateral subsidies as more of the benefits spill over to foreign consumers. The Nash subsidies simulated here will be lower with lower trade costs for similar reasons. This is consistent with the current situation in the world economy, where trade protection and higher subsidies seem to go hand in hand.

Here we introduce subsidies for the development of new varieties in a model with heterogeneous firms with endogenous product scope that can trade and move between countries (see Forslid and Okubo 2023). The marginal cost of a new product increases with the number of products in the model. Thus, the further away from a firm's core product, the higher the marginal cost. We also assume that firms are heterogeneous in the marginal costs of their core product, à la Melitz (2003). More productive firms - with lower marginal costs - will find it profitable to introduce products further away from their core product and will therefore have a wider product range. Subsidies that target the fixed costs of introducing new varieties will increase the profit-maximizing product range. We show in the paper that positive subsidies are socially optimal in our model. This is because there tends to be too little variety due to imperfect competition. This result is very similar to the original Dixit-Stiglitz model. In fact, we show that the optimal subsidy (for symmetric countries) will be the same in our model and in the original Dixit-Stiglitz model. The original Melitz model will also have the same optimal subsidy under certain assumptions on entry costs.

While the present paper focuses on the effects of R&D subsidies (subsidies for the development of new product varieties), the trade literature on multi-product firms has focused on how, for a given firm's location, trade liberalization affects product scope. A number of papers find that trade liberalization reduces the product scope of firms, as firms concentrate on their core products when trade is liberalized. This effect tends to occur in oligopolistic settings, where the firm has a core product and new products with higher marginal costs compete with those already produced by the firm (the so-called "cannibalization effect").[1]

There is also research showing that trade liberalization affects high and low productivity firms differently. The former expand their product scope due to improved access to foreign markets, while the latter reduce their product scope due to increased competition in the domestic market (Dhingra 2013; Nocke and Yeaple 2014; and Qiu and Zhou 2013).[2]

There is research on firms that produce many products in a closed economy. Chisholm and Norman (2004) study where these firms choose to locate within a Hotelling framework that accounts for different types of consumers. Flach and Irlacher (2018) examined companies that produce many products and can invest in both product and process innovation using a framework that considers nonhomothetic preferences and linear demand, following the example of Eckel et al. (2015). A larger market implies more R&D investment for both types.

The literature on subsidies and trade is largely concerned with oligopolistic models in a partial equilibrium framework. Brander and Spencer (1985) show how countries benefit from capturing most of the rents in oligopolistic industries and use export subsidies to implement such "rent shifting" policies. However, Eaton and Grossman (1986) found that the effect of export subsidies depends on whether competition is based on price or quantity. Haaland and Kind (2008) examined trade costs and concluded that it is best for the government to provide larger R&D subsidies when trade costs are lower, which is the opposite of what we find here.[3]

Kondo (2012) uses a new economic geography model with endogenous growth. He shows that as globalization progresses, competition for R&D subsidies becomes less intense and economic growth declines. This result is similar in spirit to ours, although the setting is different.

In the next section, we present some stylized evidence, and in section 2, we outline the model. The paper concludes in section 3.

2 Model

Subsidies can induce firms to relocate to the subsidizing country. To analyze R&D subsidies for new product development, we need a model of differentiated firms that can trade and relocate.

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[1] See, for example, Blanchard et al. (2012), Eckel and Neary (2010) and Ju (2003). Similar results are found by Mayer et al. (2014 and 2021), who use a monopolistically competitive model with heterogeneous firms and linear demand. Bernard et al. (2011) have heterogeneous firms that match their "capabilities" to different product attributes (or consumer preferences). Here, trade liberalisation can lead to a wider or narrower range of products. Feenstra and Ma (2007) use standard CES preferences but relax the large group assumption. By allowing firms to account for their own effect on the aggregate price index, they obtain a cannibalisation effect from new products. Eckel et al. (2015) allow for both vertical (quality) and horizontal (scope) upgrading. Here, trade liberalisation (tariff reductions) leads to a narrower product range for all firms.

[2] Bernard et al. (2018) present a very general model where firms chose multiple production locations, multiple export markets, and countries to source from. They show how more productive firms participate more intensively in the world economy along each margin.

[3] Aside from using a model with oligopolistic competition, Haaland and Kind (2008) differ in that they use a partial equilibrium framework. Thus, they do not consider the effect of taxation on subsidies.

However, existing models of this type have a fixed number (mass) of single-product firms, making them inadequate for our purposes. Therefore, we introduce subsidies here in the model of Forslid and Okubo (2023), which has heterogeneous firms with endogenous product scope, making it possible to analyze the effect of subsidies on product variety in a trade and geography framework with heterogeneous firms.

2.1 Basics

There are two markets with different populations. The larger one is the core market and the smaller one is known as the periphery and is marked with an asterisk (*). There are two primary factors of production: physical capital represented by K and labor represented by L. There is a set amount of capital and labor available globally, K^W and L^W. Each firm needs one unit of capital to operate, so there is a fixed number of companies, $N^W = K^W$. All countries have the same proportion of of workers and firms (capital). That is, the markets are identical except for size. The primary market comprises more than half ($s > 0.5$) of the overall labor force and of capital. Firms have the freedom to relocate between markets. Labor is free to move across sectors but not between markets. Firms are reborn each time period, and all policy trials are conducted only after the regeneration of firms.[4]

A homogeneous good is produced with a constant-returns technology, and differentiated manufactures are produced with increasing-returns technologies. Firms are heterogeneous in marginal costs, and there are no market-entry costs, which will mean that all firms sell in both markets. A firm produces a range of varieties. It has one core product, and as the firm moves away from its core product, each new variety has an increasingly higher marginal cost.

All individuals have the utility function

$$U = \mu \ln C_M + C_A, \quad C_M = \left[\int_{l \in \Psi} c_l^{(\sigma-1)/\sigma} dl \right]^{\sigma/(\sigma-1)}, \tag{1}$$

where $\mu \in (0,1)$, and $\sigma > 1$ are constants. Ψ is the set of consumed varieties. C_M is a consumption index of manufacturing goods, and C_A is the amount consumed of the homogenous good. c_l is the amount consumed of variety l.

Each consumer spends μ of their income on manufactures. The total demand for a variety j from firm i in market ν is

$$x_{ij\nu} = \frac{p_{ij\nu}^{-\sigma}}{P_\nu^{1-\sigma}} \cdot \mu, \tag{2}$$

where $p_{ij\nu}$ is the consumer price of variety j from firm i in market ν, P_ν is the CES price index, and Y_ν is the income.

[4] This assumption simplifies the analysis of the relocation of firms, since we do not have to account for exotic distributions of firms (with gaps) that can arise when firms move back and forth between countries.

On the supply side, the homogeneous-good sector has constant returns and perfect competition. The unit factor requirement of the homogeneous good is one unit of labor. The good is freely traded, and since we also choose it as the numeraire, we have

$$p_A = w = 1, \tag{3}$$

where w is the wage of workers in all markets. We assume trade balance, so that income equals expenditure in each market.

Manufacturing Firms

The fixed number of firms are distributed among the two regions $n + n^* = N^W$. Without loss of generality, we normalize so that $N^W \equiv 1$. We assume that ownership of firms is fully diversified; that is, each individual owns an equal share of the world stock market. The aggregate return to all firms equals aggregate operating profit in equilibrium $\mu L^w/\sigma$. Therefore, the average return to capital (aggregate operating profit divided by the fixed number of firms) remain the same regardless of the location of firms. We choose units so that $L^w \equiv 1$.

The ex ante value of all firms is constant and determined by the expected operating profit. To enter the market, firms must pay r, which will equal the expected operating profit in equilibrium. Having entered a firm draws its core marginal cost $a_i \in [0, 1]$ from a cumulative distribution function $G(a)$. The firm thereafter chooses a range of varieties to produce (the product scope), $[0, \overline{m_i}]$, where $\overline{m_i} \geq 0$.[5] Each variety m_{ij} requires an additional fixed cost f in terms of labor (the composite primary factor of production). Additionally, the government provides subsidies for product development, which results in a fixed cost $\overline{m_i}(1 - S)f$ for firm i, where $S > 0$ is the subsidy rate. It is also the case that, as the firm moves further away from it's core product, each new variety has an increasingly higher marginal cost.

The total cost function of a firm i is assumed to be:

$$TC_i = r + (1 - S)\overline{m_i}f + a_i \int_0^{\overline{m_i}} z^\theta x_i(z)dz, \tag{4}$$

where r is the entry cost and z is an integration dummy. The parameter $\theta > 0$ determines how fast the marginal cost increases as a firm expands its product scope.

Profit maximization by manufacturing firm i leads to a constant mark-up over the marginal cost of each product variety j,

$$p_{ij} = \frac{\sigma}{\sigma - 1} m_{ij}^\theta a_i. \tag{5}$$

Geographical distance is represented by trade costs. Shipping the manufactured good involves a frictional trade cost of the "iceberg" form: for one unit of good from market j to arrive in market k, $\tau_{jk} > 1$ units must be shipped. Trade costs are symmetric between markets $\tau_{jk} = \tau$ $\forall\ j, k$, and the export price is therefore $p_{ij}\tau$.

[5] Firms continue to have zero measure in this set-up. The aggregate CES price index will here be an integral over a surface, where one dimension is the continuum of firms and the other is the product scope. The limiting most productive firm (with a zero marginal cost) will have an infinite product scope, but this firm has a zero measure. The large group assumption of the monopolistically competitive framework therefore still applies.

The total profit of a firm is given by

$$\pi_i = \int_0^{\overline{m_i}} \frac{p_i(z)x_i(z)}{\sigma} dz - (1-S)\overline{m_i} f - r. \tag{6}$$

Using (5) and (2), we can write the profit as

$$\pi_i = \frac{\overline{m_i}^{1-\theta(\sigma-1)}}{1-\theta(\sigma-1)} a_i^{1-\sigma} B + \frac{\overline{m_i}^{1-\theta(\sigma-1)}}{1-\theta(\sigma-1)} a_i^{1-\sigma} \phi B^* - (1-S)\overline{m_i} f - r, \tag{7}$$

where $B \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{\mu s L^w}{\Delta}$, $B^* \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{\mu(1-s)L^w}{\Delta^*}$ are measures of market potential that are exogenous from the point of view of an individual firm. $\phi \equiv \tau^{1-\sigma} \in [0,1]$ is the freeness of trade. It is seen from (7) that positive profits require the following assumption:

Assumption 1 $\theta < \frac{1}{\sigma-1}$

The assumption relates the decreasing returns to scale in introducing new varieties to the substitutability of varieties, which governs the mark-ups.

The price indices when firms aglomerate in the large country are

$$\begin{aligned}\Delta &\equiv P^{1-\sigma} = s\int_0^1 \left(\int_0^{\overline{m_i}} p_i(z)^{1-\sigma} dz\right) dG(a) + (1-s)\,\phi \int_{a_R}^1 \left(\int_0^{\overline{m_i}^*} p_i^*(z)^{1-\sigma} dz\right) dG(a) \\ &\quad + (1-s)\int_0^{a_R} \left(\int_0^{\overline{m_i}} p_i(z)^{1-\sigma} dz\right) dG(a),\end{aligned} \tag{8}$$

$$\begin{aligned}\Delta^* &\equiv P^{*(1-\sigma)} = s\phi\int_0^1 \left(\int_0^{\overline{m_i}} p_i(z)^{1-\sigma} dz\right) dG(a) + (1-s)\int_{a_R}^1 \left(\int_0^{\overline{m_i}^*} p_i^*(z)^{1-\sigma} dz\right) dG(a) \\ &\quad + (1-s)\,\phi\int_0^{a_R} \left(\int_0^{\overline{m_i}} p_i(z)^{1-\sigma} dz\right) dG(a),\end{aligned} \tag{9}$$

where a_R is the marginal cost of the firm, which is indifferent between locating in the two markets. We use $s > 0.5$ for the core's endowment (expenditure) share, and $1-s$ for the periphery's share.

Subsidies imply that firms not nessecarily agglomerate in the large economy. We will use the term *reverse sorting*, when firms agglomerate in the small country. This sorting pattern will imply that the equations or the price indices become:

$$\begin{aligned}\widetilde{\Delta} &\equiv \widetilde{P}^{1-\sigma} = \phi s\int_0^{\widetilde{a_R}} \left(\int_0^{\overline{m_i^*}} p_i^*(z)^{1-\sigma} dz\right) dG(a) + s\int_{\widetilde{a_R}}^1 \left(\int_0^{\overline{m_i}} p_i(z)^{1-\sigma} dz\right) dG(a) \\ &\quad + (1-s)\,\phi\int_0^1 \left(\int_0^{\overline{m_i^*}} p_i^*(z)^{1-\sigma} dz\right) dG(a),\end{aligned} \tag{10}$$

$$\widetilde{\Delta}^{*} \equiv \widetilde{P}^{*(1-\sigma)} = s\int_{0}^{\widetilde{a_R}}\left(\int_{0}^{\overline{m_i^*}} p_i^*(z)^{1-\sigma}dz\right)dG(a) + \phi s\int_{\widetilde{a_R}}^{1}\left(\int_{0}^{\overline{m_i}} p_i(z)^{1-\sigma}dz\right)dG(a)$$
$$+ (1-s)\int_{0}^{1}\left(\int_{0}^{\overline{m_i^*}} p_i^*(z)^{1-\sigma}dz\right)dG(a). \qquad (11)$$

We will denote the marginal cost of the indifferent firm $\widetilde{a_R}$ in the case of *reverse sorting*.

We can now calculate the profit-maximizing product scope of a firm i in each market from (7):

$$\overline{m_i}^{opt} = ((1-S)f)^{-\frac{1}{\beta}}(B+\phi B^*)^{\frac{1}{\beta}}a_i^{-\frac{1}{\theta}}, \qquad (12)$$

$$\overline{m_i}^{*opt} = ((1-S^*)f)^{-\frac{1}{\beta}}(\phi B+B^*)^{\frac{1}{\beta}}a_i^{-\frac{1}{\theta}}, \qquad (13)$$

where $\beta \equiv \theta(\sigma-1) > 0$. As noted there is no separate market entry cost, and a firm will therefore sell all it's varieties in both markets.[6] The expressions (12) and (13) show how the optimal product scope of a firm expands with the market size (B and B^*), and the subsidy level S, and it decreases with the firms' core marginal cost, a_i, and with the fixed cost, f. Firms trade off the increase in fixed cost against the additional operating profit of an extra variety when choosing product scope. More productive firms have lower marginal costs and higher operating profits. Their break-even fixed cost is consequently higher, meaning that such firms will opt for a wider product range. A lower fixed cost or a higer subsidy will for similar reasons lead to a wider product scope. Furthermore, a higher market potential implies that an extra variety adds more to the operating profit, meaning a higher break-even fixed cost, and consequently a wider product scope.[7]

2.2 Parametrization

In order to analytically solve for the equilibrium, we impose a Pareto distribution for firm productivity:

$$G(a) = \left(\frac{a}{a_0}\right)^k, \quad a \in [0,1] \qquad (14)$$

where a_0 is a scale parameter, which we normalize to one, and k is a shape parameter.

[6] The range of produced varieties might differ in the two markets, if firms could pay an additional fixed cost to produce abroad (conduct FDI) as in Baldwin and Ottaviano (2001). We do not allow for this possibility here. Forslid and Ekholm (2001) analyzes the location patterns in a single-product core-periphery model with horizontal and vertical FDI.

[7] Multiproduct firm models with cannibalization effects normally have a negative association between product scope and market size. However, Qiu and Zhou (2013) finds a positive relationship between product scope and firm level productivity and market size as in the present paper. Dhingra (2013) and Nocke and Yeaple (2014) finds that high productive firms expand their product scope in large markets whereas the opposite is true for low productive firms.

2.3 A benchmark: Two symmetric countries

Before turning to simulations of the general case we will here analytically solve the model when countries are symmetric. This provides a useful benchmark. Thus, we ere derive optimal policies when $s = 0.5$, and $S = S^*$. It is easy to show that the same results would hold in a closed economy. The importance of the symmetry assumption is that there is in this case no relocation of firms between the two countries.

Indirect per capita utility, given (1), is

$$W = \frac{\mu}{\sigma - 1} \ln \Delta + \mu \ln \mu + \overline{Y} - \mu - \frac{T}{s}, \tag{15}$$

where $\overline{Y} = \frac{\mu}{\sigma} + 1$, given that $L^w \equiv 1$. Taxes are determined by the subsidy level:

$$T = s \int_0^1 S f \overline{m}^{opt}(a) dG(a) = s \int_0^1 S f ((1-S) f)^{-\frac{1}{\beta}} (1+\phi)^{\frac{1}{\beta}} B^{\frac{1}{\beta}} a^{-\frac{1}{\theta}} dG(a), \tag{16}$$

where $B = B^*$. Solving the integral for $G(a) = a^k$, gives

$$T = s \frac{k}{k - \frac{1}{\theta}} S f ((1-S) f)^{-\frac{1}{\beta}} (1+\phi)^{\frac{1}{\beta}} B^{\frac{1}{\beta}}, \tag{17}$$

and ´solving for $\Delta = \int_0^1 \left(\int_0^{\overline{m}_i} p_i(z)^{1-\sigma} dz \right) dG(a)$ gives

$$\Delta = \xi (1+\phi)^{\frac{1}{\beta}} B^{\frac{1}{\beta}-1} f^{1-\frac{1}{\beta}} (1-S)^{1-\frac{1}{\beta}} \frac{k}{k - \frac{1}{\theta}}, \tag{18}$$

where $\xi \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{1}{1-\beta}$. Using $B = \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{\mu}{\sigma \Delta}$ and (18) gives

$$B^{\frac{1}{\beta}} = \frac{\mu (1-\beta)}{\sigma (1+\phi)^{\frac{1}{\beta}} f^{1-\frac{1}{\beta}} (1-S)^{1-\frac{1}{\beta}} \frac{k}{k-\frac{1}{\theta}}}, \tag{19}$$

which together with (17) gives

$$T = s \frac{S}{1-S} \frac{\mu (1-\beta)}{\sigma}$$

We can derive the optimal subsidy rate from the first order condition $\frac{\partial W}{\partial S} = 0$ (see Appendix 4.1):

$$S^{opt} = 1 - \frac{\sigma - 1}{\sigma}, \tag{20}$$

A higher elasticity of substitution implies that the utility value of more variety is lower, and the optimal subsidy consequently falls with σ.

The total available product variety in a country, N, is endogenous, despite the fixed mass of firms, because of the endogenous product scope. It can be calculated from (12):

$$N = 1 \cdot \int_0^1 \overline{m_i} dG = \frac{\mu}{\sigma} \frac{1 - \theta (\sigma - 1)}{f} \frac{1}{1 - S}. \tag{21}$$

Substituting $S = S^{opt}$ from (20) we get

$$N^{opt} = \frac{\mu}{\sigma - 1} \frac{1 - \beta}{f}. \tag{22}$$

This can be compared to variety in this economy without a subsidy. Substituting $S = 1$ into (21) gives $N = \frac{\mu}{\sigma} \frac{1-\theta(\sigma-1)}{f}$. Clearly $N < N^{opt}$. This is very similar to the original Dixit-Stiglitz (D-S) model, where variety also tends to be suboptimally low because $p > mc$ due to monopolistic competition. Variety in the D-S model with two symmetric countries is

$$N_{DS} = \frac{\mu}{f^E \sigma}, \tag{23}$$

where f^E is the fixed entry cost of firms. Allowing for subsidies of f^E gives $N_{DS} = \frac{\mu}{\sigma f^E (1-S)}$, and calculating the welfare maximizing subsidies using (15) gives

$$N_{DS}^{opt} = \frac{\mu}{f^E(\sigma - 1)}, \text{ and } S_{DS}^{opt} = 1 - \frac{\sigma - 1}{\sigma}. \tag{24}$$

Thus, the optimal subsidy is exactly the same in both models. Endogenous product scope in our model, with a fixed mass of firms, plays much the same role as the endogenous entry in the original D-S. However, the parameter θ, which determines the cost of increasing the product scope naturally enters the expression for optimal variety in our model (22).

The role of firm heterogeneity can be seen by calculating optimal variety in a Melitz type of model, N_{Mz}, where all firms produce and export (i.e., have no additional market entry costs when exporting). Allowing for subsidies of the firm entry costs in this model gives

$$N_{Mz} = \frac{\mu}{f^E \sigma} \frac{1 - \sigma + k}{k} \frac{1}{(1 - S)}. \tag{25}$$

The shape parameter in the productivity distribution is now important. Note, that this model approaches the standard case of homogeneous firms as k goes to infinity. Interestingly, the optimal subsidy is again the same in this model (see Appendix 2):

$$N_{Mz}^{opt} = \frac{\mu}{f^E (\sigma - 1)} \frac{1 - \sigma + k}{k}, \text{ and } S_{Mz}^{opt} = 1 - \frac{\sigma - 1}{\sigma} \tag{26}$$

Thus the optimal subsidies are the same in all three models as long as there is no relocation of firms (form the symmetry assumption). However the resulting optimal variety differs between the models. The shape parameter in the Pareto distribution plays a role in the Melitz model, where a higher k leads to higher optimal variety, because a higher k implies a larger share of low-productivity firms that are smaller. In our model the difficulty of of increasing the product scope, θ, matters, with a lower θ implying a higher optimal product scope. Interestingly, however, the shape parameter k turns out not to matter in our case.

2.4 Trade and relocation between asymmetric countries

We now acknowledge that firms can move between countries, once we allow for asymmetries in market size and in subsidy rates S and S^* . We keep subsidies exogenous for now. The incentive to move varies across firms. Substituting (12) into (7) gives the difference in profits in the two markets:

$$\pi_i - \pi_i^* = \left((1-S)^{1-\frac{1}{\beta}} (B+\phi B^*)^{\frac{1}{\beta}} - (1-S^*)^{1-\frac{1}{\beta}} (\phi B + B^*)^{\frac{1}{\beta}}\right) \cdot \tag{27}$$

$$a_i^{-\frac{1}{\theta}} f^{1-\frac{1}{\beta}} \left(\frac{\theta(\sigma-1)}{1-\theta(\sigma-1)}\right) \tag{28}$$

This expression reveals that the increase in profit from relocation is higher for a more productive firm (with lower a_i). We apply Baldwin and Okubo's (2006) assumption that the cost of relocation decreases as the relocation pressure falls.[8] This means that the most productive firm, which has the most to gain from relocation, will move first, and firms thereafter will relocate in order of productivity. Relocation will continue until profits are equalized or all firms are located in one country:

$$\pi_i = \pi_i^* \neg a_R \; or \; \widetilde{a_R} = 1 \tag{29}$$

This expression defines the marginal cost of a firm that is indifferent between locations for internal solutions, a_R. Firms with $a_i < a_R$ will find it profitable to relocate. Figures 6a, 6b, and 6c illustrate the intitial location pattern as well as the pattern after firms have relocated (sorted). In Figure 6b, the most productive firms move to the larger market. These are also, from (12) and (13), the firms with the widest product scope.[9] Figure 6c illustrates reverse sorting, where firms agglomerate in the small country. This happens when the subsidies in the small country are sufficiently higher than those in the large economy, so that the difference in subsidies outweighs the disadvantage of the smaller market.

[8] This assumption can be thought of as congestion at the border, or as a representation of a relocation service sector with a fixed capacity. The price of these services will fall as demand falls. Baldwin and Okubo (2014) provided more discussion on dynamic aspects of this. The relocation path in Baldwin and Okubo (2006) and the free entry model of Melitz model are different, but coincident in the long-run equilibrium.

[9] These firms will be the ones that are willing to pay the most for relocation services.

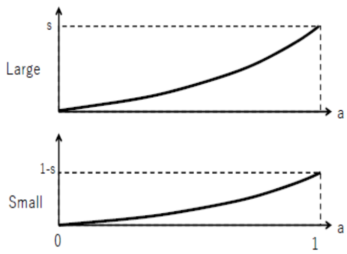

Figure 1a: Initial equilibrium

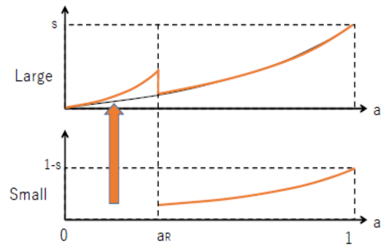

Figure 1b: Sorting to the large country

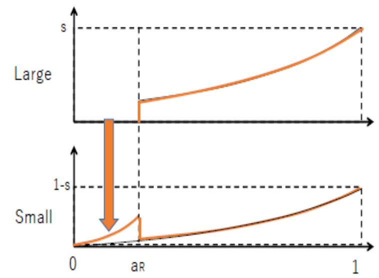

Figure 1c: Sorting to the small country

2.4.1 Internal equilibrium ($0 < a_R < 1$)

Relocation costs fall to zero in equilibrium where relocation stops, which implies that $\pi_i = \pi_i^*$. This, in turn - from (27) - means that

$$(1-S)^{\beta-1}(B+\phi B^*) = (1-S^*)^{\beta-1}(\phi B + B^*) \tag{30}$$

Solving (30) using (5) and (12) gives[10]

$$a_R^{k-\frac{1}{\theta}} = \frac{\phi(2s-1) - s\Omega^{\beta-1} + (1-s)\Omega^{1-\beta}}{(1-s)(-\frac{1}{\phi} - \phi + \Omega^{\beta-1} + \Omega^{1-\beta})} \tag{31}$$

where $\Omega \equiv \frac{1-S}{1-S^*}$, and in the case of reverse sorting we have

$$\widetilde{a_R}^{k-\frac{1}{\theta}} = \frac{(\phi(1-2s) + s\Omega^{\beta-1} - (1-s)\Omega^{1-\beta})}{s\{-\frac{1}{\phi} - \phi + \Omega^{\beta-1} + \Omega^{1-\beta}\}} \tag{32}$$

The tendency to relocate (agglomerate) depends, as usual, on the relative market size s and the level of trade costs ϕ. Here it also depends on the relative subsidy level Ω and on the cost of increasing the product scope θ, and we will concentrate on these more novel effects.

The effect of θ can be illustrated by the following proposition

Proposition 1 $\left.\frac{\partial a_R}{\partial \theta}\right|_{S=S^*, s>\frac{1}{2}} < 0$, $\left.\frac{\partial a_R}{\partial \theta}\right|_{S>S^*, s=\frac{1}{2}}, < 0$

Proof. *See Appendix 4.3 and 4.4* ■

The proposition illustrates that a higher θ leads to less relocation due to differences in market size and in subsidies.

The effect of higher subsidies is straightforward

Proposition 2 $\frac{\partial a_R^{k-\frac{1}{\theta}}}{\partial \Omega} < 0 \; for \quad s > 0.5$

[10] See Appendix 4.2 for a derivation of this expression.

Proof. See Appendix 4.5 ■

Thus, higher unilateral subsidies attract firms. Note also that for $\Omega = 1$, $a_R^{k-\frac{1}{\theta}} = \frac{\phi(2s-1)}{(1-s)(1-\phi)}$. That is, location is always unaffected by symmetric subsidies.

2.5 Welfare effects of symmetric (cooperative) subsidies

Subsidies attract firms but they also imply higher taxes. Welfare increases as a result of more variety (higher Δ), but it decreases with taxes, as can be seen from (15). The net effect therefore depends, and here we resort to simulations to illustrate the welfare effect of subsidies.

First, consider symmetric or cooperative subsidies. Figure 1 ($\sigma = 3, k = 7, \theta = 0.35, f = 2, \mu = 0.5, s = 0.55$) plots the effect of symmetrically increasing subsidies for the two countries. The figure shows the welfare in the two countries for two.levels of trade freeness.

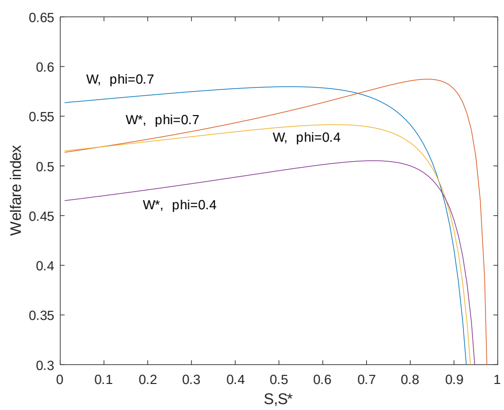

Figure 2: Welfare as a function of S,S^* ($s = 0.55, \sigma = 3, k = 7, \theta = 0.35, \mu = 0.5, f = 2$)

It can be seen that the large country initially has a higher welfare index. This is because it has a greater mass of domestic product variety, which translates into a lower price index. Higher subsidies increase welfare as long as they are below the optimal level. However, the optimal level of subsidies is higher in the smaller country, and welfare in the small country actually exceeds welfare in the large country for sufficiently large subsidies. The reason for this is that most varieties are produced in the large country, which has also attracted the most productive firms from the small economy ($a_R > 0$), and the subsidies for all these varieties are financed by a tax on consumers in the large economy. Thus, the small country is free-riding on the subsidies in the large economy through it's imports. The value of more foreign variety is higher for the importer the freer trade is. This effect is seen in Figure 2, where the utility in the small economy continues to increase even after the welfare in the large country has started to decrease in S.

Thus, up to a point, welfare increases in both economies with higher cooperative subsidies. However, the optimal cooperative subsidy level tends to be higher in the small economy due to the spillovers.

2.6 Welfare effects of unilateral subsidies in the large country

We here consider unilateral subsidies in the large economy. Figure 3a shows the welfare effects in the two countries for two different levels of trade costs, and Figure 3b illustrates the corresponding degree of agglomeration as shown by a_R. Welfare is typically higher in the large economy, but it is seen how welfare declines as subsidies are raised above what is optimal. The small country experiences lower welfare as industry moves away. However, higher subsidies begin to increase welfare in the small economy once all industry is agglomerated in the large country. This is because the small country benefits from more variety due to higher subsidies without having to bear any of the tax associated with the subsidy. High enough unilateral subsidies can therefore at some point ($S > \widetilde{S}$) lead to a higher welfare in the small deindustrialized country than in the large one.

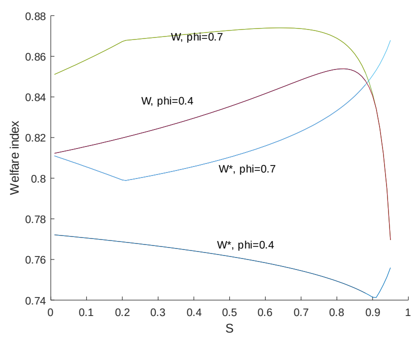

Figure 3a: Welfare as a function of S ($s = 0.55, S^* = 0.01, \sigma = 3.5, k = 7, \theta = 0.35, \mu = 0.5, f = 2$)

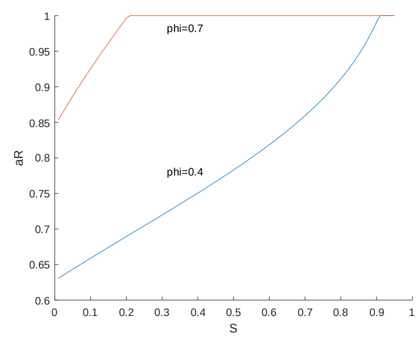

Figure 3b: a_R as a function of S ($s = 0.55, S^* = 0.01, \sigma = 3.5, k = 7, \theta = 0.35, \mu = 0.5, f = 2$)

The sustain point where full agglomeration occurrs is found by setting $a_R = 1$ in (31).

$$\Omega^{sust} = \left(s\phi + \frac{1-s}{\phi} \right)^{\frac{1}{\beta}} \tag{33}$$

This equation defines subsidy levels that lead to full agglomeration (the sustain point) for given s and ϕ. That is, it defines the kinks where $a_R = 1$ in Figure 3b. It may also be noted that for symmetric subsidies, $\Omega = 1$, we get $\phi^{sust} = \frac{1-s}{s}$, which is the standard sustain point in trade and geography models.

It is possile to derive the optimal subsidy S^{opt}, in the case when the optimum uccurs after full agglomeration, as for example for $\phi = 0.7$ in Figure 3à:

$$S^{opt} = 1 - \frac{1}{s} \frac{\sigma - 1}{\sigma} \tag{34}$$

This expression is very similar to the one we derived in the symmetric case in (20), but now the size of the economy also matters. A larger economy tends to have a higher optimal subsidy, since it can spread the tax on a larger population.

The difference in per capita welfare at S^{opt} is given by

$$W - W^* = \frac{\mu}{(\sigma-1)} \left((\frac{1-\sigma(1-s))}{s\sigma}) (1-\theta(\sigma-1)) - \ln\phi \right) \geq 0, \tag{35}$$

and this difference depends on ϕ, and θ accorting to:

Proposition 3 $\frac{\partial(W-W^*)}{\partial\phi} < 0$ *and* $\frac{\partial(W-W^*)}{\partial\theta} < 0$ *when there is full aglomeration in the large country,* $a_R = 1$.

Proof. *The result is seen directly from equation (35).* ■

The welfare difference decreases with ϕ, because the advantage of having all industry located in the large country disappears as trade costs go to zero, and a higher θ shrinks the welfare difference, because the subsidy is less efficient at generating more variety at larger θ.

2.7 Welfare effects of unilateral subsidies in the small country

Next, consider unilateral subsidies by the small country. A higher S^* will dampen the agglomeration of firms into the large market and if large enough reverse the agglomeration pattern. If the subsidies are large enough, firms in the large economy will begin to relocate to the small country.[11] Again, from (27), the relocation starts with the most productive firms. Figure 4a,b shows the effect of increasing subsidies.

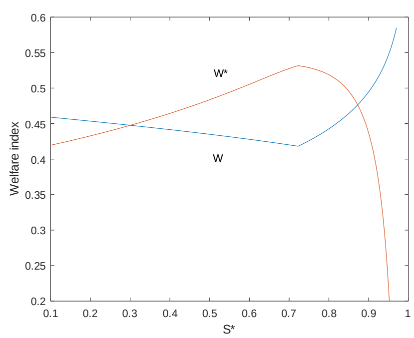

Figure 4a: Welfare as a function of S^* ($s = 0.55, S = 0.01, \sigma = 3.5, k = 7, \theta = 0.35, \mu$

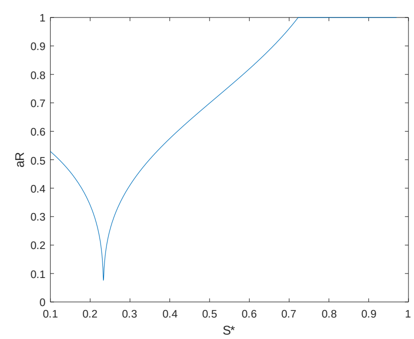

Figure 4b: a_R and $\widetilde{a}_R$ as a functions of S^* ($s = 0.55, S^* = 0.01, \sigma = 3.5, k = 7, \theta = 0.35, \mu = 0.5, f = 2$)

Some firms have already relocated to the large economy at the starting point ($a_R = 0.52$) with symmetric subsidies $S = S^* = 0.1$ due to the difference in market size. As S^* increases fewer and fewer of the small country firms will find it optimal to relocate to the core, and as S^* becomes high enough the two regions become equally attractive.[12] This point, where $a_R = 0$,

[11] Note that each new subsidy level is analyzed after firms have been reborn.

[12] The difference in subsidies at this point is found by setting $a_R = 0$ in (31): $\Omega^\beta = \frac{-(2s-1)\phi+\sqrt{(2s-1)^2+4s(1-s)}}{2(1-s)}$

is shown in the right hand side panel of Figure 4b. Further increases in subsidies in the small economy cause sorting from large to small (reverse sorting), and $\widetilde{a}_R$ now increases as more and more firms relocate from large to small. Finally, all firms agglomerate in the small economy ($a_R = 1$) if S^* is sufficiently high. Welfare in the small economy overtakes the large economy relatively early, but higher subsidies start to benefit the deindustrialized large country after full agglomeration to the small country, and further increases in subsidies lead to another reversal of welfare levels. This again depends on the positive spillovers from foreign subsidies, which lead to a lower price index, while all taxes are paid in the industrialized economy.

The sustain point for full agglomeration in the small country is found by setting $\widetilde{a}_R = 1$ in (32):

$$\Omega^{sust} = \left((1-s)\,\phi + \frac{s}{\phi} \right)^{\frac{1}{1-\beta}} \tag{36}$$

The simulations show that unilateral subsidies are attractive as long as they do not become too large, but that they often hurt the other country by causing firms to relocate to the country with higher subsidies. It may also be difficult for the countries to agree on a common subsidy rate, because countries of different sizes have different optimal rates (as seen in Figure 2). We therefore next turn to simulations of a Nash game of subsidies.

2.8 Nash subsidies

It is seen, in Figure 3a, how S^* is a strategic complement to S before the point of full agglomeration ($a_R = 1$). We can also see that this is not the case after full agglomeration in Home. A higher S, after full agglomeration, also benefits the foreign country. However, the game disappears at this point, since S^* does not matter if there are no firms in Foreign, and Home with all firms will simply set subsidies optimally. The same thing happens when all firms agglomerate in Foreign: subsidies in Home cease to matter and Foreign sets its optimal subsidy rate. Therefore we simulate Nash reaction functions when we do not have full agglomeration ($a_R < 1$). The simulated reaction functions show the welfare optimizing subsidy rate of one country for each level of subsidy in the other country.

In the simulations, we will focus on the effect of the new parameter θ as well as the effect of lower trade costs. Turning first to the effect of θ, Figure 5 plots the reaction functions for the two values of θ ($s = 0.55, k = 7, \sigma = 3, \mu = 0.5, \theta = 0.25, f = 2, \phi = 0.4$). It is shown how a lower theta leads to lower equilibrium subsidies. A lower θ shifts the entire reaction function inward, implying that the effect on the product scope outweighs the substitution effect (the substitution of higher taxes for higher variety (product scope). As a result, Nash susbsidies increase with θ.

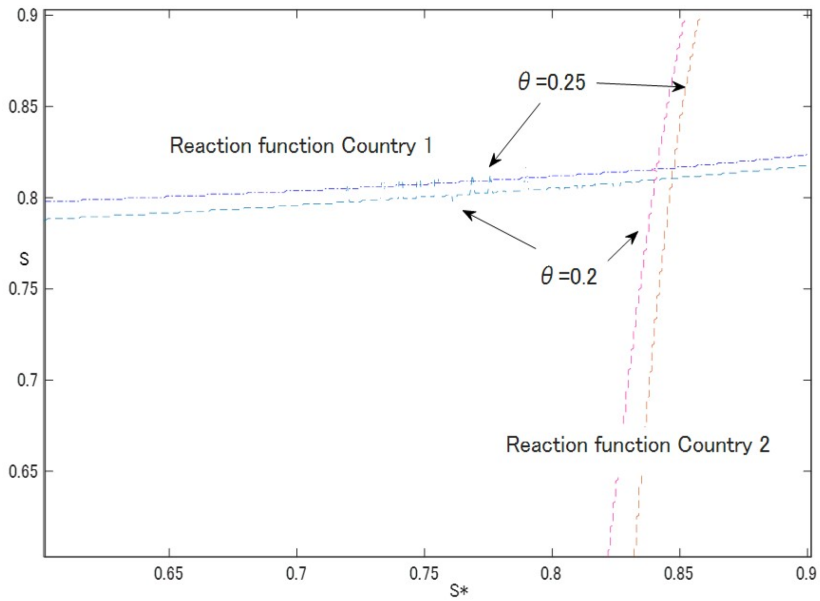

Figure 5: Simulated reaction functions for $\theta = 0.2$ and $\theta = 0.25$
$(s = 0.55, k = 7, \sigma = 3, \mu = 0.5, f = 2, \phi = 0.4)$

The effect of higher trade costs (lower ϕ) is shown in Figure 6. Higher trade costs (lower ϕ) imply higher Nash subsidies. Higher trade costs imply that less of the subsidy benefits "leak" abroad, and it becomes increasingly important to have the subsidized firms located in the domestic economy. This implies that trade wars and subsidy wars would tend to go hand in hand.

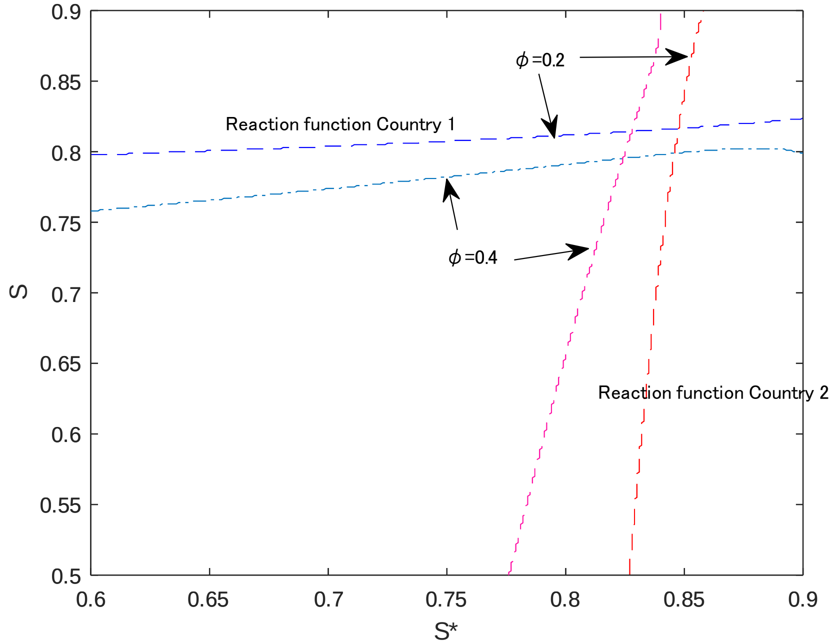

Figure 6: Simulated reaction functions for
$s = 0.55, k = 7, \sigma = 3, \mu = 0.5, \theta = 0.25, f = 2$

3 Conclusions

This paper analyzes subsidies that target the fixed costs of developing new product varieties. We use a setting with international trade and multiproduct heterogeneous firms, which allows us to analyze both the relocation of heterogeneous firms and changes in product variety.

We show that our model has the same optimal subsidies as a standard Dixit-Stiglitz model and as a Melitz model as long as countries are symmetric so that no relocation of firms takes place. However the optimal product variety resulting from these subsidies is different in the models.

Our simulations show how unilaterally higher subsidies tend to benefit a country and attract firms. Subsidies also spill over to the other country in the form of increased product variety (higher product scope). Higher domestic subsidies always benefit the foreign country once all firms are agglomerated in the home country. Increasing subsidies after full agglomeration can even lead to a reversal where the country without firms has higher welfare becasue it benefits from more variety while not having to pay any taxes to finance the subsidies.

The model provides a theoretical rationale for why higher subsidies would tend to be introduced simultaneously with higher trade costs. Higher trade costs make it more important from a welfare perspective to have firms located in the domestic economy since domestically produced varieties are sold without trade costs. We also simulate how higher trade costs lead to higher Nash subsidies. This is consistent with recent developments in the world economy.

References

[1] Baldwin, R.E., and Okubo, T. (2006). Heterogeneous firms, agglomeration and economic geography: spatial selection and sorting. Journal of Economic Geography, 6, pp. 323-346. https://doi.org/10.1093/jeg/lbi020

[2] Baldwin, R. E., & Okubo, T. (2014). International trade, offshoring and heterogeneous firms. Review of International Economics, 22(1), 59-72.

[3] Baldwin, Richard and Ottaviano, Gianmarco, (2001), multi-product multinationals and reciprocal FDI dumping, Journal of International Economics, 54, issue 2, p. 429-448. https://doi.org/10.1016/S0022-1996(00)00099-4

[4] Bernard, A. B., Jensen, J. B., Redding, S. J., & Schott, P. K. (2018). Global firms. Journal of Economic Literature, 56(2), 565-619. https://doi.org/10.1257/jel.20160792

[5] Bernard, A.B. and Okubo, T. (2016). Product switching and the business cycle (No. w22649). National Bureau of Economic Research. https://doi.org/10.3386/w22649

[6] Bernard, A.B., Redding, S.J. and Schott, P.K. (2010). Multiple-product firms and product switching. American Economic Review, 100(1), pp.70-97. https://doi.org/10.1257/aer.100.1.70

[7] Bernard, A.B., Redding, S.J. and Schott, P.K. (2011). multi-product firms and trade liberalization. The Quarterly Journal of Economics, 126(3), pp.1271-1318. https://doi.org/10.1093/qje/qjr021

[8] Blanchard, P., Gaigné, C. and Mathieu, C. (2012). Trade costs and international strategy of firms: the role of endogenous product differentiation. Regional Science and Urban Economics, 42(6), pp.1023-1036. https://doi.org/10.1016/j.regsciurbeco.2012.07.008

[9] Brander, J. A., & Spencer, B. J. (1985). Export subsidies and international market share rivalry. Journal of International Economics, 18(1-2), 83-100.

[10] Chisholm, D.C. and Norman, G. (2004). Heterogeneous preferences and location choice with multi-product firms. Regional Science and Urban Economics, 34(3), pp.321-339. https://doi.org/10.1016/j.regsciurbeco.2003.05.001

[11] Dhingra, S. (2013). Trading Away Wide Brands for Cheap Brands, American Economic Review vol. 103(6), pages 2554-2584, October. https://doi.org/10.1257/aer.103.6.2554

[12] Eaton, J., & Grossman, G. M. (1986). Optimal trade and industrial policy under oligopoly. The Quarterly Journal of Economics, 101(2), 383-406.

[13] Eckel, C., Iacovone, L., Javorcik, B. and Neary, J.P. (2015). Multi-product firms at home and away: Cost-versus quality-based competence. Journal of International Economics, 95(2), pp.216-232. https://doi.org/10.1016/j.jinteco.2014.12.012

[14] Eckel, C. and Neary, J.P. (2010). Multi-product firms and flexible manufacturing in the global economy. The Review of Economic Studies, 77(1), pp.188-217. https://doi.org/10.1111/j.1467-937X.2009.00573.x

[15] Feenstra, Robert, and Hong Ma. (2007). "Optimal Choice of Product Scope for multi-product Firms under Monopolistic Competition." National Bureau of Economic Research Working Paper 13703. https://doi.org/10.3386/w13703

[16] Flach, L. and Irlacher, M. (2018). Product versus Process: Innovation Strategies of multi-product Firms. American Economic Journal: Microeconomics, 10(1), pp.236-77. https://doi.org/10.1257/mic.20150272

[17] Forslid, R., & Okubo, T. (2023). Trade, location, and multi-product firms. Regional Science and Urban Economics, 100, 103891.

[18] Haaland, J. I., & Kind, H. J. (2008). R&D policies, trade and process innovation. Journal of International Economics, 74(1), 170-187.

[19] Hoekman, B., & Nelson, D. (2020). Rethinking international subsidy rules. The World Economy, 43(12), 3104-3132.

[20] Ju, Jiandong. (2003). "Oligopolistic Competition, Technology Innovation, and multi-product Firms." Review of International Economics 11 (2): 346–59.

[21] Kondo, H. (2013). International R&D subsidy competition, industrial agglomeration and growth. Journal of International Economics, 89(1), 233-251.

[22] Martin, P. and C.A. Rogers (1995) "Industrial location and public infrastructure," Journal of International Economics 39, pp. 335–351. https://doi.org/10.1016/0022-1996(95)01376-6

[23] Mayer, T., Melitz, M. J., & Ottaviano, G. I. (2014). Market size, competition, and the product mix of exporters. The American Economic Review, 104(2), 495-536. https://doi.org/10.1257/aer.104.2.495

[24] Mayer, T., Melitz, M. J., & Ottaviano, G. I. (2021). Product mix and firm productivity responses to trade competition. Review of Economics and Statistics, 103(5), 874-891. https://doi.org/10.1162/rest

[25] Melitz, M. J. (2003). "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity", *Econometrica*, 71(6), pp. 1695–1725. https://doi.org/10.1111/1468-0262.00467

[26] Melitz, M.J. and G.I.P.Ottaviano, (2007). "Market Size, Trade, and Productivity", *Review of Economic Studies*, 75 (1), pp. 295-316. https://doi.org/10.1111/j.1467-937X.2007.00463.x

[27] Melitz, M. J., & Redding, S. J. (2015). New trade models, new welfare implications. *American Economic Review*, 105(3), pp.1105-46. https://doi.org/10.1257/aer.20130351

[28] Nocke, V., & Yeaple, S. (2014). Globalization and multi-product firms. International Economic Review, 55(4), 993-1018. https://doi.org/10.1111/iere.12080

[29] Qiu, Larry D. & Zhou, Wen, (2013). multi-product firms and scope adjustment in globalization, Journal of International Economics vol. 91(1), pages 142-153. https://doi.org/10.1016/j.jinteco.2013.04.006

[30] Schoar, A. (2002). Effects of corporate diversification on productivity. The Journal of Finance, 57(6), pp.2379-2403. https://doi.org/10.1111/1540-6261.00500

4 Appendix

4.1 Equilibrium with symmetric countries

4.1.1 Our model

Social welfare is given as quasi-linear

$$W = \frac{\mu}{\sigma - 1} \ln \Delta + \overline{Y} - \mu - T,$$

$$\begin{aligned}
\Delta^{\frac{1}{\theta(\sigma-1)}} &= \frac{1}{1-\theta(\sigma-1)} \frac{k}{k-\frac{1}{\theta}} \left((1+\phi)\frac{\mu}{\sigma}\right)^{\frac{1}{\theta(\sigma-1)}-1} f^{1-\frac{1}{\theta(\sigma-1)}} (1-S)^{1-\frac{1}{\theta(\sigma-1)}} (1+\phi) \\
\Delta &= \left(\frac{1}{1-\theta(\sigma-1)} \frac{k}{k-\frac{1}{\theta}}\right)^{\theta(\sigma-1)} (1+\phi) f^{\theta(\sigma-1)-1} \left(\frac{\mu}{\sigma}\right)^{1-\theta(\sigma-1)} (1-S)^{\theta(\sigma-1)-1}
\end{aligned}$$

$$\frac{d \ln \Delta}{dS} = -(\theta(\sigma-1)-1)\frac{1}{1-S} = (1-\theta(\sigma-1))\frac{1}{1-S}$$

$$\begin{aligned}
T &= \frac{k}{k-\frac{1}{\theta}} \frac{1}{0.5} S((1-S)f)^{-\frac{1}{\theta(\sigma-1)}} f((1+\phi)B)^{\frac{1}{\theta(\sigma-1)}} \\
&= \frac{k}{k-\frac{1}{\theta}} S((1-S)f)^{-\frac{1}{\theta(\sigma-1)}} f(1+\phi)^{\frac{1}{\theta(\sigma-1)}} \left(\frac{1}{1-\theta(\sigma-1)} \frac{k}{k-\frac{1}{\theta}}\right)^{-1} (1+\phi)^{-\frac{1}{\theta(\sigma-1)}} f^{\frac{1}{\theta(\sigma-1)}-1} \frac{\mu}{\sigma} (1-S)^{\frac{1}{\theta(\sigma-1)}-1} \\
&= \frac{S}{1-S} \frac{\mu(1-\theta(\sigma-1))}{\sigma}
\end{aligned}$$

$$\frac{dT}{dS} = \frac{\mu(1-\theta(\sigma-1))}{\sigma} \frac{1}{(1-S)^2}$$

Thus, the differentiation of social welfare is is given as

$$\frac{\mu}{\sigma-1}\frac{d\ln\Delta}{dS} - \frac{dT}{dS} = \frac{\mu}{\sigma-1}(1-\theta(\sigma-1))\frac{1}{1-S} - \frac{\mu(1-\theta(\sigma-1))}{\sigma}\frac{1}{(1-S)^2}$$

$$\frac{1}{\sigma-1}\frac{1}{1-S} = \frac{1}{\sigma}\frac{1}{(1-S)^2}$$

Substituting (18),(19) and (**??**) into (15), and thereafter maximizing w.r.t. S gives

$$S^{opt} = 1 - \frac{\sigma-1}{\sigma}, \tag{37}$$

$$T^{opt} = \frac{S}{1-S}\frac{\mu(1-\theta(\sigma-1))}{\sigma} = \frac{1-\frac{\sigma-1}{\sigma}}{\frac{\sigma-1}{\sigma}}\frac{\mu(1-\theta(\sigma-1))}{\sigma} = \frac{\mu(1-\theta(\sigma-1))}{\sigma(\sigma-1)}$$

$$\begin{aligned} N^{opt} &= \overline{m_i} = ((1-S)f)^{-\frac{1}{\theta(\sigma-1)}} ((1+\phi)B)^{\frac{1}{\theta(\sigma-1)}} \\ &= \frac{\mu(1-\theta(\sigma-1))}{f(\sigma-1)} \end{aligned}$$

4.1.2 DS model

Free entry condition is given by

$$\pi = \frac{\mu}{\sigma}\left(\frac{s}{\Delta} + \phi\frac{1-s}{\Delta^*}\right) = \frac{\mu}{\sigma}\left(\frac{s}{n+\phi n^*} + \phi\frac{1-s}{\phi n + n^*}\right) = F(1-S)$$

Two symmetric countries, $N = N^*$, gives us:

$$N = \frac{\mu}{\sigma}\left(\frac{s}{F(1-S)}\right)$$

$$\begin{aligned} W &= \frac{\mu}{\sigma-1}\ln\Delta + \overline{Y} - \mu - T = \frac{\mu}{\sigma-1}\ln(n+\phi n^*) + \overline{Y} - \mu - T \\ &= \frac{\mu}{\sigma-1}\ln\frac{\mu}{\sigma}(1+\phi)\left(\frac{s}{F}\right) + \overline{Y} - \mu - \frac{\mu}{\sigma-1}\ln(1-S) - \frac{\mu}{\sigma}\left(\frac{S}{1-S}\right) \end{aligned}$$

$$\frac{dW}{dS} = \frac{\mu}{\sigma-1}\frac{1}{1-S} - \frac{\mu}{\sigma}\frac{1}{(1-S)^2} = 0$$

$$S^{opt} = 1 - \frac{\sigma-1}{\sigma}$$

$$N^{opt} = \frac{\mu}{\sigma}\left(\frac{1}{F(1-S)}\right) = \frac{\mu}{F(\sigma-1)}$$

Note that the number of firms at equilibrium, $S = 0$, is given as $N = \frac{\mu}{F\sigma}$, which is well-known as market equilibrium in DS model.

Tax is

$$T^{opt} = \frac{\mu}{(\sigma-1)}\left(1 - \frac{\sigma-1}{\sigma}\right) = \frac{\mu}{\sigma(\sigma-1)}$$

4.1.3 Melitz model

Free entry condition is given as

$$\pi = \frac{\mu}{\sigma}\left(\frac{s}{\Delta}\right)a_D^{1-\sigma} = \frac{\mu}{\sigma}\frac{1}{\frac{k}{1-\sigma+k}(a_D^{1-\sigma+k}+\phi a_D^{1-\sigma+k})}a_D^{1-\sigma} = F(1-S)$$

$$\frac{\mu}{\sigma s}\frac{1-\sigma+k}{k}\frac{1}{F(1-S)} = a_D^k$$

$$\Delta = 0.5\frac{k}{1-\sigma+k}\left(\frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{1}{F(1-S)}\right)^{\frac{1-\sigma+k}{k}}$$

Tax is given as

$$T = 0.5a_D^k FS/s = \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{S}{(1-S)}$$

$$\frac{dT}{dS} = \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{1}{(1-S)^2}$$

Welfare is given by

$$\frac{dW}{dS} = \frac{\mu}{\sigma-1}\frac{1-\sigma+k}{k}\frac{1}{1-S} - \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{1}{(1-S)^2} = 0$$

Solving this, we get

$$S^{opt} = 1 - \frac{\sigma-1}{\sigma}$$

$$T^{opt} = \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{S}{(1-S)} = \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{1}{\sigma-1}$$

$$\begin{aligned} N^{OPT} &= \frac{\mu}{\sigma}\frac{1-\sigma+k}{k}\frac{1}{F(1-S)} \\ &= \frac{\mu}{(\sigma-1)F}\frac{1-\sigma+k}{k} \end{aligned}$$

If S=0 (no subsidies), then $N = \frac{\mu}{\sigma F}\frac{1-\sigma+k}{k}$, which is well-known as market equilibrium in Melitz model.

4.2 Deriving a_R

$$\pi_i = \frac{\mu}{\sigma}\delta\left(\frac{\overline{m_i}^{1-\theta(\sigma-1)}}{1-\theta(\sigma-1)}a_i^{1-\sigma}B + \frac{\overline{m_i}^{1-\theta(\sigma-1)}}{1-\theta(\sigma-1)}a_i^{1-\sigma}\phi B^*\right) - (1-S)\overline{m_i}f - r, \tag{38}$$

$$\overline{m_i}^{opt} = ((1-S)f)^{-\frac{1}{\theta(\sigma-1)}}(B+\phi B^*)^{\frac{1}{\theta(\sigma-1)}}a_i^{-\frac{1}{\theta}}. \tag{39}$$

Substituting (39) into (38) gives after simplification

$$\pi_i = \left(\frac{\theta(\sigma-1)}{1-\theta(\sigma-1)}\right)((1-S)f)^{1-\frac{1}{\theta(\sigma-1)}}(B+\phi B^*)^{\frac{1}{\theta(\sigma-1)}}a_i^{-\frac{1}{\theta}} \tag{40}$$

The profit differential between markets is

$$\pi_i - \pi_i^* = \left((1-S)^{1-\frac{1}{\theta(\sigma-1)}}(B+\phi B^*)^{\frac{1}{\theta(\sigma-1)}} - (1-S^*)^{1-\frac{1}{\theta(\sigma-1)}}(\phi B+B^*)^{\frac{1}{\theta(\sigma-1)}}\right)a_i^{-\frac{1}{\theta}}f^{1-\frac{1}{\theta(\sigma-1)}}\left(\frac{\theta(\sigma-1)}{1-\theta(\sigma-1)}\right) \tag{41}$$

a_R is found by solving (41) for $\pi_i - \pi_i^* = 0$.

For $\pi_i - \pi_i^* = 0$, (41) may be written as

$$\left(\frac{1-S}{1-S^*}\right)^{\theta(\sigma-1)-1} = \left(\frac{\phi B + B^*}{B+\phi B^*}\right) \tag{42}$$

Now $B \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\frac{\mu s L^w}{\Delta}$, $B^* \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}\frac{\mu(1-s)L^w}{\Delta^*}$, which gives

$$\left(\frac{1-S}{1-S^*}\right)^{\theta(\sigma-1)-1} = \left(\frac{\phi s\Delta^* + (1-s)\Delta}{s\Delta^* + \phi(1-s)\Delta}\right) \tag{43}$$

To solve the RHS of this expression note that Δ, Δ^* are defined by:

$$\begin{aligned}\Delta \equiv P^{1-\sigma} &= s\int_0^1\left(\int_0^{\overline{m_i}}p_i(z)^{1-\sigma}dz\right)dG(a) + (1-s)\phi\int_{a_R}^1\left(\int_0^{\overline{m_i}^*}p_i^*(z)^{1-\sigma}dz\right)dG(a)\\ &+ (1-s)\int_0^{a_R}\left(\int_0^{\overline{m_i}}p_i(z)^{1-\sigma}dz\right)dG(a),\end{aligned} \tag{44}$$

$$\begin{aligned}\Delta^* \equiv P^{*(1-\sigma)} &= s\phi\int_0^1\left(\int_0^{\overline{m_i}}p_i(z)^{1-\sigma}dz\right)dG(a) + (1-s)\int_{a_R}^1\left(\int_0^{\overline{m_i}^*}p_i^*(z)^{1-\sigma}dz\right)dG(a)\\ &+ (1-s)\phi\int_0^{a_R}\left(\int_0^{\overline{m_i}}p_i(z)^{1-\sigma}dz\right)dG(a).\end{aligned} \tag{45}$$

Solving these integrals and substituting (12) and (13) gives

$$\Delta = \xi(B+\phi B^*)^{\frac{1}{\theta(\sigma-1)}-1}f^{1-\frac{1}{\theta(\sigma-1)}}(1-S)^{1-\frac{1}{\theta(\sigma-1)}}\left(s+\phi(1-s)\left(\frac{1-S}{1-S^*}\right)^{1-\theta(\sigma-1)}\left(1-a_R^{k-\frac{1}{\theta}}\right)+(1-s)a_R^{k-\frac{1}{\theta}}\right) \tag{46}$$

$$\Delta^* = \xi \left(B + \phi B^*\right)^{\frac{1}{\theta(\sigma-1)}-1} \left(\left(1-S\right)f\right)^{1-\frac{1}{\theta(\sigma-1)}} \left(\phi s + (1-s)\left(\frac{1-S}{1-S^*}\right)^{1-\theta(\sigma-1)} \left(1 - a_R^{k-\frac{1}{\theta}}\right) + \phi\left(1-s\right) a_R^{k-\frac{1}{\theta}}\right) \tag{47}$$

where $\xi \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{1}{1-\theta(\sigma-1)}$

Next defining $\Psi \equiv \xi \left(B + \phi B^*\right)^{\frac{1}{\theta(\sigma-1)}-1} f^{1-\frac{1}{\theta(\sigma-1)}} \left(1-S\right)^{1-\frac{1}{\theta(\sigma-1)}}$ gives

$$\begin{aligned} & s\Delta^* + \phi(1-s)\Delta = \Psi\phi s\left(\phi s + (1-s)\left(\frac{1-S}{1-S^*}\right)^{1-\theta(\sigma-1)}\left(1-a_R^{k-\frac{1}{\theta}}\right) + \phi\left(1-s\right)a_R^{k-\frac{1}{\theta}}\right) \qquad (48) \\ & +\Psi(1-s)\left(s + \phi\left(1-s\right)\left(\frac{1-S}{1-S^*}\right)^{1-\theta(\sigma-1)}\left(1-a_R^{k-\frac{1}{\theta}}\right) + \left(1-s\right)a_R^{k-\frac{1}{\theta}}\right) \\ = \; & \Psi(1-s)\left(\frac{\phi^2 s^2}{1-s} + s + \phi\left(\frac{1-S}{1-S^*}\right)^{-\beta} + (\phi^2 s + 1 - s - \phi\left(\frac{1-S}{1-S^*}\right)^{-\beta})a_R^{k-\frac{1}{\theta}}\right) \end{aligned}$$

where $\beta \equiv \theta\left(\sigma-1\right) - 1$ and similarly

$$\begin{aligned} & s\Delta^* + \phi(1-s)\Delta \qquad (49) \\ = \; & \Psi(1-s)\left(\frac{\phi s^2}{1-s} + \phi s + \left(s + \phi^2\left(1-s\right)\right)\left(\frac{1-S}{1-S^*}\right)^{-\beta} + \left(\phi - \left(s+\phi^2\left(1-s\right)\right)\left(\frac{1-S}{1-S^*}\right)^{-\beta}\right)a_R^{k-\frac{1}{\theta}}\right) \end{aligned}$$

Substituting (48) and (49) into (43) ad solving for a_R gives the expression in (31):

$$a_R^{k-\frac{1}{\theta}} = \frac{\phi(2s-1) - s\left(\frac{1-S}{1-S^*}\right)^{\beta} + (1-s)\left(\frac{1-S}{1-S^*}\right)^{-\beta}}{(1-s)(-\frac{1}{\phi} - \phi + \left(\frac{1-S}{1-S^*}\right)^{\beta} + \left(\frac{1-S}{1-S^*}\right)^{-\beta})} = \frac{\phi(2s-1) - s\Omega^{\beta} + (1-s)\Omega^{-\beta}}{(1-s)(-\frac{1}{\phi} - \phi + \Omega^{\beta} + \Omega^{-\beta})} \tag{50}$$

where $\Omega \equiv \left(\frac{1-S}{1-S^*}\right)$.

4.3 Proof that $\frac{\partial a_R}{\partial \theta}\Big|_{S=S^*,\ s>\frac{1}{2}} < 0$

Substituting $S = S^*$ into (31) gives

$$a_R = \left(\frac{\left(2s-1\right)\phi}{\left(1-\phi\right)\left(1-s\right)}\right)^{\frac{1}{k-\frac{1}{\theta}}} \tag{51}$$

It is seen diretly from this expression that $\frac{\partial a_R}{\partial \theta} < 0$ ■

4.4 Proof that $\frac{\partial a_R}{\partial \theta}\Big|_{S>S^*,\ s=\frac{1}{2}} < 0$

Substituting $s = \frac{1}{2}$ into (31) gives

$$a_R^{k-\frac{1}{\theta}} = \frac{-\Omega^{1-\beta} + \Omega^{\beta-1}}{(-\frac{1}{\phi} - \phi + \Omega^{1-\beta} + \Omega^{\beta-1})} \tag{52}$$

which can be written as

$$a_R = \left(\frac{-\Omega^{2(\beta-1)} + 1}{(\Omega^{(\beta-1)}(-\frac{1}{\phi} - \phi) + \Omega^{2(\beta-1)} + 1)}\right)^{\frac{1}{k-\frac{1}{\theta}}} \tag{53}$$

It is seen diretly from this expression that $\frac{\partial a_R^{k-\frac{1}{\theta}}}{\partial \theta} < 0$ ■

4.5 Proof that $\frac{\partial a_R^{k-\frac{1}{\theta}}}{\partial \Omega} < 0 \quad for \quad s > \frac{1}{2}$

$$a_R^{k-\frac{1}{\theta}} = \frac{\phi(2s-1) - s\Omega^{\beta-1} + (1-s)\Omega^{1-\beta}}{(1-s)(-\frac{1}{\phi} - \phi + \Omega^{\beta-1} + \Omega^{1-\beta})} \tag{54}$$

First we take log:

$$\ln a_R^{k-\frac{1}{\theta}} = \ln(\phi(2s-1) - s\Omega^{\beta-1} + (1-s)\Omega^{1-\beta}) - \ln(1-s) - \ln(-\frac{1}{\phi} - \phi + \Omega^{\beta-1} + \Omega^{1-\beta})$$

Differentiated by Ω

$$\frac{d \ln a_R^{k-\frac{1}{\theta}}}{d\Omega} = \beta\Omega^{-1}\left(\frac{s\Omega^{\beta-1} + (1-s)\Omega^{1-\beta}}{(-s(\Omega^{\beta-1} - \phi) + (1-s)(\Omega^{1-\beta} - \phi))} - \frac{-\Omega^{\beta-1} + \Omega^{1-\beta}}{-\frac{1}{\phi} - \phi + \Omega^{\beta-1} + \Omega^{1-\beta}}\right)$$

where $\Omega < 1$, because of $S > S^*$. $-\Omega^{\beta-1} + \Omega^{1-\beta} < 0$ always holds.
First, we derive

$$(-s(\Omega^{\beta-1} - \phi) + (1-s)(\Omega^{1-\beta} - \phi)) < 0$$

because

$$\begin{aligned} & (-s(\Omega^{\beta-1} - \phi) + (1-s)(\Omega^{1-\beta} - \phi)) \\ = \;& -s\Omega^{\beta-1} + (1-s)\Omega^{1-\beta} - (1-2s)\phi \\ < \;& (1-2s)\Omega^{\beta-1} - (1-2s)\phi \\ = \;& (1-2s)(\Omega^{\beta-1} - \phi) < 0 \end{aligned}$$

where $\Omega^{\beta-1} > 1$ due to $\Omega < 1$ and s>0.5.

Next, we have $-\frac{1}{\phi}-\phi+\Omega^{\beta-1}+\Omega^{1-\beta}<0$. This is because $a_R^{k-\frac{1}{\theta}}=\frac{\phi(2s-1)-s\Omega^{\beta-1}+(1-s)\Omega^{1-\beta}}{(1-s)(-\frac{1}{\phi}-\phi+\Omega^{\beta-1}+\Omega^{1-\beta})}>0$ and $(\phi(2s-1)-s\Omega^{\beta-1}+(1-s)\Omega^{1-\beta})<(1-2s)\Omega^{\beta-1}+(1-2s)\phi<0$.

Using these results, we have

$$\frac{d\ln a_R^{k-\frac{1}{\theta}}}{d\Omega}=\beta\Omega^{-1}\left(\underbrace{\frac{s\Omega^{\beta-1}+(1-s)\Omega^{1-\beta}}{(-s(\Omega^{\beta-1}-\phi)+(1-s)(\Omega^{1-\beta}-\phi))}}_{-}-\underbrace{\frac{-\Omega^{\beta-1}+\Omega^{1-\beta}}{-\frac{1}{\phi}-\phi+\Omega^{\beta-1}+\Omega^{1-\beta}}}_{+}\right)<0$$

Therefore It is seen diretly from this expression that $\frac{da_R^{k-\frac{1}{\theta}}}{d\Omega}<0$ ■

4.6 Proof that $\frac{\partial a_R^{k-\frac{1}{\theta}}}{\partial S}>0 \quad for \quad s>\frac{1}{2}$

First, we note that $\left(\frac{1-S}{1-S^*}\right)^{1-\beta}$ is less than 1 due to $S>S^*$. Here for simplicity, we suppose that S and S are no substantial difference, Ω close to 1.

We now define $\frac{F}{G}=\frac{\phi(2s-1)-s\left(\frac{1-S}{1-S^*}\right)^{\beta-1}+(1-s)\left(\frac{1-S}{1-S^*}\right)^{1-\beta}}{(1-s)(-\frac{1}{\phi}-\phi+\left(\frac{1-S}{1-S^*}\right)^{\beta-1}+\left(\frac{1-S}{1-S^*}\right)^{1-\beta})}$

$$\begin{aligned}
F &= \phi(2s-1)-s\left(\frac{1-S}{1-S^*}\right)^{\beta-1}+(1-s)\left(\frac{1-S}{1-S^*}\right)^{1-\beta}>0\\
\frac{dF}{dS} &= -\frac{1}{1-S^*}(1-\beta)\,s\left(\frac{1-S}{1-S^*}\right)^{\beta-2}-\frac{1}{1-S^*}(1-\beta)(1-s)\left(\frac{1-S}{1-S^*}\right)^{-\beta}\\
&= -\frac{1}{1-S^*}\beta\left[s\left(\frac{1-S}{1-S^*}\right)^{-\beta-1}+(1-s)\left(\frac{1-S}{1-S^*}\right)^{\beta-1}\right]<0
\end{aligned}$$

$$\begin{aligned}
G &= (1-s)(-\frac{1}{\phi}-\phi+\left(\frac{1-S}{1-S^*}\right)^{-\beta}+\left(\frac{1-S}{1-S^*}\right)^{\beta})<0\\
\frac{dG}{dS} &= (1-s)\frac{1}{1-S^*}\beta\left(\frac{1-S}{1-S^*}\right)^{-\beta-1}-(1-s)\beta\frac{1}{1-S^*}\left(\frac{1-S}{1-S^*}\right)^{\beta-1}\\
&= (1-s)\frac{1}{1-S^*}(1-\beta)\left[\left(\frac{1-S}{1-S^*}\right)^{\beta-2}-\left(\frac{1-S}{1-S^*}\right)^{-\beta}\right]>0
\end{aligned}$$

$$\frac{da_R^{k-\frac{1}{\theta}}}{dS}=\frac{F'G-G'F}{G^2}>0$$