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Optimal Government Debt Policy in the Overlapping Generations Model with Idiosyncratic Capital Return Risk[*]

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Abstract

In this paper, we study the two-period overlapping generations model in which individuals are subject to idiosyncratic risks and study the optimal provision of government debt. In our model, individuals are ex ante homogeneous, and hold risky capital and safe government bonds in the first period. They are subject to idiosyncratic capital return risk in the second period. It is well-known that in deterministic overlapping generations models, when government debt is provided to maximize steady state welfare, the interest rate (r) is equal to the economic growth rate (g). However, in our model with idiosyncratic risks, the risk-free rate is less than the economic growth rate in the optimal steady state. This implies that even when the rollover of government debt is sustainable, increases in debt may reduce welfare. When the return on capital accumulation is risky, the level of safe assets the individuals hold is inefficiently high. Setting the risk-free rate below the economic growth rate reduces demand for government bonds and enhances capital accumulation, which is welfare-improving.

Keywords: overlapping generations, government debt, capital income risk
JEL classification: E5

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1 Introduction

A number of developed countries including Japan and the United States have accumulated huge amount of government debt for decades. The desirability of the government debt in these countries are therefore becoming an important macroeconomic issue these days. The arguments on the government debt are often related to the difference between the (risk-free) interest rate (r) and the growth rate (g).

In the canonical deterministic overlapping generations (OLG) model with exogenous productivity growth of Diamond (1956), the steady state welfare is maximized when the interest rate (i.e, the marginal product of capital) is equal to the economic growth rate. This is because when the marginal product of capital is equal to the economic growth rate, the aggregate consumption is maximized. It is known as a golden rule of capital (see Phelps(1961)). If the interest rate is lower than the growth rate, then the transfer policy from the young the old raises welfare. In the OLG model, the government debt policy works as the intergenerational transfer. Therefore, the increases of the government debt are welfare-improving (see Tirole (1985)).

A seminal paper of Blanchard (2019) uses the OLG model with aggregate productivity shocks to study the optimal government debt policy. In his model, capital returns are risky, and then the rate of return on capital exceeds the risk-free rate, that is the rate of return on the government bonds (also see Abel et al. (1989) and Barro (2020)). He finds that the equilibrium welfare is maximized when the risk-free rate is equal to the economic growth rate, and that issuing government debt is beneficial as long as the risk-free rate is less than the growth rate. [1]

Recently, Abel and Panageas (2022) use the OLG model with the aggregate shocks on capital depreciation rate to study the sustainable government debt policy. Here the policy is sustainable if the government bonds can be rolled over forever. They first show that the level of sustainable government debt is maximized when the risk-free rate is equal to

[1] Blanchard (2019) point out that in many countries including Germany and the United States, the risk-free rate has been often below the economic growth rate for the last twenty years. Blanchard (2019) then claims that issuing government bonds in these countries generates little cost.

the economic growth rate. They also show that in that case, welfare is also maximized. The result on welfare is very close to the one obtained by Blanchard (2019). Equating the risk-free rate with the economic growth rate therefore seems to be a robust optimal fiscal policy rule. However, Blanchard (2019) and Abel and Panageas (2022) use the models with the aggregate shocks, and it is not clear whether their results continue to hold in a economy with heterogenous agents.

In this paper, we study the government debt policy in the two-period OLG model with heterogeneous agents. We incorporate idiosyncratic capital return risk into the model of Diamond (1956). Idiosyncratic capital risk has recently been theoretically studied by many authors including Benhabib et al. (2011), Gottardi et al. (2014, 2015), Biswas et al. (2018), Krueger et al. (2021) and Brumm et al. (2022), and its importance is empirically confirmed by Fagereng et al. (2020).

In each period, two-period lived individuals with constant population are born. They are ex-ante identical. When they are young, they work, receive wage income, consume, invest on the risky capital and hold the safe government bond. We assume that the amount of capital which the individuals hold when they are old is equal to the amount of the investment they make when they are young multiplied by the idiosyncratic shock, as modeled by Krueger et al. (2021) and Brumm et al. (2022). The level of the idiosyncratic shock is not known until the individuals become old. The old individuals rent capital to the representative firm, receive capital income and interest on the government debt, consume, and die.

In our model, the representative firm uses capital and labor to produce the good. The production function is strictly concave as opposed to Brumm et al. (2022). Here the factor market is competitive, and the capital rental rate is equal to the marginal product of capital and the wage income is equal to the marginal product of labor. Therefore, the return on investing the risky capital is equal to the marginal product of capital times the idiosyncratic shock which each individual receives. The government issues bonds and also transfers money to the young and the old by lump-sum so as to satisfy the government budget constraints. We assume that the lump-sum transfers are independent of the idiosyncratic shock.

We first characterize the stationary competitive equilibrium allocation which is determined by the budget constraints of the individuals, the first order condition (FOC) on risky capital and the one on the safe government bonds. Second, we consider the constrained efficient allocation in which the benevolent social planner choose the level of capital to maximize welfare. We find that it violates the FOC on risky capital. This implies that in our economy with idiosyncratic risk, the constrained efficient allocation cannot be decentralized. Third, we then study the optimal government policy which maximizes the steady state welfare. The problem is to choose the best stationary allocation which satisfies the budget constraints and the FOC on risky capital. We show that in the optimal equilibrium allocation, the risk-free rate, that is the interest rate on the government bonds is *less* than the economic growth rate. This implies that the introduction of the government debt might reduce welfare even when the risk-free rate is less than the economic growth rate. The result is crucially different from Blanchard (2019) and Abel and Panageas (2022).

In the OLG models, capital accumulation affect welfare, either positively or negatively, by changing the aggregate consumption. However, in our model with idiosyncratic capital return risk, it has also a negative welfare effect by raising the volatility of capital income that the old individuals receive. From the planner's perspective, this effect is diminishing because the aggregate capital income, which is a product of the capital and the capital rental rate, is concave in capital as long as the production function is strictly concave. However, for the equilibrium consumer, the effect is constant, not diminishing, because the consumer takes the capital rental rate as given. This implies that the equilibrium capital level is insufficiently low in terms of welfare. In other words, the consumers hold excessively large amount of the government bond. Setting the risk-free rate below the economic growth rate and reducing the incentive to hold the government bond is therefore welfare-improving. It is true that when the risk-free rate is different from the economic growth rate, the wedge between the marginal utility of consumption of the young and the one of the old is created and it reduces the welfare. However, its first order effect is almost zero when the risk-free rate is close to the economic growth rate.

Literature review:

Our paper is related to Brumm et al. (2022) who study the OLG model with idiosyncratic capital return risk but without economic growth. They use the linear production function, and they find that the pay-go policy, that is, the lump-sum transfer from the young to old always improves stationary welfare as long as the risk-free rate is negative. They also show that the transfer from the high income earners to low income earners always Pareto-improves the allocation with pay-go policy. Brumm et al. (2023) study a similar policy in the model with aggregate productivity shock.

In Brumm et al. (2022), the expansion of the pay-go policy always improves welfare as long as the risk-free rate is less than the economic growth rate, but the result does not hold in our paper. The main difference between our results and Brumm et al. (2022) lies in the shape of the production function. When the production function is linear, the constrained efficient allocation can be decentralized. In that case, setting the risk-free rate to the economic growth rate and equalizing the marginal utility of the consumption of the young with the one of the old maximizes welfare. In our paper, however, the constrained efficient allocation cannot be decentralized and then setting the risk-free rate below the growth rate is optimal.

Our paper is also related to Krueger et al. (2021) who study optimal capital taxation in the OLG models with idiosyncratic risks on skill and capital return. Krueger et al. (2021) derive the optimal saving rate in the competitive equilibrium allocation and also the capital tax rate which implements the best saving rate. Krueger et al. (2021) show that the optimal saving rate and the optimal capital tax rate is unaffected by the introduction of the government bonds. Our paper abstracts from capital tax and study the level of the optimal interest rate on the government bond.

Some authors investigate the effect the government debt policy in the heterogeneous agent models. Aiyagari and McGrattan (1998), and more recently Aguiar et al. (2021) study the the infinite horizons incomplete market models in which the individuals are subject to the idiosyncratic income risk. Amol and Luttmer (2022) study the endogenous growth model with idiosyncratic risk on capital depreciation. Reis (2021) studies a continuous time infinite horizons endogenous growth model in which households are subject to the idiosyncratic capital depreciation risk. Miao and Su (2023) study the New Keynesian model in which the entrepreneurs are subject to the idiosyncratic capital return shock. Our paper differ from them in that we show the relationship between the welfare-maximizing risk-free rate with the the economic growth models with capital return risk.

Our conclusion on the desirability of setting the risk-free rate below the economic growth rate is very much similar to the one obtained by Ball and Mankiw (2022) who study the deterministic OLG model in which the firms have market power. They show that when the government bond is optimally accumulated, the interest rate is less than the economic growth rate. This is mainly because in their model, the interest rate deviates from the marginal product of capital. As Abel and Panageas (2022) point out, when the interest rate is less than the economic growth, the infinite rollover of the government debt is possible. It is also called Debt-Ponzi game (see Blanchard and Weil (2001)). However, Ball and Mankiw (2022) warn that it may hurt welfare. Their warning applies to our model with idiosyncratic risks.

Finally, our argument on welfare is related to the literature on the dynamic efficiency. Cass (1972) uses the deterministic OLG model and shows that the equilibrium allocation is dynamically inefficient if and only if the interest rate is less than the economic growth rate. He shows the proposition by proving that one can increase the aggregate consumption in some period without reducing the consumption in a different period. Abel et al. (1989) and Zilcha (1990, 1991) extend Cass (1972) to the models of aggregate shocks and derive conditions on the dynamic efficiency by studying the maximization of the aggregate consumption. Investigating the dynamic efficiency in our model with the uninsured idiosyncratic risks is beyond our scope, and here we focus on the steady state welfare. However, we at least show that the maximization of the aggregate consumption does not necessary mean that the steady state welfare is maximized.

Organization of the paper

This paper is organized as follows. Section 2 set-up the model. Section 3 characterizes the equilibrium allocation. Section 4 studies the optimal government policy. Section 5 performs a numerical analysis. Section 6 concludes the paper. Proofs of the lemmas are in Appendix.

2 Model

In this section, we describe the model.

2.1 Consumer

Time is discrete ($t = 0, 1, ...$) and has infinite horizons. In each period, a continuum of individuals with measure one is born. They are ex-ante identical and live for two periods. In period 0, there are initial old individuals, with measure one and live only one period. We call the individuals born in period t ($t \geq 0$) *generation* t.

When the individuals are young, they supply one unit of labor when they are young, and receive W_t units of labor income. They consume, and save the rest of the income. As a way of saving, the individual hold the risky capital and the risk-free government debt. The budget constraints of generation t when the individual is young is,

$$C_{y,t} = W_t - B_{t+1} - I_t + \Phi_{y,t}. \tag{1}$$

Here, $C_{y,t}$ is the consumption of the young, B_{t+1} is the government debt the individual holds, I_t is the investment of on physical capital in period t, and $\Phi_{y,t}$ is the lump-sum transfer from the government to the young. We let $R_{f,t} = 1+r_{f,t}$ denote the gross risk-free rate on the government bond in period t.

The individuals face the uninsurable idiosyncratic shocks on capital return. We follow Gottardi et al. (2015), Biswas et al. (2020) and Krueger et al. (2021), and assume that an investment of I units in the current period yields θI units of the physical capital in the next period, where θ is the stochastic return shock. We let $R_t = 1 + r_t$ denote the gross rate of return on capital in period t. The rate of return on investment in period t when the idiosyncratic risk is θ is equal to θR_{t+1}. We assume that capital is fully depreciated. The ex-ante identical individuals do not know the level of θ when they determine their portfolio. Therefore the amount of the investment I_t and bond holdings B_{t+1} are the same across individuals.

Let $J \geq 2$ denote the number of states. The variable θ takes a value $\theta^j > 0$ with probability $\pi^j > 0$ $(j = 1, 2, ..., J)$, where $\theta^j < \theta^{j+1}$ $(j = 1, 2, ..J - 1)$ and $\sum_{j=1}^{J} \pi^j = 1$. Without loss of generality, we can assume that the expected value of return shock is equal to one: $\mathbb{E}[\theta] = \sum_{j=1}^{J} \pi^j \theta^j = 1$.

When the individuals are old, they do not work, and they use the gross returns on capital and on the government bonds to consume. Let $C_{o,t+1}(\theta)$ denote the consumption of generation t when he is old and his state is θ. The budget constraint of the old individual whose state is θ is written as

$$C_{o,t+1}(\theta) = \theta R_{t+1} I_t + R_{f,t+1} B_{t+1} + \Phi_{o,t+1}, \tag{2}$$

where $\Phi_{o,t+1}$ is the lump-sum transfer from the government to the old in period $t + 1$. Here we assume that the transfer

By the law of large numbers, the measure of the individual whose state is θ^j is equal to π^j, and the aggregate capital provided at time $t + 1$, K_{t+1}, is equal to its expected value. This means that when the investment of the ex-ante individual in period t is I_t, the aggregate capital in period $t+1$ is $K_{t+1} = \mathbb{E}[\theta] I_t$. By assumption, the expected value of the variable θ equals $\mathbb{E}[\theta] = 1$. Therefore the investment of the young is equal to the aggregate capital stock;

$$I_t = K_{t+1}. \tag{3}$$

In the following, the we write the investment of the young in period t as K_{t+1}.

In period 0, there exists a unit measure of the initial old agents who are endowed with $K_0 > 0$ units of the capital stocks and $B_0 > 0$ units of the government bond, that are fixed. They simply consume and do not make any choices. Their consumptions are equal to $C_{o,0} = R_0 K_0 + R_{f,0} B_0$. For $t \geq 0$, generation t has the expected utility

$$U_t = u(C_{y,t}) + \mathbb{E}[v(C_{o,t+1}(\theta))],$$

where u is the period utility function of the young, v is the utility function of the old. The individual chooses I_t and B_{t+1} to maximize the utility function U_t subject to the budget constraints (1) and (2). Here we assume that the utility functions are logarithmic.

Assumption 1: $u(C_y) = (1-\beta)\ln C_y$ and $v(C_o) = \beta \ln C_o$ where $\beta \in (0,1)$ is the preference parameter.

2.2 Government

In period t, the government uses lump-sum transfer $\Phi_{y,t}$ to the income of the young individual, and $\Phi_{o,t}$ to the income of the old individuals. The government also issues the government bonds by B_{t+1} units. However, the government has to repay the debt issued on the previous period B_t plus the interest $r_{f,t}B_t$ to the bond holders. The government budget constraint is given by

$$B_{t+1} = R_{f,t}B_t + \Phi_{y,t} + \Phi_{o,t}. \tag{4}$$

Substitution of equation (4) into the budget constraint of the young, (1) yields

$$C_{y,t} = W_t - K_{t+1} - (R_{f,t}B_t + \Phi_{o,t}).$$

On the other hand, the budget constraint of generation $t-1$ when they are old is

$$C_{o,t}(\theta) = \theta R_t K_t + (R_{f,t}B_t + \Phi_{o,t}).$$

These two equations imply that the term $\mathcal{T}_t \equiv R_{f,t}B_t + \Phi_{o,t}$ can be interpreted as a intergenerational transfer from the young to the old.

2.3 Production

There is a representative firm. The production function F exhibits constant returns to scale and is given by the Cobb-Douglas production function

$$F(K, AL) = K^\alpha (AL)^{1-\alpha},$$

where α is capital share, K is aggregate capital, L is aggregate labor, and A is the labor-augmenting productivity level. In equilibrium, $L = 1$. The growth rate of the technology is $g > 0$ where g is constant. The productivity level at time t, is equal to $A_t = (1+g)^t$ where $g \geq 0$ is an index of productivity growth and $A_0 > 0$ is the initial technology level. The resource constraint is written as

$$F(K_t, A_t L) = K_{t+1} + C_{y,t} + \sum_{j=1}^{J} \pi^j C_{o,t}(\theta^j).$$

Let $k = \frac{K}{AL} = \frac{K}{A}$ denote the ratio of capital to effective labor. Also, let $f(k) = k^\alpha$ denote the per-capita production function. The factor prices are competitive and the capital return and the wage rate are respectively determined by the marginal product. If we let $k_t = K_t/(A_t L)$ denote the capital to effective labor ratio in period t, the equilibrium wage rate in period t, is equal to $W_t = w_t A_t$ where w_t is the function of k_t:

$$w_t = w(k_t) \equiv f(k_t) - k_t f'(k_t) = (1-\alpha) k_t^\alpha, \tag{5}$$

Similarly, the gross capital rental rate in period t is also a function of k_t:

$$R_t = R(k_t) \equiv f'(k_t) = \alpha k_t^{\alpha - 1}. \tag{6}$$

This is also equal to the average return on holding capital for the individual.

3 Equilibrium

This section characterizes the competitive equilibrium allocation

3.1 Competitive equilibrium

To make the model stationary, let us define new variables $c_{y,t} = C_{y,t}/A_t$, $c_{o,t}(\theta) = C_{o,t}(\theta)/A_t$, $b_t = B_t/A_t$, $k_t = K_t/A_t$, $\phi_{y,t} = \Phi_{y,t}/A_t$ and $\phi_{o,t} = \Phi_{o,t}/A_t$. Along a balanced growth path, these variables are constant. Since the period utility functions are logarithmic, the intertemporal utility is re-written as

$$U_t = u(c_{y,t}) + \mathbb{E}[v(c_{o,t+1}(\theta))] + \text{constant},$$

where the constant term is equal to $\ln A_t + \delta \ln A_{t+1}$. In the following, we ignore the constant term and simply express the utility of the individual as $\mathcal{U}_t \equiv u(c_{y,t}) + \mathbb{E}[v(c_{o,t+1}(\theta))]$.

Also, the budget constraints of the individual are re-written with respect to $c_{y,t}$ and $c_t^o(\theta)$:

$$
\begin{aligned}
c_{y,t} &= w_t - G(k_{t+1} + b_{t+1}) + \phi_t^y, &(7)\\
c_{o,t+1}(\theta) &= \theta R_{t+1} k_{t+1} + R_{f,t+1} b_{t+1} + \phi_{t+1}^o, &(8)
\end{aligned}
$$

where $G = 1 + g$ is the gross rate of economic growth. From the government budget constraint,

$$
Gb_{t+1} = R_{f,t} b_t + \phi_t^y + \phi_t^o. \tag{9}
$$

The individual who is born in period t chooses k_{t+1} and b_{t+1} and maximizes the utility U_t^* subject to equations (7) and (8). The FOCs on k_{t+1} and b_{t+1} are respectively

$$
\begin{aligned}
Gu'(c_{y,t}) &= R_{t+1}\mathbb{E}[\theta v'(c_{o,t+1}(\theta))]. &(10)\\
Gu'(c_{y,t}) &= R_{f,t+1}\mathbb{E}[v'(c_{o,t+1}(\theta))]. &(11)
\end{aligned}
$$

We now formally define the competitive equilibrium.

Definition 1 *A competitive equilibrium consists of an allocation, $\{c_{y,t}, c_{o,t+1}(\theta), k_t\}_{t=0}^{\infty}$, factor prices $\{w_t, R_t\}_{t=0}^{\infty}$and the government policy, $\{\phi_t^y, \phi_t^o, R_{f,t}, b_{t+1}\}_{t=0}^{\infty}$ that satisfy equations (5) (6), (7), (8),* (9)*, (*10) *and (*11)*.*

The resource constraint in intensive forms is given by

$$
f(k_t) = Gk_{t+1} + c_{y,t} + \sum_{j=1}^{J} \pi^j c_{o,t}(\theta^j).
$$

One can easily check that the equilibrium allocation satisfies the resource constraint.

To compare the expected rate of gross return on capital R with the gross risk-free rate R_f, first note that the variable θ satisfies $\mathbb{E}[\theta] = 1$. Therefore

$$
\begin{aligned}
\mathbb{E}[\theta v'(c_{o,t+1}(\theta))] - \mathbb{E}[v'(c_{o,t+1}(\theta))] &= \mathbb{E}[(\theta - \mathbb{E}[\theta])v'(c_{o,t+1}(\theta))] &(12)\\
&= \mathrm{Cov}(\theta, v'(c_{o,t+1}(\theta))) < 0. &(13)
\end{aligned}
$$

In the second line, the covariance term is negative because the consumption of the old $c_{o,t+1}(\theta)$ increases with the returns shock θ and the marginal utility v' is a decreasing function. On the other hand, from equations (10) and (11), we have $R_{t+1}\mathbb{E}[\theta v'(c_{o,t+1}(\theta))] = R_{f,t+1}\mathbb{E}[v'(c_{o,t+1}(\theta))]$. Therefore $R_{t+1} > R_{f,t+1}$.

3.2 Stationary equilibrium

We now consider the stationary equilibrium in which the government policy, $\{\phi^y, \phi^o, R_f, b\}$ is constant. From equation (9), the transfer to the young is denoted as

$$\phi^y + \phi^o = (G - R_f)b.$$

Government Ponzi scheme is possible and the government bonds can be rolled over forever when $G > R_f$.

The steady state allocation $\{c_y, c_{o,t+1}(\theta), k\}$ satisfies the budget constraints:

$$\begin{aligned} c_y &= w(k) - Gk - T, \quad (14) \\ c_o(\theta) &= \theta R(k)k + T, \quad (15) \end{aligned}$$

where $T = \frac{\mathcal{T}_t}{A_t} = \phi^o + R_f b$.

Substituting equation (14) into equation (15), we can express the old period consumption c_o as a function of the young period consumption c_y and the capital k:

$$c_o(\theta) = \{f(k) - Gk - c_y\} + \tilde{\theta}R(k)k. \quad (16)$$

Here the term $\tilde{\theta} = \theta - \mathbb{E}[\theta]$ denotes the deviation of the variable θ from the mean $\mathbb{E}[\theta](= 1)$. The term $f(k) - Gk - c_y$ equals the aggregate consumption of the old, $\mathbb{E}[c_o(\theta)]$. The term $\tilde{\theta}R(k)k$ represents the volatility in the capital income. The capital accumulation affects steady state welfare by changing the aggregate consumption $f(k) - Gk$ just as the deterministic OLG models. However, in this model, it has another effect on welfare by increasing the volatility in capital income. The effect is negative since the average capital income $R(k)k$ increases with k.

The FOCs in the stationary equilibrium are

$$\begin{aligned} Gu'(c_y) &= R(k)\mathbb{E}[\theta v'(c_o(\theta))]. \quad (17) \\ Gu'(c_y) &= R_f\mathbb{E}[v'(c_o(\theta))]. \quad (18) \end{aligned}$$

Given the government policy $\{\phi^y, \phi^o, R_f, b\}$ the stationary equilibrium allocation $\{c_y, c_o(\theta), k\}$ is characterized by equations (7), (8), (17), and (18).

The aggregate resource constraint in the stationary equilibrium is

$$f(k) = Gk + c_y + \sum_{j=1}^{J} \pi^j c_o(\theta^j).$$

In the following, we let k^{gold} denote the Golden Rule level of capital which maximizes the aggregate consumption $c \equiv c_y + \sum_{j=1}^{J} \pi^j c_o(\theta^j)$. It is determined by $f'(k^{gold}) = G$.

4 Optimal government debt policy

In this section, we follow Abel and Panageas (2022), and study the government debt policy that maximizes the utility in the stationary equilibrium.

4.1 Constrained efficient allocation

Before we study the optimal debt policy, we first derive the *constrained efficient* allocation in which a benevolent social planner make an investment decision and choose k on behalf of the consumers to maximize the steady state utility, while the factor prices are still determined competitively and the budget constraints (25) and (26) are satisfied. In other words, the constrained efficient allocation maximizes the steady state utility of the consumer

$$\mathcal{U} = u(c_y) + \mathbb{E}[v(c_o(\theta))],$$

subject to equations (14) and (15). Similar analysis is done by Davila et al. (2012). If the constrained efficient allocation violate the FOC on capital (17), then it cannot be implemented as a competitive equilibrium allocation.

the functions f, w and R satisfy $f(k) - w(k) = R(k)k$, which simply says that capital income is equal to total income minus labor income. Thus the intertemporal utility is written as

$$\mathcal{U} = u(w(k) - Gk - T) + \mathbb{E}[v(\theta R(k)k + T)].$$

The social planner maximizes $\mathcal{U}$ by choosing capital k and the transfer T. Let $\{c_y^*, c_o^*(\theta), k^*\}$ denote the constrained efficient allocation. It satisfies the FOCs on T and k as below:

$$u'(c_y^*) = \mathbb{E}[v'(c_o^*(\theta))] \tag{19}$$

$$(G - w'(k^*))u'(c_y^*) = (R(k^*) - w'(k^*))\,\mathbb{E}[\theta v'(c_o^*(\theta))] \tag{20}$$

Here we use the equality $(R(k)k)' = R(k) - w'(k)$ which holds because the functions f, w and R satisfy $f(k) - w(k) = R(k)k$. The first condition (19) says that in the constrained efficient allocation, the social planner equalizes the marginal utility of young with the expected marginal utility of the old. This is possible through the intergenerational transfer. Substituting equations (19) and (13), into equation (19), we have the following inequality:

$$Gu'(c_y^*) - R(k^*)\mathbb{E}[\theta v'(c_o^*(\theta))] = -w'(k^*)\mathrm{Cov}\left(\theta, v'(c_o^*(\theta))\right). \tag{21}$$

The term $\mathrm{Cov}(\theta, v'(c_o^*(\theta)))$ is negative. Therefore it implies the following inequality:

$$Gu'(c_y^*) > R(k^*)\mathbb{E}[\theta v'(c_o^*(\theta))]. \tag{22}$$

Inequality (22) implies that the constrained efficient allocation never satisfy the FOC on capital in the stationary equilibrium, (17), because it requires that the left hand side and the right hand side of (22) are equal. We have the following conditions.

Proposition 1 *Constrained efficient allocation cannot be implemented as a competitive equilibrium allocation.*

Equations (22) and (17) imply that the social planner consume less and invest more on capital when he is young than the equilibrium consumer. To see why, let us re-write intertemporal utility as a problem of finding k and c_y :

$$\mathcal{U} = u(c_y) + \mathbb{E}[v(f(k) - Gk - c_y + \tilde{\theta}R(k)k)], \tag{23}$$

where we use equation (16) to express $c_o(\theta)$ by c_y and k.

In the constrained efficient allocation, the utility is maximized with respect to c_y and k. The FOC on c_y is the same as (19). The FOC on k is written as

$$0 = (R(k^*) - G)\mathbb{E}[v'(c_o^*(\theta))] + (R(k^*)k^*)' \cdot \mathrm{Cov}\left(\theta, v'(c_o^*(\theta))\right). \tag{24}$$

The condition (24) says that the net marginal effect of capital accumulation on welfare is zero at the optimal. The first term $(R(k) - G)\mathbb{E}[v'(c_o(\theta))]$ shows the welfare effect of accumulating capital by changing the aggregate consumption. The second term $(k^*R(k^*))' \cdot \text{Cov}\,(\theta, v'(c_o^*(\theta)))$ shows the effect of capital accumulation on welfare, which is always negative, by raising the volatility of the old consumption. Equation (24) implies that the marginal product of capital is greater than the growth rate G and therefore capital in the constrained efficient allocation, is less than the consumption-maximizing level of capital k^{gold}.

Equation (24) implies that the social planner fully recognizes the negative effect of capital holdings by increasing the volatility of the old period consumption. It is true that the equilibrium consumer also recognizes the effect, but the consumer does this only partially. For the social planner, the marginal effect of capital accumulation on the volatility term is equal to $(R(k)k)'\tilde{\theta} = \alpha^2 k^{\alpha-1}\tilde{\theta}$, which is diminishing in k. On the other hand, for the equilibrium consumer, the marginal effect is equal to $R(k)\tilde{\theta}$, which is constant, not diminishing, because the consumer takes the capital rental rate R as given. This is why the equilibrium consumer holds capital less than the social planner. [2]

When there is no shock, i.e., $\theta = 1$ or equivalently $\tilde{\theta} = 0$, the intertemporal utility is simplified as $\mathcal{U} = u(c_y) + v(f(k) - Gk - c_y)$. To maximize the utility, the social planner simply maximizes the aggregate consumption $f(k) - Gk$ and equalizes the marginal utility of young consumption $u'(c_y)$ with that of the old individual $v'(c_o)$. Therefore the capital level equals to the Golden rule level k^{gold}, and the marginal product of capital $f'(k)$ equals to the economic growth rate G. This implies that the constrained efficient allocation satisfies the equilibrium FOC on capital (17), which is simplified as $Gu'(c_y) = R(k)v'(c_o)$. In other words, the constrained efficient allocation can be decentralized as a competitive equilibrium allocation in which the capital rental rate, the interest rate on the government bond, and the economic growth rate are all equal. However, when there is uncertainty, the

[2] In Brumm et al. (2022), the production function is linear and then the marginal product of capital $R(k)$ is constant. In our paper, the dependency of R on k is crucial because it is the reason why the constrained efficient allocation differs from the equilibrium allocation.

constrained efficient allocation deviates from the equilibrium allocation, and the capital level does not satisfy the golden rule. In the next section, we investigate the equilibrium allocation in detail.

4.2 Welfare maximizing government policy

In this section, we characterize the optimal stationary equilibrium and compares the risk-free rate R_f with the economic growth rate G. Without loss of generality, here we assume that the government chooses the transfer-GDP ratio $\tau \equiv \frac{T}{f(k)} = \frac{\mathcal{T}_t}{Y_t}$ (not the transfer T itself) to maximizes the equilibrium welfare. A stationary allocation $\{c_y, c_o(\theta), k\}$ can be implemented as a competitive equilibrium allocation if it and only if the allocation satisfies the budget constraints

$$\begin{aligned} c_y &= w(k) - Gk - \tau f(k), && (25) \\ c_o(\theta) &= \theta R(k)k + \tau f(k), && (26) \end{aligned}$$

and the equilibrium FOC on capital (17) for some transfer-GDP ratio τ. The risk-free rate is determined by $R_f = \frac{Gu'(c_y)}{\mathbb{E}[v'(c_o(\theta))]}$. For given k and τ, the government policy $\{\phi^y, \phi^o, R_f, b\}$ implements the allocation $\{c_y, c_o(\theta), k\}$ if it satisfies $R_f = \frac{Gu'(c_y)}{\mathbb{E}[v'(c_o(\theta))]}$, and

$$\begin{aligned} \phi^o &= \tau f(k) - R_f b, \\ \phi^y &= \tau f(k) + Gb. \end{aligned}$$

The next lemma shows the uniqueness of the stationary equilibrium allocation and derives the capital level given τ.

Lemma 1 *For a fixed value of τ, the stationary equilibrium allocation $\{c_y, c_o(\theta), k\}$ is uniquely determined. The amount of equilibrium capital is given by*

$$k = \left(\frac{\phi}{1+\phi} \frac{1-\alpha-\tau}{G} \right)^{1/(1-\alpha)},$$

where $\phi = \frac{\beta}{1-\beta}\alpha\mathbb{E}[\frac{\theta}{\theta\alpha+\tau}]$.

Proof. See the Appendix. ■

The problem of finding the welfare-maximizing stationary equilibrium is written as

$$\max_{(k,\tau)} u(c_y) + \mathbb{E}[v(c_o(\theta))] \tag{27}$$

subject to equation (17), where c_y and $c_o(\theta)$ are defined in equations (17) and (17), respectively. Once we find the solution path to the problem, we can always choose the risk-free rate R^f such that equation (18) is satisfied.

As we show in the previous section, when we ignore equation (17) and simply maximizes the utility, the solution violates the equilibrium condition (17), and its left hand side $Gu'(c_y)$ exceeds the right hand side $R\mathbb{E}[\theta v'(c_o(\theta))]$ (see equation (22)). This means that to obtain the optimal equilibrium allocation, we must take into account the following inequality constraint:

$$Gu'(c_y) \leq R\mathbb{E}[\theta v'(c_o(\theta))], \tag{28}$$

In the following, we first investigate the *relaxed* problem of maximizing the intertemporal utility $u(c_y) + \mathbb{E}[v(c_o(\theta))]$ subject to the inequality constraint (28), not the equality constraint (17). Later we show that the solution to the relaxed problem always satisfy the inequality constraint (28) with equality.

The Lagrangian on the relaxed problem is

$$L = u(c_y) + \mathbb{E}[v(c_o(\theta))] + \lambda \left[R\mathbb{E}[\theta v'(c_o(\theta))] - Gu'(c_y)\right].$$

where λ is the Lagrange multiplier. The optimal value of (k, τ), if it exists, satisfies $\frac{\partial L}{\partial k} = \frac{\partial L}{\partial z} = 0$ for some nonnegative λ, and if $\lambda > 0$, the inequality constraint (28) holds with equality.

The following lemma shows that the solution exists and the Lagrange multiplier λ is strictly positive.

Lemma 2 *The solution to the relaxed problem exists. The multiplier λ is strictly positive.*

Proof. See the appendix. ■

The lemma implies that the solution to the relaxed problem coincides with the solution to the original problem of finding the optimal equilibrium allocation. The optimal allocation satisfies $\frac{\partial L}{\partial k} = \frac{\partial L}{\partial z} = 0$. The FOC on τ is written as

$$\frac{1}{f'(k)}\frac{\partial L}{\partial \tau} = -u'(c_y) + \mathbb{E}[v'(c_o(\theta))] + \lambda\left[R\mathbb{E}[\theta v''(c_o(\theta))] + Gu''(c_y)\right] = 0. \tag{29}$$

The functions u and v are strictly concave and then $u''(c_y) < 0$ and $v''(c_o(\theta)) < 0$. Moreover, from the lemma above, we have $\lambda > 0$. Therefore we have

$$u'(c_y) - \mathbb{E}[v'(c_o(\theta))] < 0. \tag{30}$$

We now have our main proposition on the risk-free rate $r_f = Gu'(c_y)/\mathbb{E}[v'(c_o(\theta))] - 1$.

Proposition 2 *The interest rate on the government debt r_f is smaller than the growth rate g along the optimal steady state.*

Proof. The gross risk-free rate is denote as $R_f = G\frac{u'(c_y)}{\mathbb{E}[v'(c_o(\theta))]}$. Since $u'(c_y) < \mathbb{E}[v'(c_o(\theta))]$, $R_f < G$, or equivalently $r_f < g$. ■

If the constrained efficient allocation could be implemented as the competitive equilibrium allocation, the risk-free rate $R_f = G\frac{u'(c_y)}{\mathbb{E}[v'(c_o(\theta))]}$ would be equal to the growth rate G from equation (19). However, as we point out in the previous section, the allocation violates the FOC on risky capital, equation (28). If we reduce the risk-free rate, the government bond becomes less attractive as a financial asset and the individual holds less amount of the government bond holdings. In other words, the consumer increases the current period consumption c_y, which reduces the current period marginal utility $u'(c_y)$. This means that inequality constraint (28) is relaxed. It is true that when the risk-free rate differs from the economic growth rate, the intertemporal wedge between the marginal utility of the young consumption and that of the old consumption is generated, and it reduces welfare. However, its first order effect is negligible as long as r_f is close to g. Therefore reduction of the risk-free late below the growth rate is always welfare-improving in our model with idiosyncratic risks.

As many authors including Abel and Panageas (2022) and Ball and Mankiw (2022) have pointed out, when the interest rate is less than the economic growth rate, the rollover of the government debt is possible. The proposition implies that when the individuals are subject to the idiosyncratic capital returns shock, such a rollover might reduce welfare.

4.3 Welfare change by transfer increases

Blanchard (2019) studies the welfare effect of small increases in the intergenerational transfer and finds that it consists of the partial equilibrium effect and the general equilibrium effect. He then shows that both terms are positive if and only if the interest rate is below the economic growth rate. This implies that as long as the interest rate is less than the economic growth rate, the expansion of the intergenerational transfer raises welfare and the welfare is maximized when the interest rate is equal to the economic growth rate.

Here we follow Blanchard (2019) to decompose the welfare effect of increasing the intergenerational transfer and show how our result differ from Blanchard (2019). The stationary welfare is written as

$$\mathcal{U} = u(w(k) - Gk - T) + \mathbb{E}[v(\theta R(k)k + T)].$$

Suppose that the transfer increases by a small value $dT > 0$. Let dw and dR denote the change in the factor prices w and R, respectively. One can easily check that the increase in T reduces capital. Therefore it raises the capital rental rate R and reduces the wage rate w, that is, $dR > 0$ and $dw < 0$. Since $dw = -kdR$, the welfare change by increasing T is written as

$$d\mathcal{U} = \{-u'(c_y) + \mathbb{E}[v'(c_o)]\}dT + \{-u'(c_y) + \mathbb{E}[v'(c_o)]\}kdR + \mathrm{Cov}(\theta, v'(c_o(\theta))kdR.$$

Blanchard (2019) calls the first term $d\mathcal{U}_a \equiv \{-u'(c_y)+\mathbb{E}[v'(c_o)]\}dT$ the *partial equilibrium effect*, and the second term $d\mathcal{U}_b \equiv \{-u'(c_y) + \mathbb{E}[\theta v'(c_o)]\}kdR$ as the *general equilibrium effect.* Let us call the third term $d\mathcal{U}_c \equiv \mathrm{Cov}(\theta, v'(c_o(\theta))kdR$ the *idiosyncratic effect.* As Blanchard (2019) shows, the first term and the second terms are is written as

$$\begin{aligned} d\mathcal{U}_a &= \frac{G - R_f}{G}\mathbb{E}[v'(c_o)]dT, \\ d\mathcal{U}_b &= \frac{G - R_f}{G}\mathbb{E}[v'(c_o)]kdR. \end{aligned}$$

Here the terms dU_a and dU_b are positive if and only if the risk-free rate is less than the growth rate, that is, $r_f < g$. In Blanchard (2013), the third term $d\mathcal{U}_c$ does not exist, and then the increase in transfer increases welfare if and only if $r_f < g$. As Blanchard (2013), the intergenerational transfer generates higher returns than capital and then reduction of capital through the transfer improves welfare.

However, in our model, the utility change has another term $d\mathcal{U}_c = \mathrm{Cov}(\theta, v'(c_o(\theta))kdR$, which is always negative. Therefore the fiscal transfer may worse welfare even when the the risk-free rate is below the economic growth rate.

4.4 Numerical example

We now calculate how the welfare changes as the risk-free rate deviates from the growth rate in a simple numerical example. With respect to the preference parameter, we follow Abel and Panageas (2022) and set $\beta = 0.35, \alpha = 0.33$, and $G = 1.35$. We interpret one period in our model as 30 years. In terms of the capital income risk θ, we follow Brumm et al. (2022, Section 2) and assume that θ takes two values and $\theta = 1.9$ with probability 0.5, and $\theta = 0.1$ with probability 0.5.

Let $\bar{c}_y$ and $\bar{c}_o(\theta)$ respectively denote the consumption of the young and the old in state θ when the risk-free rate R_f is equal to the economic growth rate G. We measure the welfare along the equilibrium consumption allocation $\{c_y, c_o(\theta)\}$ by consumption equivalent Δ:

$$u(c_y) + \mathbb{E}[v(c_o(\theta))] = u(\Delta\bar{c}_y) + \mathbb{E}[v(\Delta\bar{c}_o(\theta))].$$

This equation shows that If we increase the consumption allocation under the condition $r_f = g$ by a factor of Δ, the intertemporal utility is unchanged. When $\Delta > 1$, then it implies that the equilibrium welfare is higher than welfare in the allocation with $r_f = g$. Figure 1 shows the welfare Δ as a function of the difference between the risk-free rate and the growth rate ($R^f - G = r_f - g$). At the utility maximizing steady state, the gross the risk-free rate is around 0.68, which is significantly less than the economic growth rate. On an annual base, the risk-free rate is less than the economic growth rate by in the optimal steady state by 2.2%.

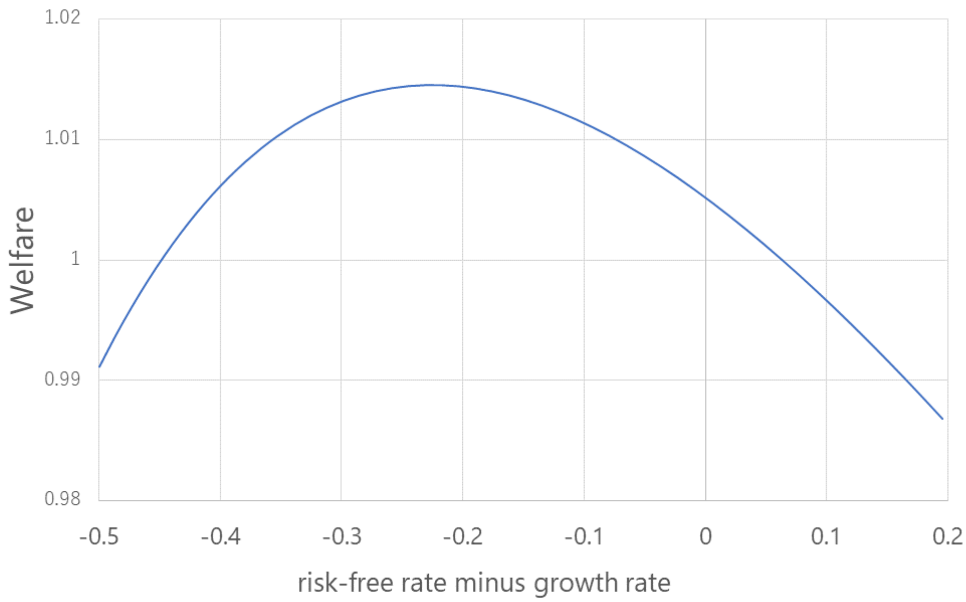

Figure 1: Welfare

4.5 Relationship with existing literature

In this section, we compare our result with some two papers which are closely related to our model. First, our conclusion is very close to that of Ball and Mankiw (2022) who study the optimal government debt policy in the overlapping generations models with mark-up and show that setting the interest rate below the economic growth rate is optimal. Ball and Mankiw (2022) show their proposition by decomposing the welfare effect of raising debt level into the aggregate effect and generational effect. One can show the same proposition by using the Lagrangian just as we did here.

In Ball and Mankiw (2022), there is no uncertainty in capital accumulation, but the gross interest rate R is still not equal to the marginal product of capital $f'(k)$ because of the price mark-up. In their model, the interest rate R is equal to the marginal product of capital divided by the mark-up rate $\mu > 1$, that is, $R = f'(k)/\mu$. Therefore the FOC on capital holdings, $Gu'(c_y) = Rv'(c_o)$ are written as

$$Gu'(c_y) = \frac{f'(k)}{\mu} v'(c_o). \tag{31}$$

where $c_o = f(k) - Gk - c_y$. The first best allocation which simply maximizes the intertemporal utility $u(c_y) + v(c_o)$ subject to the resource constraint $c_o = f(k) - Gk - c_y$ satisfies $f'(k) = G$ and $u'(c_y) = v(c_o)$. Therefore the allocation does not satisfy equation (31), and the left hand side $Gu'(c_y)$ exceeds the right hand side $\frac{f'(k)}{\mu} v'(c_o)$.

The problem of finding the optimal steady state equilibrium is therefore to maximize the utility $u(c_y) + v(c_o)$ subject to the inequality constraint $Gu'(c_y) \leq \frac{f'(k)}{\mu} v'(c_o)$. The Lagrangian of the problem is written as

$$L^{BM} = u(c_y) + v(f(k) - Gk - c_y) + \lambda^{BM} \left(Rv'(f(k) - Gk - c_y) - Gu'(c_y) \right).$$

where the multiplier λ^{BM} is positive. The FOC on c_y is written as $u'(c_y) - v'(c_o) - \lambda^{BM} \left(Rv''(c_o) + u''(c_y) \right) = 0$. This implies that $u'(c_y) < v'(c_o)$ and the interest rate $R = G\frac{u'(c_y)}{v'(c_o)}$ in the optimal steady state is less than G. In both Ball and Mankiw (2022) and our paper, to find out the best equilibrium allocation, we have to set the marginal utility of the young individual relative to the old individual, $\frac{u'(c_y)}{v'(c_o)}$ below the constrained efficient level. When the interest rate decreases, the incentive to save decreases and then this condition can be more easily satisfied.

Second, our paper is also related to Brumm et al. (2022) who also study the idiosyncratic capital return risk in the overlapping generations model. Brumm et al. (2022) show that as long as the gross risk-free rate (denoted as S in Brumm et al. (2022)) is less than the economic growth rate, which is equal to one in their model, the additional transfer from the young individual to the old individual always improves welfare. Although the model is similar to our model, the result is different from ours in the sense that the expansion of the intergenerational transfer may reduce welfare even when the risk-free rate is lower than the economic growth rate,

The crucial difference between Brumm et al. (2022) and ours is the (in)dependence of the marginal product of capital R on capital k. Here we study the neoclassical production function and R depends negatively on k. As we explain in section 3, this is the main reason why the constrained efficient allocation cannot be implemented as a competitive equilibrium allocation, and the constraint on capital, (28) is binding when we obtain the welfare-maximizing steady state. Setting the risk-free rate below the economic growth rate relaxes the constraint (28) and raises the welfare. On the other hand, in Brumm et al. (2022), the production function is linear and R is constant. Therefore the constraint (28) does not bind in obtaining the optimal steady state, and there is no benefit from deviating the risk-free rate from the economic growth rate. A similar argument can be applicable to Abel and Panageas (2022), although they consider aggregate shocks, not the idiosyncratic shocks.

5 Conclusion

In this paper, we study the two-period overlapping generations model in which individuals are subject to the idiosyncratic capital return risks and study the optimal risk-free rate. In our model, individuals are ex ante homogeneous, but are subject to capital income risk in the second period. It is well-known that in the deterministic OLG models, when the government debt is provided as to maximize steady state welfare, the real interest rate is equal to the economic growth rate. However, in our model with idiosyncratic risks, this equality does not imply the optimality. If they are subject to the idiosyncratic risk on capital return, then the risk-free rate is less than the economic growth rate in the optimal steady state.

Appendix

The Appendix provides proofs for Lemmas.

A Proof of Lemma 1

The consumption allocation is

$$c_y = (1-\alpha-\tau)k^\alpha - Gk, \tag{32}$$

$$c_o(\theta) = (\theta\alpha+\tau)k^\alpha \tag{33}$$

The FOC on capital is written as

$$\frac{Gk}{(1-\alpha-\tau)k^\alpha - Gk} = \frac{\beta}{1-\beta}\alpha\mathbb{E}\left[\frac{\theta}{\theta\alpha+\tau}\right],$$

Therefore $k^{1-\alpha} = \frac{\phi}{1+\phi}\frac{1-\alpha-\tau}{G}$. ■

B Proof of Lemma 2

(1)Existence of the solution

We first show that the solution exists. In the following, we let $K = Gk^{1-\alpha}$ and rewrite the optimization problem as the problem of choosing K and τ. The consumption is written as

$$\ln c_y = \ln(1-\alpha-\tau-K) + \frac{\alpha}{1-\alpha}\ln K + \frac{1}{1-\alpha}\ln G, \tag{34}$$

$$\ln c_o(\theta) = \frac{\alpha}{1-\alpha}\ln K + \ln(\theta\alpha+\tau) \tag{35}$$

The objective function $u(c_y) + \mathbb{E}[v(c_o(\theta))]$ is therefore written as a function of K and τ:

$$V(K,\tau) \equiv (1-\beta)\ln(1-\alpha-\tau-K) + \beta\mathbb{E}[\ln(\theta\alpha+\tau)] + \frac{\alpha}{1-\alpha}\ln K + \text{constant}$$

To ensure that the function $V(K,\tau)$ is finite, we need to put the following restriction on the value of (K, τ).

$$
\begin{aligned}
K &> 0, & (36)\\
1-\alpha-\tau-K &> 0, & (37)\\
\tau &> -\theta_1\alpha. & (38)
\end{aligned}
$$

Here, θ_1 is the lowest value which the variable θ can take. Thus if $\tau > -\theta_1\alpha$, then $\mathbb{E}[\ln(\theta\alpha+\tau)]$ is finite.

The constraint $R\mathbb{E}[\theta v'(c_o(\theta))] - Gu'(c_y) \geq 0$ is written as

$$\alpha\mathbb{E}[\frac{\theta}{\theta\alpha+\tau}] - G\frac{1}{(1-\alpha-\tau)k^{\alpha-1}-G} \geq 0.$$

This is simplified as

$$K \leq H(\tau) \equiv \frac{1}{\frac{1}{1-\alpha-\tau}\frac{1}{\alpha\mathbb{E}[\frac{\theta}{\theta\alpha+\tau}]}+1}. \tag{39}$$

The function $H(\tau)$ is a decreasing and a continuous function and is well-defined as long as $\tau > -\theta_1\alpha$.

Let S be the set of (K, τ) which satisfies (36), (37), (38), and (39). The optimization problem is re-expressed

$$\max_{(K,\tau)\in S} V(K, \tau).$$

We have

$$H(0) = \frac{1-\alpha}{2-\alpha} < 1-\alpha.$$

Thus $(K, \tau) = (0, H(0)) \in S$. Let $V_0 = V(H(0), 0)$. From equations (36), (37) and (38), we have

$$
\begin{aligned}
0 &< K < 1-\alpha.\\
-\theta_1\alpha &< \tau < 1-\alpha.
\end{aligned}
$$

Note that the function $\ln x$ converges to $-\infty$ as x goes to zero. Therefore, there exists a sufficiently small and positive $\varepsilon_1 > 0$ such that $\varepsilon_1 < H(0)$ and that for any $(K, \tau) \in S$ such that $K < \varepsilon_1$, $V(K, \tau) < V_0$. Similarly, there exists a sufficiently small and positive $\varepsilon_2 > 0$ such that for any $(K, \tau) \in S$ such that $1-\alpha-\tau-K < \varepsilon_2$, $V(K, \tau) < V_0$.

Moreover, there exists $\varepsilon_3 > 0$ such that $-\theta_1\alpha + \varepsilon_3 < 0$ and that for any $(K, \tau) \in S$ such that $\tau < -\theta_1\alpha + \varepsilon_3$, $V(K, \tau) < V_0$. Therefore the optimization problem is re-written as

$$\max_{K,\tau} V(K, \tau)$$

subject to

$$\begin{aligned} K &\geq \varepsilon_1, &(40)\\ 1 - \alpha - \tau - K &\geq \varepsilon_2, &(41)\\ \tau &\geq -\theta_1\alpha + \varepsilon_3, &(42)\\ K &\leq H(\tau). &(43) \end{aligned}$$

Let S^* be the set of (K, τ) which satisfies all the four inequalities. Clearly the set S^* is compact and is included in the set S. The function $V(K, \tau)$ is continuous, and well-defined for any $(K, \tau) \in S$. Also, a point $(K, \tau) = (H(0), 0)$ is included in S. Thus the maximum exists.

(2) Positivity of the multiplier λ

We next show that the Lagrange multiplier λ is strictly positive. To the contrary, suppose $\lambda = 0$. The optimal path satisfies $\partial U(k, \tau)/\partial k = \partial U(k, \tau)/\partial \tau = 0$. Thus we have

$$\begin{aligned} -u'(c_y) + \mathbb{E}[v'(c_o(\theta))] &= 0 \\ \{w'(k) - G - \tau f'(k)\}u'(c_y) + \mathbb{E}[(\theta\{R(k) - w'(k) + \tau f'(k)\}v'(c_o(\theta))] &= 0 \end{aligned}$$

Therefore we have

$$\{w'(k) - G\}u'(c_y) + \{R(k) - w'(k)\}\mathbb{E}[\theta v'(c_o(\theta))] = 0$$

Thus we have

$$Gu'(c_y) - R\mathbb{E}[\theta v'(c_o(\theta))] = -w'(k)\,\mathrm{Cov}(\theta, v'(c_o(\theta)) > 0$$

This violates constraint (28). This means that $\lambda > 0$, and then from the KKT condition, constraint (28) holds with equality.■

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