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Human Capital Accumulation According to HANK

INOSE, Junya
Mitsubishi Research Institute, Inc.

Research Institute of Economy, Trade & Industry, IAA
The Research Institute of Economy, Trade and Industry
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Human Capital Accumulation According to HANK[i]

Junya INOSE
Mitsubishi Research Institute, Inc.

Abstract

The heterogeneous model in macroeconomics has produced great developments in recent decades. One major development which includes heterogeneity relates to consumer behavior, especially in describing income and wealth inequality. More powerful and sophisticated computing technologies and the increasing availability of microdata have fueled these developments. Among these developments is the invention of the Heterogeneous agent New Keynesian (HANK) models. We advanced the Huggett model of income and wealth distribution to include human capital accumulation. The inclusion of human capital accumulation into a heterogeneous agent model enables us to capture not only wealth, but skill inequality and its dynamics. This paper provides two main contributions. We (i) construct a mathematical tool to analyze models with non-linearity, and (ii) provide implications for the policy of wealth redistribution, especially basic income. The conclusions of this analysis can be again summarized by the following three points: (i) the introduction of basic income may increase the share of liquidity constrained households, (ii) the introduction of basic income results in a decrease of the aggregate share of time spent investing in human capital, and (iii) the introduction of basic income may increase consumption and this may result in an increase in the interest rate.

Keywords: HANK, Human Capital, Asset and Income Distribution
JEL classification: D31, E12, E21, E43

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1 Introduction

The heterogeneous model in macroeconomics showed great developments for current decades. One of the major developments to include heterogeneity was the consumer behavior, especially to describe the income and wealth inequality. The more powerful and sophisticated computing technologies and increasing availability of microdata have fueled these developments. Among these developments, the invention of the Heterogeneous agent New Keynesian (HANK) models was one of the key issues raised widespread interests. Kaplan et al. [2018] described HANK literature to understand the monetary transmission mechanism under two types of asset: liquid and illiquid, and Achdou et al. [2021] described the marginal propensity to consume (MPC) heterogeneity and precautionary savings under the HANK economy in detail.

The HANK literature locates themselves very close to the established literature on the income and wealth heterogeneity such as Krusell and Smith [1998], Hubmer et al. [2021], etc. Krusell and Smith [1998] includes a large number of consumers who face idiosyncratic income shocks in its stochastic growth model. One of the major difference between the two literature is the treatment of the time. The HANK literature treat the continuous-time model whereas Krusell and Smith [1998] and related literature uses the discrete-time. There are several advantages/disadvantages for each model, but the continuous-time model has much advantage on the computation performance, or namely the computation speed.

Both in HANK and Krusell and Smith [1998] literature, the income process is set exogenous like $y_t \in \{y_1, y_2\}$ or $dy_t = \mu(y_t)dt + \sigma(y_t)dW$ where y_t is the income of the households. It is natural and suitable to set such income process to analyze the effect of the income shock. On the other hand, it is also worthwhile to set the origin of the income level, the human capital, into the model to capture how the income shock propagates in the long run. So in this paper we introduce another important literature on the inequality: the Human Capital Accumulation into the HANK model and analyze the dynamics of not only the marginal propensity to consume but human capital accumulation.

2 The Model

2.1 Setup

2.1.1 Individuals

There is a continuum of heterogeneous households with respect to their wealth a and human capital h. Each household maximises the expected utility with strictly increasing and concave utility function u and future consumption c_t:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \tag{1}$$

Here we assume CRRA for the utility function, i.e., $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. In this economy we consider the standard bond a_t with corresponding interest rate r_t and the household income y_t. The household wealth evolves according to

$$\dot{a_t} = y_t + r_t a_t - c_t. \tag{2}$$

Here we set the lower limit for a_t as $a_t \geq \underline{a}$, where $-\infty < \underline{a} < 0$.

2.1.2 Human Capitals

The introduction of the human capital to the household income is the major contribution of this model. The income of each household y_t is determined by its level of human capital as Ben-Porath:

$$\begin{aligned} y_t &= (1 - s_t)h_t + \sigma \frac{dW_t}{dt}, & (3) \\ \varphi(s_t h_t) &= \theta(s_t h_t)^\alpha, & (4) \\ dh_t &= \{\varphi(s_t h_t) - \delta_h h_t\} dt & (5) \end{aligned}$$

where h_t is the human capital, $s_t \in [0, 1]$ is the share of time spent investing in human capital, and δ_h is a depreciation rate of the human capital. Individuals maximizes (1) under (2) and (3)-(5).

2.1.3 Hamilton-Jacobi-Bellman Equation and Kolmogorov Forward Equation

Then the value function $v(a_t, h_t)$ can be calculated through the dynamics of the 2 state variables:

$$\begin{aligned} da_t &= \{(1-s_t)h_t + r_t a_t - c_t\}dt + \sigma dW_t, &(6)\\ dh_t &= \{\theta(s_t h_t)^\alpha - \delta_h h_t\}dt. &(7)\end{aligned}$$

Individual's maximization problem under the 2 state variable's dynamics can be described with two differential equations. The first equation is a Hamilton-Jacobi-Bellman equation which describes the dynamics of the value function $v(a_t, h_t)$, and the second is a Kolmogorov Forward Equation which describes the dynamics of the density function of individuals in wealth-human capital phase space: $g(a_t, h_t)$.

$$\begin{aligned} \rho v(a_t,h_t) &= \max_{c_t,s_t} u(c_t) + \partial_a v(a_t,h_t)\{(1-s_t)h_t + r_t a_t - c_t\} + \partial_h v(a_t,h_t)\{\theta(s_t h_t)^\alpha - \delta_h h_t\} &(8)\\ &+\frac{1}{2}\partial_{aa} v(a_t,h_t)\sigma^2 + \partial_t v(a_t,h_t), \\ \partial_t g(a_t,h_t) &= \partial_a[\{(1-s_t)h_t + r_t a_t - c_t\}g(a_t,h_t)] + \partial_h[\{\theta(s_t h_t)^\alpha - \delta_h h_t\}g(a_t,h_t)] &(9)\\ &+\frac{1}{2}\partial_{aa} g(a_t,h_t)\sigma^2 \end{aligned}$$

Here $\partial_a v$ is the short-hand notation of $\partial v/\partial a$ and so on. The price in this economy is only the interest rate r_t and the price is determined to match the supply and demand of the bonds:

$$\int_0^\infty \int_{\underline{a}}^\infty a g(a_t,h_t) da_t dh_t = B \qquad (10)$$

where $0 \leq B < \infty$. B can be positive when there is an external bond supplier such as a government. In this paper we only assume the Huggett economy (Huggett [1993]) and the net supply of bond is kept constant (positive).

2.2 Stationary Equilibrium

2.2.1 Equations

The stationary equilibrium can be obtained with setting $\partial_t v$ of (8) and $\partial_t g$ of (9) as zero:

$$\begin{aligned} \rho v(a,h) &= \max_{c,s} u(c) + \partial_a v(a,h)\{(1-s)h + r_t a - c\} + \partial_h v(a,h)\{\theta(sh)^\alpha - \delta_h h\} &(11)\\ &+\frac{1}{2}\partial_{aa} v(a,h)\sigma^2, \\ 0 &= \partial_a[\{(1-s)h + r_t a - c\}g(a,h)] + \partial_h[\{\theta(sh)^\alpha - \delta_h h\}g(a,h)] &(12)\\ &+\frac{1}{2}\partial_{aa} g(a,h)\sigma^2 \end{aligned}$$

The basic concept for calculating the stationary equilibrium is to utilize the Mean Field Game (MFG) Systems that was initiated by Lasry and Lions [2007]. The detailed explanation of the MFG system under the HANK literature is written in Achdou et al. [2021]. In general, the solutions of these partial differential equations in HANK are basically not classical, but weak or viscosity solutions. This is mainly because the Hamilton-Jacobi-Bellman equation does not have classical (continuous and differentiable) solutions in general. We let the detailed explanation of the difference of the classical and weak / viscosity solution for other mathematical works (e.g., Bardi et al. [1997]), but these calculations are very common in calculating the solution of the Hamilton–Jacobi–Bellman equation in mathematics.

2.2.2 Solving HJB and KF Equation

In this paper we calculate the solution of HJB and KF equation through the implicit method. First define the backward and forward derivative of the value function with respect to a and h as

$$\begin{aligned} \partial_{a,B} v_{i,j} &= \frac{v_{i,j} - v_{i-1,j}}{\Delta a} &(13)\\ \partial_{a,F} v_{i,j} &= \frac{v_{i+1,j} - v_{i,j}}{\Delta a} &(14)\end{aligned}$$

and

$$\partial_{h,B}v_{i,j} = \frac{v_{i,j} - v_{i,j-1}}{\Delta h} \tag{15}$$

$$\partial_{h,F}v_{i,j} = \frac{v_{i,j+1} - v_{i,j}}{\Delta a}. \tag{16}$$

Also, the second order derivative of $v_{i,j}$ with respect to a is caliculated as

$$\partial_{aa}v_{i,j} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta a)^2}. \tag{17}$$

Then the implicit scheme evolves the value function $v_{i,j}^n$ to $v_{i,j}^{n+1}$ with the following relation:

$$\frac{v_{i,j}^{n+1} + v_{i,j}^n}{\Delta} + \rho v_{i,j}^{n+1} = u(c_{i,j}^n) + \partial_a v_{i,j}^{n+1} f_{i,j}^{n,a} + \partial_h v_{i,j}^{n+1} f_{i,j}^{n,h} + \frac{1}{2}\sigma^2 \partial_{aa} v_{i,j}^{n+1} \tag{18}$$

where $f_{i,j}^{n,a} = (1 - s_{i,j}^n)h_j + ra_i - c_{i,j}^n$ and $f_{i,j}^{n,h} = \theta(s_{i,j}^n h_j)^\alpha - \delta_{h_j} h_j$.
The following calculation uses the "upwind scheme", this is an idea to use the forward difference approximation when the drift of the state variable (a and h) is positive and vice versa. The implementation of the upwind scheme can be easily achieved by replacing:

$$\partial_a v_{i,j}^{n+1} f_{i,j}^{n,a} \rightarrow \partial_{a,F} v_{i,j}^{n+1} f_{i,j}^{n,a+} + \partial_{a,B} v_{i,j}^{n+1} f_{i,j}^{n,a-} \tag{19}$$

$$\partial_h v_{i,j}^{n+1} f_{i,j}^{n,h} \rightarrow \partial_{h,F} v_{i,j}^{n+1} f_{i,j}^{n,h+} + \partial_{h,B} v_{i,j}^{n+1} f_{i,j}^{n,h-} \tag{20}$$

where $f_{i,j}^{n,x+}$ $(x = a, h)$ means to use this value when the drift of the related state variables are positive and vice versa. Then we can calculate $I \times J$ matrix to see the conversion of the implicit scheme. The detailed explanation for calculating matrix and any other variables are written in Achdou et al. [2021]. Substituting (13) (17), (19) and (20) into (18) and further calculation leads to

$$\frac{1}{\Delta}(\boldsymbol{v}^{n+1} + \boldsymbol{v}^n) + \rho \boldsymbol{v}^{n+1} = \boldsymbol{u}^n + A^n \boldsymbol{v}^{n+1} \tag{21}$$

where

$$A^n = \begin{bmatrix}
y_{1,1} & z_{1,1} & 0 & \cdots & 0 & \alpha_{1,1} & 0 & 0 & 0 & \cdots \\
x_{2,1} & y_{2,1} & z_{2,1} & 0 & \cdots & 0 & \alpha_{2,1} & 0 & \cdots & \cdots \\
0 & x_{3,1} & y_{3,1} & z_{3,1} & 0 & \cdots & 0 & \alpha_{3,1} & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \cdots \\
0 & \ddots & \ddots & x_{I,1} & y_{I,1} & 0 & 0 & 0 & \alpha_{I,1} & \cdots \\
\beta_{1,2} & 0 & 0 & 0 & 0 & y_{1,2} & z_{1,2} & 0 & 0 & \cdots \\
0 & \beta_{2,2} & 0 & 0 & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & \cdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & 0 & \beta_{I,2} & 0 & \cdots & x_{I,2} & y_{I,2} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \cdots
\end{bmatrix} \tag{22}$$

and $y_{i,j} = -f_{i,j}^{n,a+}/\Delta a + f_{i,j}^{n,a-}/\Delta a - f_{i,j}^{n,h+}/\Delta h + f_{i,j}^{n,h-}/\Delta h - \sigma^2/\Delta a^2$, $x_{i,j} = -f_{i,j}^{n,a-}/\Delta a + \sigma^2/2\Delta a^2$, $z_{i,j} = f_{i,j}^{n,a+}/\Delta a + \sigma^2/2\Delta a^2$, $\alpha_{i,j} = f_{i,j}^{n,h+}/\Delta h$, $\beta_{i,j} = -f_{i,j}^{n,h-}/\Delta h$.

2.2.3 Solving KF Equation

The Kolmogorov Forward Equation (or Fokker-Planck Equation) can also be calculated through the upwind scheme. The calculation starts from equqation (9) and (10). If we conduct similar calculation with the HJB equation, the final result corresponding to (21) becomes

$$A^T g = 0 \tag{23}$$

where A^T is the transpose of the matrix A (A^n from the final HJB iteration).

2.2.4 Computational Strategy

The basic idea for the computation is the same as Achdou et al. [2021], using a bisection algorithm on the stationary interest rate. We iterate the following procedure until the change in value function $|v^l - v^{l-1}|$ becomes adequately small with initial guess interest rate r^0 and l increases as $l = 0, 1, 2, ...$:

- Given r^l, solve the Hamilton-Jacobi-Bellman equation in equilibrium (11) to determine c^l, s^l and v^l.
- Given s^l, solve the Kolmogorov Forward Equation equation in equilibrium (12) using a Finite Difference Method.
- Given $g^l(a, h)$, compute the net supply of bonds $B = S(r^l) = \int_{\underline{a}}^{\infty} a g^l(a, h) dadh$ and update the interest rate.

We update the interest rate negatively when $S(r^l) > B$ and positively when $S(r^l) < B$. For more detail, we set r^l_{max} and r^l_{min} and update r^{l+1}_{max} as r^l when $S(r^l) > B$ and update r^{l+1}_{min} as r^l when $S(r^l) < B$. We stop the calculation if the change in value function $|v^l - v^{l-1}|$ becomes less than a criteria.

2.3 Transition Dynamics Calculations

Before discussing the technical features for the transition dynamics calculation, let us define the transition in this paper. Here we assume the transition from natural state (without any external endowment) to the Basic Income state without any prior announcement. The Basic Income state can be summarized by changing wealth dynamics (2) and income process (3) to

$$\dot{a}_t = y_t + r_t a_t - (1+\tau) c_t \tag{24}$$

$$y_t = (1 - s_t) h_t + BI + \sigma \frac{dW_t}{dt} \tag{25}$$

where τ is the consumption tax rate and BI is the amount of the Basic Income distributed to all households, defined as $BI = \tau \times \int_0^\infty \int_{\underline{a}}^\infty cg(a, h) dadh$. It is clear that if we consider the RANK (representative agent new Keynesian), the introduction of BI does not affect any consumption profiles because the basic income only take a tax from the representative agent and re-distribute it for the same amount. However, the introduction of the human capital accumulation allows us to consider the non-linear effect with respect to the level of the human capital in heterogeneous households.

We assume that this economy transit from natural state to the BI state at $t = 0$ without any previous notice. The method to solve HJB and KF equation is almost the same as the equilibrium calculation. The time-dependent HJB equation can be rewritten with using A as:

$$\rho \boldsymbol{v}_t = \boldsymbol{u}_t + A^{t+1} \boldsymbol{v}_t + \frac{1}{\Delta t}(\boldsymbol{v}_{t+1} - \boldsymbol{v}_t) \tag{26}$$

and KF equation as

$$\frac{g_{t+1} - g_t}{\Delta t} = (A^t)^T g_t. \tag{27}$$

The calculation of the transition dynamics from the initial state (economy without Basic Income) is straight forward by using Newton method or any other differential method.

3 Computational Results

This section presents computational results about the continuous-time Huggett economy model. Our main result can be summarized as the following 3 folds: (i) the introduction of the basic income may increase the share of the liquidity constrained households, (ii) the introduction of the basic income results in decrease of the aggregated share of time spent investing in Human Capital, and (iii) increase the interest rate due to less wealth demand and more consumption. The summary of the key parameter values are as shown in Table 1. Here it should be emphasized that these parameters are not calibrated based on certain economy, as the parameter calibration is much more complicated compared to the standard New Keynesian model.

Figure 1 shows the initial (without basic income, equilibrium state) and terminal (with basic income, equilibrium state) distribution of households ($g(a, h)$), and Figure 2 shows its differences. Figure 2 explains the human capital (or the income itself) seems to increase for most households as the distribution in low human capital area decreases and vise

Table 1: Summary of Key Parameters

Parameters	Value
α	0.5
θ	0.1
γ	1.5
ρ	0.04
δ	0.032
h_{max}	e^4
h_{min}	$e^{2.5}$
σ^2	$(h_{max} + h_{min})/4$
S_0	20
τ	0.1

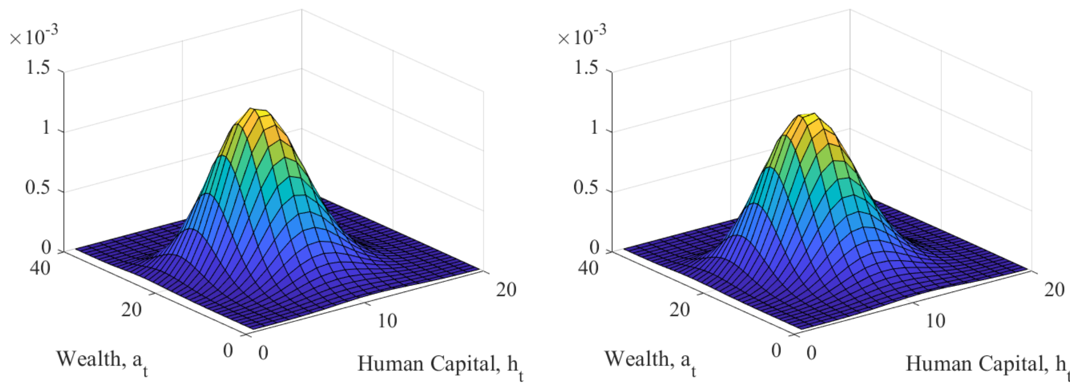

Figure 1: Initial (left) and terminal (right) Distribution of Households.

versa. Also, if we focus on the distribution of the liquidity constrained ($a = \underline{a}$), the difference value is positive and this means that the introduction of the basic income may increase the share of the liquidity constrained households. This result also contradict to some empirical results, such as Daidone et al. [2014]. Actually the share of the liquidity constrained decreases at the beginning, but the share again increases with changing interest rate. Mainly the decrease in wealth demand affects the increase in share of the liquidity constrained.

The dynamics of interest rate is described in Figure 3. The interest rate shows impulse decrease at first, but after that recovers and equilibrium interest rate increases at the end. This is mainly because the economy has less wealth demand, more consumption and less human capital invests. To explain this result, let us first show what is the shape of s_t and

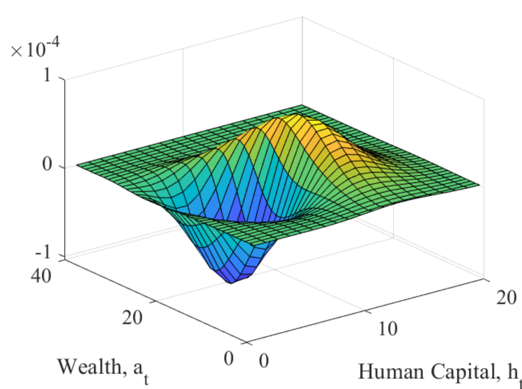

Figure 2: Difference in initial and terminal Distribution.

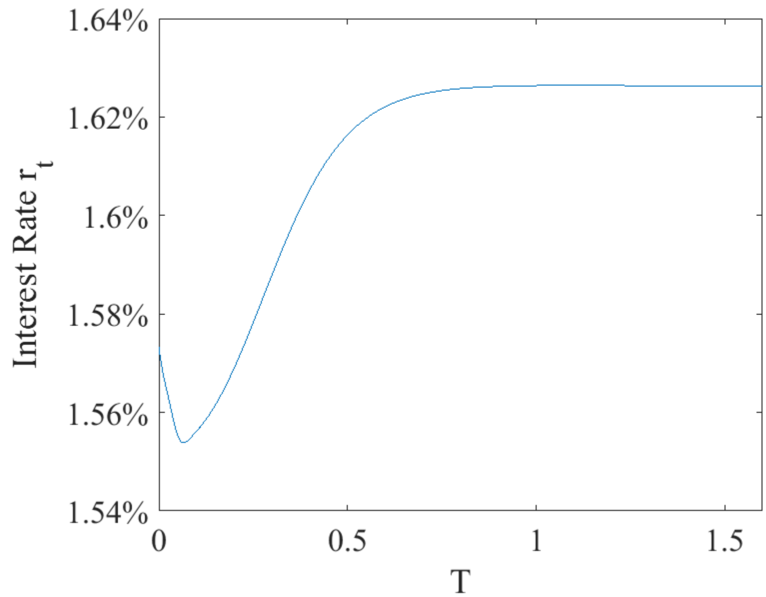

Figure 3: Dynamics in Interest Rate (year).

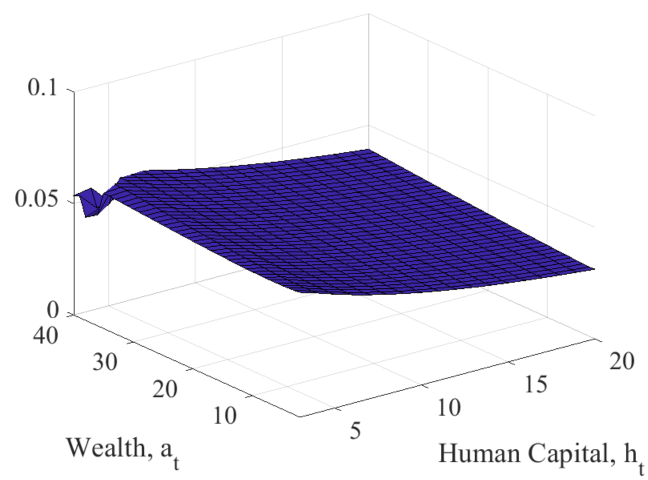

Figure 4: Initial s_t for heterogeneous households.

how it changes across time. Figure 5 describes the shape of s_t at initial state and Figure 4 shows the change in s_t from initial to equilibrium state (s_∞ - s_0). As shown in Figure 4, the share of time spent investing in human capital increases for most meshes. However, the distribution of households shifts with the introduction of the Basic Income and changing interest rate according to Figure 2, and this effect is much larger than the increase of s_t for most households. The dynamics of aggregated s_t, averaged time spent investing in human capital is described in Figure 6. The main reason for the decrease in time spent investing in human capital is due to the increase of the income of the low-human capital households, which is mainly caused by the change in household distribution. In general, the marginal cost for investing human capital is much lower when the human capital of the household is low and vice versa. This non-linearity yields the deviation from the RANK model. As is described in Figure 4, the major driver for determining s_t is the value of the human capital itself (h_t).

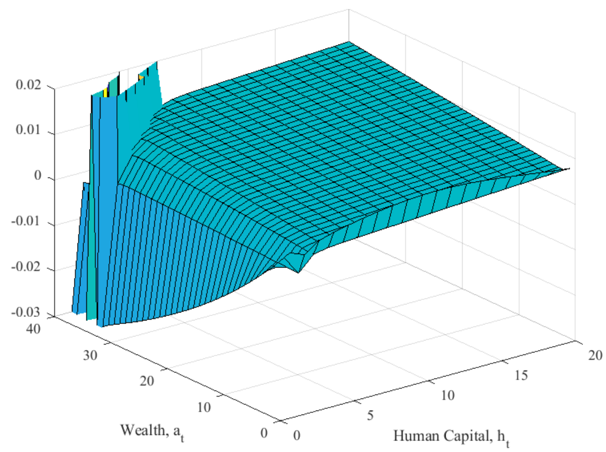

Figure 5: Difference in initial and terminal s_t.

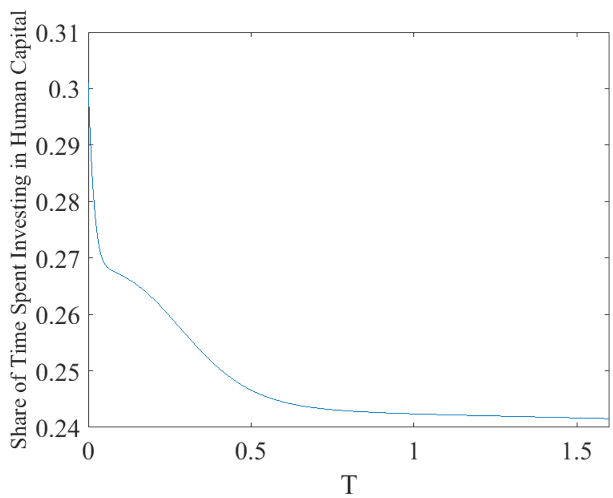

Figure 6: Dynamics of Aggregated Share of Time Spent Investing in Human Capital (year).

4 Discussion

This paper analyzes how the introduction of basic income affects the equilibrium interest rate and human capital formation in society as a whole, by using HANK model with human capital accumulation. The conclusions of this analysis can be again summarized by the following 3 folds: (i) the introduction of the basic income may increase the share of the liquidity constrained households, (ii) the introduction of the basic income results in decrease of the aggregated share of time spent investing in Human Capital, and (iii) increase the interest rate due to less wealth demand and more consumption. Some of the results, especially (ii), may seen to be contradict to empirical study results on the Basic Income, such as Gibson et al. [2018]. In most analysis, the Basic Income has effects to (a) decrease hours worked and increase hours in home, (b) increase educational participation. One big reason for this contradiction may be raised from the model assumption. We did not introduce the leisure term into the utility function and therefore we can not replicate the effect of the Basic Income to increase hours in home. On the other hand, the effect to increase educational participation should be discussed carefully. In most empirical results, the treatment group of the Basic Income is not so large in most cases and the effect to change the interest rate due to the introduction of the BI is limited. If we could establish much more large field and when the change in interest rate due to the basic income can not be negligible, the huge shift of the households may occur as described in the Figure 2. The interpretation of these results is that the introduction of basic income does not necessarily reduce the number of liquidity-constrained households, and the increase in the number of such households leads to an increase in consumption in society as a whole, which may weaken the progress of human capital formation for households. Although this fact itself appears to be consistent at first glance, it should be noted that this analysis is the result of calculations based only on a limited set of parameters. The list of parameters is shown in Table 1, but since excessive changes in the values of α and θ, for example, may cause the model itself to fail to terminate. Therefore it is necessary to verify the robustness of the above results by conducting future analyses for a variety of parameters as a first step.

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