RIETI Discussion Paper Series 21-E-058

# Deflation and Declining Business Dynamism in a Cash-in-Advance Economy

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RIETI Discussion Paper Series 21-E-058
**July 2021**

# Deflation and Declining Business Dynamism in a Cash-in-Advance Economy[1]

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Abstract

This study considers the relationship between deflation and declining business dynamism. We do this by incorporating the empirical evidence that inflationary factors essentially affect all stages of an R&D firm's life cycle—entry, exit, and survival—into an R&D-based growth model. Our model has a new feature; namely, the entry, exit, and survival of R&D firms are all endogenous and subject to a cash-in-advance constraint. The core finding is that deflation can significantly affect the nature of business dynamism. Specifically, a decrease in the inflation rate potentially encourages or discourages innovation and survival investments; however, it necessarily discourages both if the entry cost is sufficiently high. In this case, deflation stifles business dynamism, leading to lower entry and exit rates and a maturity bias in the firm age distribution. Calibrating the model to the U.S. economy, we show that deflation causes declining business dynamism under realistic values of entry, exit, and growth rates. Then, we show that deflation also causes welfare loss if the natural rate of firm exit is higher.

Keywords: Cash-in-advance constraints, inflation, firm dynamics, R&D-based growth
JEL classification: O31, O34, O41



[1]This study has been conducted as part of a research project, Social Scientific Studies on Self-replicating Natural and Technical Phenomenon, undertaken by the Research Institute of Economy, Trade, and Industry (RIETI). The authors would like to thank Chris Dai, Taiji Furusawa, Tetsugen Haruyama, Arata Ito, Kazuo Mino, Masayuki Morikawa, Rui Ota, Makoto Yano, Lex Zhao, and the seminar/conference participants at the 2020 IEFS Japan Annual Meeting, 58th NIESG Meeting, and RIETI for their valuable and constructive comments.
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# 1 Introduction

The Japanese economy has long suffered from a double downtrend. Like many other advanced economies including the U.S. economy, it now faces a declining trend in business dynamism—the process by which firms continuously entry, exit, and survive.[1] This trend is typically characterized by a decreasing entry rate and an increasing share of older firms.[2] Such declining business dynamism, as many thinks, may hurt stable long-term growth. Meanwhile, as is well perceived, Japan's monetary policy since the 1990s has struggled with deflationary pressures. This "double whammy" of downtrends can also stifle other economies that experience declining business dynamism under current deflationary pressures, which have more or less covered the world at least since the 2008 crisis;[3] as a *Bloomberg* article writes, "[d]eflation remains the bigger danger" (Moss, 2020).[4]

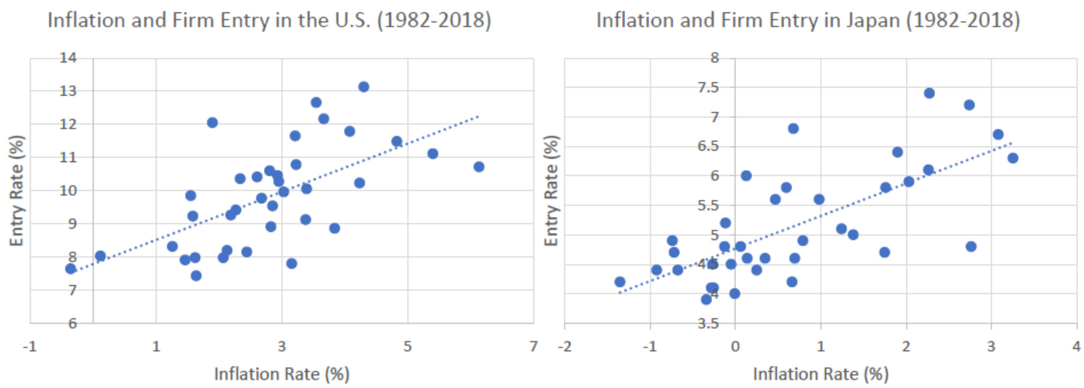


Figure 1: Inflation and Firm Entry

While existing research has considered these two issues—which have coexisted for a decade or longer—separately, facts illustrated in Figure 1 suggest a positive correlation between the inflation rate and the firm entry rate in the U.S. and Japan.[5] This implies that business dynamism more likely declines under tighter deflationary pressures. However, given that no formal analysis has ever linked these two downtrends, their relationship is still a puzzle to the literature. This study explains this relationship as a market equilibrium phenomenon.

We, therefore, consider the two downtrends in inflation and business dynamism together in a single setup. For this, we utilize the well-established evidence as a missing

[1] See Hong et al. (2019) for the Japanese case. See also Calvino, Criscuolo, and Verlhac (2020), who show that many other countries face downtrends in business dynamics.

[2] For other characteristics, see Akcigit and Ates (2021), who document ten stylized facts relating to declining business dynamism. Here, we pick up two crucial characteristics from the ten.

[3] From a more long-run perspective, it is fair to say that the inflation rate has fallen on average since 1980s, for many developed countries including the U.S.

[4] Most recently, the Consumer Price Index for the U.S. jumped higher than economists anticipated in April and May, 2021. This could imply that inflation worries, rather than deflation pressures, come back. However, some people think it is still arguable or, to say the least, they are small enough; for example, see a *Voice of America* article (Garver, 2021) and a *Slate Magazine* article (Weissmann, 2021).

[5] For the U.S., we use data for entry rates from the Business Dynamics Statistics from the U.S. Census Bureau. For Japan, we use data for entry rates from the White Paper on Small Enterprises in Japan 2020 (see Figure 1-3-5), published by Small and Medium Enterprise Agency in Japan. The inflation rate data for both countries (based on consumer prices) is from the World Development Indicators of the World Bank. Notably, at least for the U.S. economy, we can also find a significant positive relationship between inflation and entry for R&D-intensive industries like the pharmaceutical or electronics industry.

link between these two downtrends: Nominal factors such as inflation (or deflation) can essentially affect all stages of firms' life cycle—entry, exit, and survival—because they face severe financial constraints (Hall, 2008). This evidence indicates a potential linkage between inflationary factors and the nature of business dynamism. Incorporating this fact allows us to explain declining business dynamism as a market equilibrium phenomenon, caused by deflationary trends.

To this end, we propose a new theory that explains how the inflation rate affects the dynamics of firms and thereby economic growth. Specifically, we develop a new research-and-development (R&D)-based growth model, in which, following Romer (1990), firms in the entry stage invest optimally in the innovation of new goods. To incorporate money demand, which is necessary to examine the role of inflation, we closely follow Chu and Cozzi (2014), imposing a cash-in-advance (CIA) constraint on firms' R&D investments, which are used to innovate a new variety of products. In order to examine the entry stage in detail, we assume that firms that successfully innovate new products have to pay an entry cost to start production. Further, we consider a fully endogenous distribution of firm age by assuming that after the success of innovation, firms in our model invest in survival to delay the obsolescence of their innovated goods. This survival investment also faces a CIA constraint if the internal cash (from the profits at each date) fails to cover their survival investment needs, which in in line with extant empirical evidence (Musso and Schiavo, 2008).[6] This is different from existing models, where firms that successfully innovate new goods neither invest in survival nor face a CIA constraint in the survival and exit stages. In our model, monetary factors can affect three critical characteristics of business dynamism—an entry rate, exit rate, and average firm age.

The core finding is that deflation can be a fundamental cause of declining business dynamism. Potentially, a decrease in the inflation rate may encourage or discourage both innovation and survival investments. Our analysis identifies the role of entry cost in determining the direction of effect of inflation on innovation and survival. If the entry cost is higher than some threshold value, a lower inflation rate discourages entry but encourages survival, thus, leading to a lower entry and exit rate of firms. This implies that deflation decreases both entry and exit rates; as a result, it causes a maturity bias in the firm-age distribution, namely, if entries are costlier, deflation causes declining business dynamism. This result essentially relies on the empirically plausible assumption that startup firms investing in innovation face more stringent CIA constraints than incumbents counterparts (Hall, 2008). Calibrating the model to the U.S. economy, we show that deflation causes declining business dynamism under realistic values of entry, exit, and inflation rates because the calibrated value of an entry cost exceeds the threshold. We then show that the Friedman rule is suboptimal if the natural rate of firm exit is higher, suggesting that deflation can also cause welfare loss.

Our study closely relates to a traditional macroeconomic issue. Specifically, the macroeconomic literature, stemming from Tobin (1965), explored the relationship between inflation and real investment activities. For instance, Stockman (1981) and Abel (1985) initiated an influential literature stream by focusing on incorporating a CIA framework for money demand into neoclassical capital accumulation models, which was pioneered

[6] Evidence reveals that such investments in market survival are also financially constrained in several circumstances. From a broader context, financing investments in intangible assets, such as R&D, is essentially sensitive to cash/financial constraints because of the firm-specific and inalienable nature of intangible assets. For a helpful review on the empirical literature, see, for instance, Chen (2014) and Morikawa (2015).

by Lucas (1980), following Clower (1967).

More recently, a breakthrough occurred in this area, as Chu and Cozzi (2014) extended the analysis to a more recent class of growth models, that is, R&D-based models.[7] They achieved this by incorporating the aforementioned empirical evidence that R&D investments face significantly severe cash/financial constraints (Hall, 2008). Since then, this literature has examined the role of nominal interest rates or inflation in innovation, innovation-driven growth, and other economic phenomena such as unemployment and income inequality both theoretically and empirically. Such studies include those by Chu et al. (2015, 2017, 2019a, 2019b, 2020), He and Zou (2016), He (2018), Hori (2020), Huang et al. (2020), and Zheng et al. (2020a).[8] We complement these studies by examining the effects of the CIA constraint on R&D firm dynamics, not only at the entry stage but also for survival, and thereby the relationship between deflation and declining business dynamism. Our approach differs from those of existing studies, which consider neither firm exit nor survival investment were, in that firms' survival activity, exit rate, and distributions for age and size are all endogenous, depending on the different degrees of CIA severity for R&D and market survival.

In different contexts, several studies also examined the relationships between inflation, innovation, and growth; as recent examples, see Arawatari et al. (2018) and Zheng et al. (2020b). In particular, as in this study, Miyakawa et al. (2020) examined the effects of inflation on firm dynamics, but with a different source of money demand (i.e., endogenous price revisions in a new Keynesian with menu costs). Further, their focus is on a resource reallocation role of monetary policy. We complement their pioneering analysis by focusing on the linkage between deflation and declining business dynamism and providing an explanation for the mechanism behind linkage based on a CIA approach.

Some recent studies in endogenous growth theory explicitly address declining business dynamism. For example, Akcigit and Ates (2021) document ten stylized facts on declining business dynamism, and explain them by developing a new endogenous growth model featuring product market competition with strategic interaction between competing firms. Haruyama (2021) shows declining business dynamism can explain increasing trends in income inequality developing a new Schumpeterian growth model. Unlike these models, we focus on monetary factors as a fundamental cause of declining business dynamism. Therefore, we complement the existing literature by finding an essential linkage between inflation and business dynamism.

In modeling a firm's dynamic optimization for survival, we follow the growth models with endogenous firm survival developed by Dinopoulos and Syropoulos (2007) and Eicher and García-Peñalosa (2008). Subsequent studies, such as Grieben and Sener (2009) and Davis and Sener (2012), further extended the analysis to other dimensions. Further, Akiyama and Furukawa (2009), Furukawa (2013), Furukawa and Yano (2014), and Niwa (2018) considered endogenous firm survival under a so-called variety expansion growth model, similar to the present study. We contribute to this literature stream by identifying a new role of endogenous firm survival in the relationship between inflation and long-run growth, which has long been one of the most important topics in macroeconomics.

The remainder of the paper proceeds as follows. Section 2 proposes a new monetary R&D-based growth model with endogenous firm survival facing a CIA constraint. Section

[7] Romer (1990) first developed the R&D-based growth model with an expanding variety of goods. Meanwhile, Segerstrom et al. (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) developed the Schumpeterian growth model with quality improvement.

[8] See Chu (2020) for an extensive review.

3 examines the effects of inflation on the nature of business dynamism. Section 4 examines the effects of inflation on growth and welfare. Section 5 calibrates the model to U.S. data and provides the quantitative results. Section 6 entends the basic model by allowing for an endogenous distribution for firm size. Section 7 discusses the role of financial innovation to draw policy implications. Finally, Section 8 draws conclusions.

# 2 A Monetary R&D-Based Growth Model with Endogenous Firm Survival

## 2.1 Consumption

We consider a variety expansion model of endogenous growth by referencing Romer (1990) and Rivera-Batiz and Romer (1991). In this model, time is continuous and extends from 0 to $\infty$. There is a single final good, taken as numeraire. An infinitely lived representative consumer supplies inelastically one unit of labor and consumes $c_t$ units of final goods at each time point, $t$. The utility function is:

$$U = \int_0^\infty e^{-\rho t} \ln c_t dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate.

Following Chu and Cozzi (2014), money is introduced by assuming a CIA constraint on R&D firms' investment; the asset accumulation constraint in real terms is:

$$\dot{a}_t + \dot{m}_t = r_t a_t + i_t b_t + w_t L + \tau_t - c_t - \pi_t m_t, \tag{2}$$

where $a_t$ denotes the real value of financial assets (i.e., the equity of monopolistic firms), $m_t$ the real value of cash holdings, $i_t$ the nominal interest rate, $r_t \equiv i_t - \pi_t$ the real interest rate, $b_t \leq m_t$ the real-term amount of cash borrowed by firms, $w_t$ the real wage rate, $\tau_t$ the real value of transfers from the government, and $\pi_t \equiv \dot{P}_t/P_t$ the inflation rate. By solving the standard dynamic optimization, we obtain the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{3}$$

## 2.2 Production

The market for a final good is perfectly competitive. Firms convert differentiated intermediate inputs, each indexed by $j$, into $Y_t$ units of final goods. We then consider a Cobb–Douglas production function:

$$Y_t = L_t^{1-\alpha} \int_0^{N_t} x_t(j)^\alpha dj, \tag{4}$$

where $L_t$ is the amount of labor input, $N_t$ the number of intermediate inputs, and $x_t(j)$ the amount of intermediate input $j$ used. The factor share of intermediate inputs, $\alpha$, satisfies $\alpha \in (0, 1)$. Static optimization yields the demand function for good $j$ as:

$$x_t(j) = \alpha^{1/(1-\alpha)} L_t p_t(j)^{-1/(1-\alpha)}, \tag{5}$$

where $p_t(j)$ is the real price of final good $j$.

The market for differentiated intermediates is monopolistically competitive. A monopolistic firm that originally innovates good $j$ (or the firm that purchases the patent on good $j$ from the original inventor) manufactures each input $j$. We assume that manufacturing one unit of good $j$ requires one unit of final good as input and the marginal cost is equal to unity. Because the price elasticity of demand for any good $j$ is $1/(1-\alpha)$ from (5), monopolistic pricing yields $p_t(j) = 1/\alpha$. Then, substituting this equilibrium price into (5) yields

$$x_t(j) = \alpha^{2/(1-\alpha)} \equiv x \tag{6}$$

when using the labor market clearing condition, $L_t = 1$, which denotes the profit for good $j$. The equilibrium profit is:

$$\Pi_t(j) = \alpha^{\frac{1+\alpha}{1-\alpha}} (1-\alpha) \equiv \Pi. \tag{7}$$

## 2.3 Endogenous R&D and Entry

At each date $t$, there is a continuum of perfectly competitive potential R&D firms in the economy. We assume the number of potential firms, $K_t$, to be proportional to the current number of innovations, $N_t$ (as a proxy of the cumulative knowledge that is not yet obsolete), that is, $K_t = N_t$. Each R&D firm can innovate one new technology to produce a new consumption good with probability $\psi_t dt$ during a short time interval $dt$. Here, the R&D firm invests $(k_t/\kappa)dt$ units of final goods during the same time interval, where $\kappa > 0$ denotes their productivity. That is, $\psi_t$ denotes a Poisson arrival rate for innovation. We consider a concave production function for R&D, $\psi_t = (k_t)^\gamma$, where $\gamma \in (0,1)$.

In order to consider the R&D firm's life cycle as an endogenous phenomenon, we observe the role of an entry fee, departing from Chu and Cozzi (2014).[9] Thus, we assume that after a successful innovation, the R&D firm needs to pay $\varpi/\kappa > 0$ units of the final good for entry. We denote by $v_t$ the real value of an innovation after entry (without entry, their benefit is zero). Then, perfectly competitive firms face the following optimization problem:

$$\max_{k_t:\ \text{s.t. } \psi_t=(k_t)^\gamma} (\psi_t dt)(v_t - \varpi/\kappa) - (1+\zeta i)(k_t/\kappa)dt. \tag{8}$$

Here, we follow Chu and Cozzi (2014) in assuming that R&D investments face a CIA constraint. Specifically, in paying $k_t/\kappa$, a firm has to prepare $\zeta k_t/\kappa$ units of cash in advance. To do so, the R&D firm has to borrow from households because it is a startup company with no cash. Parameter $\zeta \in [0,1]$ represents the severity of the financial constraint for startup firms trying to innovate. By solving (8), we obtain the optimal levels for R&D investment and its success probability:

$$k_t^* = \left(\frac{\gamma(\kappa v_t - \varpi)}{1+\zeta i}\right)^{\frac{1}{1-\gamma}} \text{ and } \psi_t^* = \left(\frac{\gamma(\kappa v_t - \varpi)}{1+\zeta i}\right)^{\frac{\gamma}{1-\gamma}}, \tag{9}$$

assuming $v_t \geq \varpi$ (which will be ensured to hold in equilibrium).

[9] This is more in line with Chu et al. (2017), who identify the critical role of entry cost in explaining the nonmonotonic effects of inflation on innovation and growth. On the contrary, we focus on idetifying a relationship between inflation and R&D firm dynamics.

## 2.4 Endogenous Survival and Exit

To examine the relationship between inflation and business dynamism, in addition to the entry cost, we further depart from Chu and Cozzi (2014) by endogenizing the exit rate for R&D firms, which is a novel approach in the literature.[10] We thus assume that each innovation (or the relevant R&D firm) faces a risk of obsolescence. Obsolescence occurs with probability $\delta_t dt$ during a short time interval $dt$, forcing firms to exit the market. Therefore, a firm would invest resources in survival by marketing, advertisement, or protecting intellectual property rights. In describing the dynamic process of firm survival, we assume that, when the firm invests $z_t/\kappa$ units of a final good, the hazard (Poisson arrival) rate for obsolescence is $\delta_t \equiv \overline{\delta} - z_t$. Here, $\overline{\delta}$ gives the natural upper bound for the exit rate.

There are two critical considerations for the survival of innovation. The first is that, when making a survival investment, the firm also faces a CIA constraint: to facilitate the payment of $z_t/\kappa$, it has to prepare $\xi z_t/\kappa$ amount of cash by borrowing from consumers, where $\xi \in [0,1]$. As we already mentioned in the introduction, this consideration is supported empirically. Another critical factor is that, different from R&D investment, the firm already earns profit $\Pi_t$ and thereby holds some cash. In reality, firms typically use their own internal reserves for investment covering survival purposes. Naturally, we assume that the firm can use fraction $(1-\theta)$ of profit $\Pi_t$ (i.e., $(1-\theta)\,\Pi_t$) for survival investment (in terms of final goods) at each date $t$, where $\theta \in [0,1]$ is a parameter capturing corporate culture.[11] As a result, there are two types of equilibrium. If the internal reserve can cover the cash required for survival investment, there is no need to borrow cash from consumers (i.e., no borrowing case, labeled as $\omega = 0$); otherwise, the firm borrows $(\xi z_t/\kappa - (1-\theta)\,\Pi_t)$ units of cash from consumers at cost $i_t$ (i.e., borrowing case, labeled as $\omega = 1$). For simplicity, we introduce an indicator function, $\lambda^\omega$, such that $\lambda^0 = 0$ (i.e., no borrowing for survival) and $\lambda^1 = 1$ (i.e., borrowing for survival).[12] Therefore, the total payment for the survival investment becomes:

$$I_t \equiv \frac{z_t}{\kappa} + i_t \max\left\{\frac{\xi z_t}{\kappa} - (1-\theta)\,\Pi_t, 0\right\} \equiv \frac{z_t}{\kappa} + \lambda^\omega i_t\left(\frac{\xi z_t}{\kappa} - (1-\theta)\,\Pi_t\right). \tag{10}$$

Now, we can express the dynamic optimization for existing firms as:[13]

$$\max_{z_t:\ \text{s.t.}\ \delta_t \equiv \overline{\delta} - z_t} W_t \equiv (\Pi_t - I_t)\,dt + (1-\delta_t dt)\,\dot{v}_t dt - (\delta_t dt - \eta dt) v_t, \tag{11}$$

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[10] The novelty for the growth literature on endogenous survival of R&D firms (Dinopoulos and Syropoulos, 2007, and Eicher and García-Peñalosa, 2008) is incorporating the money demand via the CIA constraint on R&D investments.

[11] One can consider ratio $\theta$ to be endogenously determined through any maximization. In our model, $\theta = 0$ is optimal. Therefore, we consider $\theta$ to be exogenous because, in reality, firms do not typically invest all profit into a single plan. We are also interested in the role of corporate culture in the use of profits.

[12] We implicitly assume that firms have to pay out the remainder of their available net internal reserves for survival investment $(1-\theta)\,\Pi_t - \lambda^\omega \xi z_t/\kappa$ if it exists.

[13] See Furukawa (2013) and Niwa (2018) for a discrete-time version of a similar setup. For more details, during the short time interval $dt$, the firm obtains $(\dot{v}_t + \eta v_t)\,dt$ with probability $(1-\delta_t dt)$ and loses the current value $v_t$ with probability $\delta_t dt$. Further, in nominal terms, (11) is written as

$$W_t^n + \pi v_t^n = (\Pi_t^n - I_t^n)\,dt + (1-\delta_t dt)\,\dot{v}_t^n dt - (\delta_t dt - \eta dt) v_t^n,$$

where variables with superscript $n$ denotes nominal values. Given that $\dot{v}_t^n = \dot{P}_t v_t + P_t \dot{v}_t$. This implies that $W_t$ is defined as the real value of innovation *minus* a nominal benefit of inflation.

where $\eta$ denotes an exogenous growth factor for firms, whose interpretation is provided in Section 2.6. Solving (11),[14] we obtain the following condition:

$$v_t = \frac{1 + \lambda^{\omega}\xi i}{\kappa} \equiv v, \tag{12}$$

which holds unless $z_t = 0$ (i.e., a trivial case with no survival activity, which we ignore in the primary analysis).

## 2.5 Monetary Authority

Following the literature (Chu and Cozzi, 2014), the monetary authority exogenously sets $i_t = i$ as a stationary policy instrument. The Fisher equation is $i = \pi_t + r_t$. Denoting the aggregate *nominal* money balance as $M_t$, its growth rate is given by:

$$\mu_t \equiv \frac{\dot{M}_t}{M_t} = \pi_t + \frac{\dot{m}_t}{m_t} = i - r_t + \frac{\dot{m}_t}{m_t} = i - \rho - \frac{\dot{c}_t}{c_t} + \frac{\dot{m}_t}{m_t}, \tag{13}$$

where the last equality uses the Euler equation (3). Given a stationary nominal interest rate, $i$, real consumption $c_t$ and aggregate real money balance $m_t$ grow at the same rate on a balanced growth path, $\mu = i - \rho$. Then, the monetary authority returns the seigniorage revenue as a lump-sum transfer $\tau_t = \dot{m}_t + \pi_t m_t$. See Chu and Cozzi (2014) for details.

## 2.6 Dynamic General Equilibrium

Here, we characterize the equilibrium dynamics for the aggregate economy under three conditions. First, given that the benefit of owning a bond of price $v_t$ over the small time interval of $dt$ is $(r_t dt)v_t$, the standard Bellman equation is:

$$r_t v_t = \left(1 + \lambda^{\omega} i\left(1 - \theta\right)\right)\Pi_t - (\overline{\delta} - \eta)v_t \tag{14}$$

from using (11) and (12).

Second, the firm dynamics at the aggregate level can be summarized by the following differential equation:

$$\dot{N}_t = \psi_t K_t - \left(\overline{\delta} - z_t\right)N_t + \eta N_t, \tag{15}$$

which uses $\delta_t = \overline{\delta} - z_t$.

We now explain the meaning and role of an exogenous growth factor, $\eta$. First, we need $\eta > 0$ to ensure positive growth (and then have an endogenous evolution of firm size distribution), since our model is even more restrictive than the standard endogenous growth models, given the presence of borrowing and survival investment. Then, we facilitate positive growth by following Anderlini et al. (2013). Specifically, from (11) and (15), we use the exogenous growth factor, $\eta > 0$.[15] Firms introduce new goods in the market under a profit-motivated R&D investment and through "invention by accident,"

[14] As a standard argument, we can ignore term $(dt)^2$.

[15] Exogenous growth factors are often considered for a deeper understanding of the role of technological progress in various phenomena. For instance, see Lucas and Moll (2014) and Benhabib et al. (2017). Our model also includes endogenous growth factors of innovation and survival. In this sense, it is closer to Anderlini et al. (2013), who considered both endogenous and exogenous growth factors.

which sometimes occurs in reality.[16] Without any intended investment or effort, firms can create new ideas by accident, or even by mistake, as a byproduct of regular activities (in our case, production or survival).[17] Such an accident occurs with a Poisson arrival rate of $\eta > 0$. Further, the expected value for invention by accident is $(\eta dt)\, v_t$ during a short time interval. When this happens, the existing firm innovates another good and obtains an additional value of $v_t$ (i.e., $2v_t$ in total).[18]

Third, the final good market equilibrium condition is:

$$Y_t = c_t + N_t x_t + \left(\frac{k_t + \varpi}{\kappa}\right) K_t + \frac{z_t}{\kappa} N_t, \tag{16}$$

where the supply of final goods is $Y_t$ and demand results from consumption $c_t$, production $N_t x_t$, innovation $k_t K_t/\kappa$, and survival $z_t N_t/\kappa$.

By incorporating (3) and (12) into (14) and $K_t \equiv N_t$ into (9), we can derive the law of motion for $(c_t, N_t)$:

$$\frac{\dot{c}_t}{c_t} = \frac{1 + \lambda^{\omega}(1-\theta)\, i}{1 + \lambda^{\omega} \xi i} \kappa \Pi - (\overline{\delta} + \rho - \eta) \tag{17}$$

and

$$\frac{\dot{N}_t}{N_t} = \psi_t^* + z_t + \eta - \overline{\delta}. \tag{18}$$

We then solve (16) for $z_t$:

$$z_t = \frac{1+\alpha}{\alpha} \kappa \Pi - \left(\kappa \frac{c_t}{N_t} + (k_t^* + \varpi)\right), \tag{19}$$

which uses (4), (6), (7), and (9) with $K_t \equiv N_t$. The following lemma determines the equilibrium values for R&D investment.

**Lemma 1** *For any $t \geq 0$, the equilibrium R&D investment and success rate are given by:*

$$k_t^* = k^* \equiv \left(\frac{\gamma(1 - \varpi + \lambda^{\omega}\xi i)}{1 + \zeta i}\right)^{\frac{1}{1-\gamma}} \text{ and } \psi_t^* = \psi^* \equiv \left(\frac{\gamma(1 - \varpi + \lambda^{\omega}\xi i)}{1 + \zeta i}\right)^{\frac{\gamma}{1-\gamma}}. \tag{20}$$

**Proof.** Substituting (12) into (9) yields (20). ■

We define $\left(\eta_0, \overline{\delta}_0, \kappa_0\right)$ as the threshold values of $\left(\eta, \overline{\delta}, \kappa\right)$. Then, we have the following lemma on the uniqueness and global stability of a balanced growth path, which allows us to restrict our analysis to a nontrivial equilibrium with borrowing for survival.

[16] See, for example, Middendorf (1981) for more details on this innovation type.

[17] The representative consumer also has ownership for this sort of accidental innovation/existing firms.

[18] For simplicity, we assume that the firm sells the ownership right for the new "invention by accident" to a randomly chosen firm from a pool of potential manufacturing firms the representative consumer owns.

**Lemma 2** *Assume that* $\eta > \eta_0$, $\overline{\delta} > \overline{\delta}_0$, *and* $\kappa > \kappa_0$ *hold. Then, the economy immediately jumps onto a unique balanced-growth path at* $t = 0$ *and stays there permanently. On a balanced growth path, the existing firms borrow cash* ($\omega = 1$) *if and only if* $\xi > (1-\theta)$; *the growth rate is:*

$$g^* = \frac{1+(1-\theta)\,i}{1+\xi i}\kappa\Pi - (\overline{\delta} + \rho - \eta) > 0 \tag{21}$$

*and the firm exit rate is*

$$\delta^* = \overline{\delta} + \rho + \psi^* - \frac{1+(1-\theta)\,i}{1+\xi i}\kappa\Pi. \tag{22}$$

**Proof.** See the Appendix. ■

These two lemmas completely characterize the equilibrium behavior of R&D firms and, thereby, the dynamic general equilibrium of our model. In the subsequent section, from these two theoretical results, we will draw various economic insights concerning the relationship between inflation and R&D firm dynamics.

# 3 Inflation and Business Dynamism

Our ultimate goal is to identify the relationship between inflation/deflation and the nature of business dynamism. As mentioned in the Introduction, business dynamism is typically characterized by a firm's exit rate, entry rate, and its average age. Therefore, in this section, we examine the effects of inflation on these three factors.

From Lemmas 1 and 2 with (19), an increase in the nominal interest rate, $i$, as a monetary policy lever, affects the equilibrium level of innovation survival through changes in the profitability of R&D and survival investments. In the analysis below, we naturally focus on an equilibrium where existing firms borrow cash for survival with $\xi > (1-\theta)$ (see Lemma 2). We also consider the types of firms that face more stringent CIA constraints. There, thus, exists clear empirical evidence that startup firms in R&D-intensive sectors face more stringent financial constraints (Hall, 2008). Therefore, we should assume that the CIA severity, $\zeta$, for startup firms engaging in R&D is higher than that for older, existing firms, $\xi$. In summary, we proceed with $1-\theta < \xi < \zeta$.

A higher nominal interest, $i$, has two opposite effects on survival and firm exit. As a direct effect, a higher $i$ hurts both R&D and survival investment because it increases the interest payment (total cost) of these investments due to the CIA constraint with severities $\zeta$ and $\xi$, respectively. This discourages both investment in innovation and survival, decreasing firm entry and increasing firm exit. However, whether the negative effect on entry is stronger or weaker than that on exit is ambiguous. On the one hand, since financial constraints are more stringent for startup firms engaging in R&D than existing firms investing in market survival ($\zeta > \xi$), a higher $i$ tends to cause more damage to firms investing in R&D for entry. On the other hand, if the net benefit, $v - \varpi/\kappa$, of R&D is very low, R&D investment $k^*$ is originally very small; thus, an additional damage by an increase in $i$ on $k^*$ is not so significant (as shown in (20)). Consequently, if $v_t - \varpi/\kappa$ is very low (or if the entry cost $\varpi$ is very high), an increase in $i$ tends to cause more damage to incumbent firms' survival investment. This implies that the effect of a higher $i$ on the exit rate $\delta^*$ depends on the size of an entry cost.

The following formally characterizes the effect of a higher $i$ on firm exit $\delta^*$. We define $\hat{\Pi} \equiv (1-\gamma)(\kappa\Pi)/(\gamma^{1/(1-\gamma)})$ for capturing the potential profitability of incumbent firms.

**Proposition 1** *Assume that* $1-\theta<\xi<\zeta$. *If the entry cost is large such that* $\varpi \geq 1-\xi/\zeta$, *a higher nominal interest rate,* $i$, *leads to a smaller investment in survival,* $z^*$, *and, thereby, a higher exit rate of firms,* $\delta^*$, *in the long run. Otherwise, the effect of* $i$ *on* $\delta^*$ *can also be negative or U shaped depending on the potential profitability* $\hat{\Pi}$ *of incumbent firms.*

**Proof.** See the Appendix. ■

Proposition 1 identifies a critical role of entry cost $\varpi$ in determining the relationship between the nominal interest $i$ and the exit rate $\delta^*$. As mentioned, a higher entry cost, $\varpi$, implies a lower net benefit, $v-\varpi/\kappa$, of innovation. Thus, when $\varpi$ is higher, R&D investment is originally very small; an increase in the higher nominal interest rate, $i$, causes insignificant damage on R&D. The damage on survival is more vital, which leads to a shift of labor resource from survival to R&D. This explains why the high entry cost is a condition under a higher nominal interest, discouraging survival investment, leads to a higher exit rate.

Another factor determinng the effect of $i$ on $\delta^*$ is the profitability $\hat{\Pi}$ of incumbents. Given that the profit rate of incumbent firms, $(1+(1-\theta)i)\Pi/v = (1+(1-\theta)i)\kappa\Pi/(1+\xi i)$, is a function decreasing in $i$ (reflecting the cost-pressure effect), a higher interest rate, $i$, leads to a higher equilibrium firm value, $v$ (via (12)), which in turn decreases the profit rate. This effect of reducing the profit rate becomes more important as the *gross* profit $\Pi$ increases. This explains why the high profitability $\hat{\Pi}$ of existing firms is another condition for the positive effect of $i$ on $\delta^*$.

In our model, the inflation rate follows the Fisher equation $\pi = i - r = i - g(i) - \rho$, where the second equality follows from the Euler equation. Therefore, as long as $dg(i)/di < 1$, we have $d\pi/di = 1 - dg(i)/di > 0$, which implies a positive relationship between the nominal interest rate and inflation rate, that is, $\pi \equiv \pi(i)$ with $\pi' > 0$. As is apparent from the following, in our model, this always holds under $1-\theta<\xi$. The long-run positive relationship between $i$ and $\pi$ is supported by empirical studies such as those of Mishkin (1992) and Booth and Ciner (2001), which we have considered below. Proposition 1, thus, leads to a corollary: *inflation will increase the firm exit rate in the long run if the existing firms are profitable enough.*

Next, we examine the effects on firm entry. Intuitively, under $\zeta > \xi$, a higher interest rate, $i$, discourages R&D investment for firm entry, which, in turn, has a negative effect on the entry rate, $\varepsilon^* \equiv \psi^* K_t/N_t$. However, as mentioned above, this entry-discouraging effect of higher $i$ can be insignificant if the net benefit, $v_t - \varpi/\kappa$, of R&D is very small. Therefore, the effect of higher $i$ on entry can be positive or negative, depending on the size of an entry cost, $\varpi$. The following formally proves this.

**Proposition 2** *Assume that* $1-\theta<\xi<\zeta$. *If, and only if, the entry cost is large such that* $\varpi \geq 1-\xi/\zeta$, *a higher nominal interest,* $i$, *or inflation,* $\pi$, *leads to a higher firm entry rate,* $\varepsilon^*$, *in the long run.*

**Proof.** It is straightforward from (20), with $\xi<\zeta$. ■

We have shown that inflation, caused by higher nominal interest, encourages both firm entry and exit in the case with a higher entry cost, $\varpi \geq 1-\xi/\zeta$. The positive relationship between inflation and entry is consistent with the observed relationship between inflation and entry rates for the U.S. and Japan, which is depicted in Figure 1. Our finding also implies that a decrease in the inflation rate $\pi$, i.e., deflation, causes declining business dynamism, which is characterized by slower entry and exit. Lower entry and exit rates should result in a larger share of old firms, which is the third factor of declining business dynamism. We then verify this conjecture formally.

We first derive the density function of firm age. To do so, we denote a firm's birth date as $b \geq 0$. Then, firms with $b$ have age $t-b$. Additionally, we denote the number of firms of age $t-b$ as $n_t(t-b)$. Keeping $b$ constant, the law of motion for the evolution of $n_t(t-b)$ is:

$$\dot{n}_t(t-b) = -\delta^* n_t(t-b) \tag{23}$$

with initial condition $n_b(0) = \psi^* M_b + \eta N_b = (g^* + \delta^*) N_b = (g^* + \delta^*) (N_0 e^{g^* b})$. Deriving a particular solution to this differential equation from (23), we have $n_t(t-b) = (g^* + \delta^*) N_b e^{-\delta^*(t-b)}$. Dividing both sides by $N_t = N_0 e^{g^* t}$ yields a density function for firms of age $t - b \equiv a > 0$:

$$f_t(a) \equiv \frac{n_t(a)}{N_t} = (\varepsilon^* + \eta)\, e^{-(g^*+\delta^*)a}, \tag{24}$$

which is free from $t$, since firm age, $a$, is fixed. Consistent with empirical evidence,[19] the firm age distribution in our economy obeys an exponential function. We can therefore show that the average firm age is $1/(g^* + \delta^*) = 1/(\varepsilon^* + \eta)$.

The inspection of (24) reveals how a higher nominal interest rate, $i$, or inflation rate $\pi$, affects the firm age distribution. In the case with a higher entry cost $\varpi$, a higher $i$ leads to an increase both entries and exits (Propositions 1 and 2). This implies that more young firms are more likely to enter the market while older firms are more likely to exit. Therefore, a higher $i$ tends to decrease the average firm age. In the other case with a lower entry cost, the opposite occurs; a higher $i$ tends to increase the average firm age. This explains the mechanism behind the youth or maturity bias caused by inflation.

The following proposition characterizes this effect formally.

**Proposition 3** *Assume $1 - \theta < \xi < \zeta$. In equilibrium, firm age follows an exponential distribution. If and only if the entry cost is large such that $\varpi \geq 1 - \xi/\zeta$, a higher nominal interest, $i$, or inflation, $\pi$, decrease the average firm age, causing a yough bias in the firm age distribution.*

**Proof.** See the Appendix. ■

Proposition 3 shows that if the entry cost is relatively large, deflation leads to an increase in the average firm age or a larger share of older firms. This may explain the coexistence of deflation pressures and declining dynamism which some countries have experienced. One of the most relevant cases is Japan, which has experienced a deflationary trend and, at once, a decrease in the average firm age (Hong et al., 2019). All in all, our theoretical findings in Proposition 1–3 suggest that a larger entry cost is a key element to explain the linkage between deflation and declining business dynamism.

It is worth explaining the opposite causality, that is, from declining business dynamism to deflation. For example, suppose that the severity $\xi$ of a CIA constraint for incumbent

[19] See, for example, Coad (2010).

firms happens to decrease. Then, the exit rate $\delta^*$ and the entry rate $\varepsilon^*$ both decreases, from (20) and (22). Given the nominal interest $i$ as constant, this decline in business dynamism leads to a decrease in the inflation rate, $\pi = i - r$, because the real interest rate $r = g^* + \rho$ increases with a decrease in $\xi$ (see (3) and (21)). Therefore, the relationship between deflation and declining business dynamism in our model has bidirectional causality. Focusing on the case with a higher entry cost, we summarize our findings as follows.

**Remark 1** *Our theory, with some facts, asserts that if the entry cost is sufficiently high, a lower inflation rate, or deflation, causes declining business dynamism, with a low entry rate, low exit rate, and maturity bias in the age distribution. Given that, in reality, deflation pressures more or less have covered the world economy at least since the 2008 crisis, a lower inflation rate, or deflation, could be an explanatory factor for the declining business dynamism that many countries now experience.*

# 4 Macroeconomic Effects

If $1 - \theta < \xi < \zeta$, an increase in the nominal interest rate, $i$, or inflation rate, $\pi$, (caused by tight monetary policies) affects firm entries and exits, which, in turn, affect macroeconomic growth, $g^*$. While, as shown above, an increase in $i$ accelerates business dynamism, with a higher entry and exit rate, if the entry cost is higher, the equilibrium growth rate $g^*$ is always a decreasing function in $i$ from (21). This is because a higher entry is positive for growth but a higher exit rate is negative, which generates an ambiguous effect. In our model, the negative effect dominates.[20] Such a negative relationship between inflation and (innovation-based) growth is standard in the literature (Chu and Cozzi 2014).[21]

Because of the negative growth effect of $i$, the welfare effect could also be negative. However, monetary factors affect the equilibrium welfare through not only growth rate $g^*$ but also consumption base $c_0$:

$$U^* = \frac{1}{\rho}\left(\ln c_0 + \frac{g^*}{\rho}\right). \tag{25}$$

When a higher $i$ encourages entry but discourages survival (which occurs under a higher entry cost as shown above), the total resource used for investment in innovation and survival may or may not increase. If it decreases in total, a higher $i$ leaves more resources to produce goods. Therefore, more goods would be distributed for consumption. This resource reallocation effect affects consumption base $c_0$ positively. Therefore, the welfare effect of an increase in nominal interest rate $i$ is nontrivial.

[20] This is probably a model specific result. For example, if we would introduce heterogeneity of firm productivity, firms with lower productivity would tend to exit in equilibrium. In this case, the negative growth effect of a higher exit rate would become weaker; the total growth effect could be reversed. It is an interesting way to extend our analysis. However, we leave this for future research, given our focus is on linking deflation and declining business dynamism (which we successfully do in the previous section).

[21] Some studies also reported an inverted U-shaped relationship. See, for example, Chu and Lai (2013) and Chu and Cozzi (2014). See Chu (2020) for a review.

**Proposition 4** *Assume that $1-\theta<\xi$. A higher nominal interest, $i$, or inflation, $\pi$, leads to a lower economic growth rate, $g^*$ but an ambiguous effect on welfare around the zero bound of $i$.*

**Proof.** See the Appendix. ■

This implies that the Friedman rule ($i=0$) can only be suboptimal, which is in line with the existing studies (e.g., Chu and Cozzi, 2014; Chu et al. 2017; Oikawa and Ueda, 2018; Hori, 2020; Miyakawa et al., 2020). We can formally prove that it occurs in our model if, for example, the profitability $\Pi$ is sufficiently low. The intuition is that with lower $\Pi$, the negative growth effect of a higher $i$ is weaker, as shown in (21). In the next section, we will quantitatively demonstrate that inflation can increase or decrease welfare depending on the natural rate $\overline{\delta}$ of firm exit.

# 5 Quantitative Exercises

In order to see if the linkage between deflation and declining business dynamism occurs under realistic values of entry, exit, and inflation rates, we run quantitative exercises by calibrating our extended model to the U.S. data. The model features the structural parameters $\left\{\alpha,\gamma,\kappa,\rho,\zeta,\xi,\eta,\theta,\varpi,\overline{\delta}\right\}$ and a policy instrument $\pi$. For a benchmark model, we normalize $\gamma=0.5$, which implies a square-root production function for innovation. We then follow Acemoglu and Akcigit (2012) in setting the discount rate $\rho$ to 0.05.[22] We set the labor intensity to the value in the U.S., $1-\alpha=0.56$, as in Chu et al. (2019a); for related empirics, see Karabarbounis and Neiman (2014). We also set the share $(1-\theta)$ of internal cash available for survival investment in the total profit to 0.51, following the average dividend payout ratio in the U.S. (ap Gwilym et al., 2006). We follow Chu et al. (2019a) to consider the case with $\zeta=0.65$. In what follows, we compare two cases with high and low natural exit rates, $\overline{\delta}=1$ and $\overline{\delta}=0.5$.

| $\overline{\delta}$ | $\varpi$ | $\xi$ | $\eta$ | $\kappa$ | $i$ |
|---|---|---|---|---|---|
| 1 | 0.842 | 0.511 | 0.0009 | 15.8 | 0.0968 |
| 0.5 | 0.850 | 0.593 | 0.0009 | 8.46 | 0.0968 |

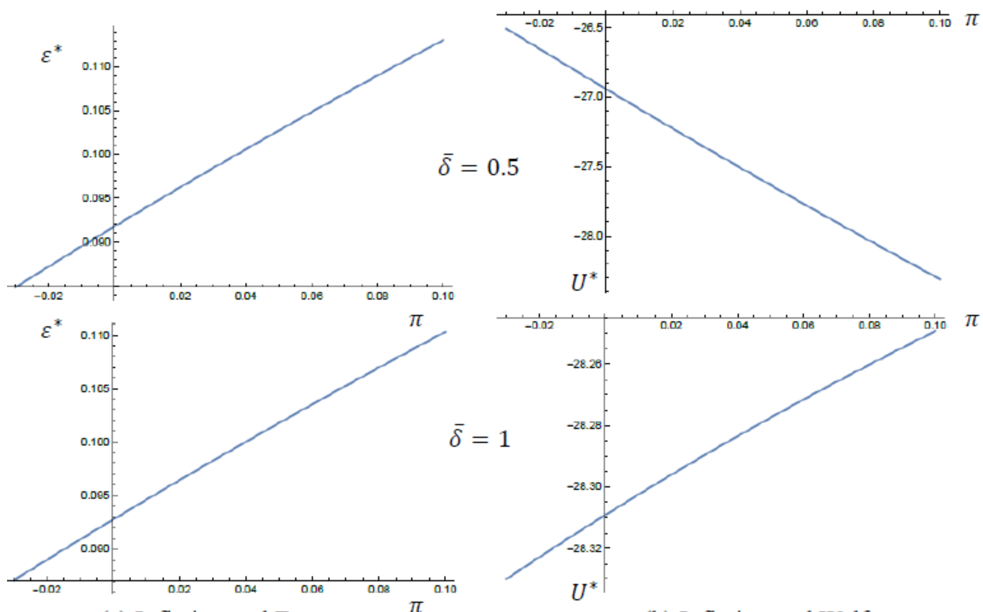
Table 1: Calibrated Parameters

We calibrate the four remaining structural parameters, $\{\xi,\eta,\kappa,\varpi\}$, by matching theoretical moments to the data. We target the entry and exit rates in 2004, 9.77% and 7.86%,[23] respectively. Using these values, we calibrate the entry cost $\varpi$ and the R&D productivity $\kappa$. We calibrate the exogenous growth rate $\eta$ to match the data on the long-run growth rate, set to 2% as standard (e.g., Chu et al., 2020). We, in turn, calibrate the CIA parameter $\xi$ to match the data on the average money-output, $m_t/Y_t$, which is around 11% in the U.S. (in 2004, the M1-GDP ratio).[24]. Using a 2004 inflation rate of 2.68%, we calibrate these parameters, as summarized in Table 1. In this calibrated model, the

[22] This value falls into a conventional range of $\rho$.

[23] Based on our own calculations from the Business Dynamism Statistics.

[24] The data are from FRED: https://fred.stlouisfed.org/graph/?g=dQQ.




(a) Inflation and Entry (b) Inflation and Welfare

Figure 2: Inflation, Entry, and Welfare

average firm age is approximately 10.14, which is reasonably close to a value of 11.7 for the average firm age in the U.S. (Fajgelbaum, 2020).[25]

Under the calibrated model, we find a positive relationship between the inflation rate and the firm entry rate, as illustrated in Figure 2a. This implies that our theoretical model is equipped to explain the observed linkage of deflation and declining business dynamism, under realistic values of entry, exit, and inflation rates. We also see that the effects of a 1% (5%) drop in inflation on the growth rate are not so significant. Specifically, a 1% or 5% increase in inflation $\pi$ causes a decrease in growth $g^*$, which is less than 0.1 in the calibrated model with $\overline{\delta} = 1$ and a larger decrease in that with $\overline{\delta} = 0.5$. Therefore, the negative effect of inflation via discouraging growth seems negligibly small for $\overline{\delta} = 1$. Therefore, the welfare effect of inflation may be more likely to be positive under $\overline{\delta} = 1$. In fact, as shown in Figure 2b, the relationship between the inflation rate $\pi$ and social welfare $U^*$ is positive when $\overline{\delta} = 1$. When $\overline{\delta} = 0.5$, the relationship is reversed. This finding suggests an important role of the natural exit rate $\overline{\delta}$. If firms are intrinsically likely to leave the market, with high $\overline{\delta}$, because of, for example, institutions that support tighter competition, deflation is a fundamental cause of not only declining business dynamism but also welfare loss. In this case, deflation hurts business dynamism and, thereby, social welfare, while slightly enhancing growth. Otherwise, if firms can more easily stay in the market, deflation causes declining business dynamism but with increased social welfare. Our quantitative result implies that the natural exit rate, $\overline{\delta}$, is a key parameter considering whether declining business dynamisms is really harmful for people's well-being.

[25] We use date for non-exporting firms because our model considers a closed economy.

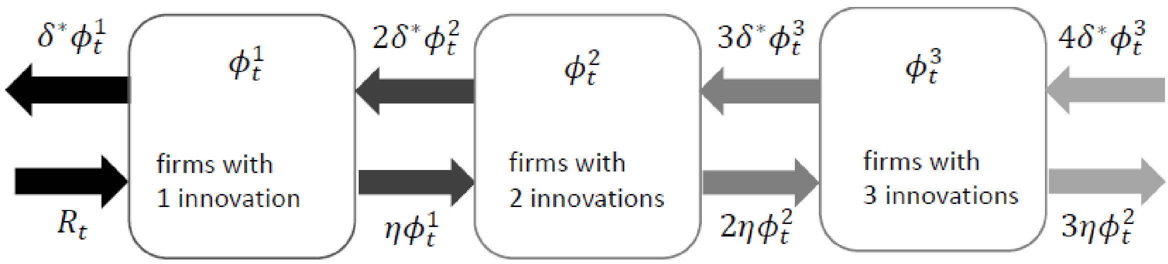



Figure 3: Flows in Firm Size Dynamics

# 6 Firm Size Distribution: An Extension

We next analyze the firm size distribution. To allow for an endogenous firm size distribution (in terms of the levels of employment or profit), we tentatively consider an extended model, where a single firm produces multiple innovated goods. First, each R&D firm develops one original new product through R&D investment. Second, it will possibly develop other new products through "inventions by accident" in the survival stage with an arrival rate of $\eta$. Because there is no interaction between the innovations of a single firm, a firm with multiple innovations solves identical dynamic optimization problems for survival, which are not interdependent.

Under this modified assumption, we introduce new variables. The number of firms, denoted as $\phi_t$, is smaller than or equal to the number of innovations, $\phi_t \leq N_t$. We denote by $\phi_t^h$ the number of firms that have $h$ innovations at date $t$, where $h = 1, 2, 3, \cdots$ is the (discrete) number of innovations that a firm produces. Let $\overline{h}$ be the maximum number of innovations that a single firm can produce at once. Note that the number of innovations, $h$, also captures the firm size in our model.

Figure 2 conceptually illustrates the size dynamics of firms. We mathematically describe these flows with the following laws of motion for the evolution of essential variables. First, the total number of firms, $\phi_t$, follows:

$$\dot{\phi}_t = R_t - \delta^* \phi_t^1, \tag{26}$$

where $R_t \equiv \psi^* K_t \equiv \varepsilon^* N_t$ is the number of new entries or equivalently investment-driven new innovations. Then, the number $\phi_t^1$ of firms with only one innovation satisfies:

$$\dot{\phi}_t^1 = R_t + 2\delta^* \phi_t^2 - (\delta^* + \eta)\, \phi_t^1. \tag{27}$$

Here, there are two inflows into $\phi_t^1$: one is from new entries, $R_t$, and the other is from the pool of firms with two innovations, $\phi_t^2$, because of the obsolescence of either innovation. There also are two outflows from $\phi_t^1$. One is an outflow $\delta^* \phi_t^1$ due to the obsolescence of innovations, in which case a firm decreasing its innovation level exits the market; the other is an outflow $\eta\phi_t^1$ to upper pool $\phi_t^2$, owing to a new invention by accident.

For firms with two or more innovations (but not reaching limit $\overline{h}$), the firm number follows:

$$\dot{\phi}_t^h = (h-1)\, \eta\phi_t^{h-1} + (h+1)\, \delta^* \phi_t^{h+1} - h\,(\delta^* + \eta)\, \phi_t^h, \tag{28}$$

for $h = 2,\ 3,\ 4, \cdots \overline{h}-1$. This is essentially the same for single-innovation firms, (27), except that multi-innovation firms face higher probabilities, for example, for the obsolescence of each innovation ($h\delta^* > \delta^*$) and the occurrence of a new invention by accident ($h\eta > \eta$).

This is because they are working with several innovations, meaning they have several times the production scale.[26]

Finally, the number of firms with the maximum innovations, $\phi_t^{\overline{h}}$, is:

$$\dot{\phi}_t^{\overline{h}} = (\overline{h} - 1)\eta\phi_t^{\overline{h}-1} - \overline{h}\delta^*\phi_t^{\overline{h}}. \tag{29}$$

There is no outflow due to invention by accident toward the upper category.

Since the dynamic system, (26)–(29), is complex, it is only analytically possible to investigate the simplest case, with $\overline{h} = 2$. In this case, (29) becomes $\dot{\phi}_t^2 = \eta\phi_t^1 - 2\delta^*\phi_t^2$. We then define a density function of a size-$h$ firm by $d_t^1 \equiv \phi_t^h/\phi_t$ and an innovation-firm ratio by $N_t^\phi \equiv N_t/\phi_t$. Using (26), (27), and (29), we can derive the steady-state values of $(d_t^1, d_t^2, N_t^\phi)$ as:

$$d^{1*} = \frac{g^* + 2\delta^*}{g^* + 2\delta^* + \eta} \text{ and } d^{2*} = \frac{\eta}{g^* + 2\delta^* + \eta}. \tag{30}$$

The density decreases with $h = 1, 2$ because $g^*+2\delta^* > g^*+\delta^* > \eta$ comes from $g^*+\delta^*-\eta \equiv \varepsilon^* > 0$.[27]

Before characterizing the effects on firm densities, $(d^{1*}, d^{2*})$, we derive a firm-size disparity and define the average firm size (measured by the number of innovations) as $\tilde{m} \equiv (\phi^{1*} + 2\phi^{2*})/\phi^* \equiv d^{1*} + 2d^{2*}$. (30) implies:

$$\tilde{m} = 1 + \frac{\eta}{g^* + 2\delta^* + \eta} < 1.5, \tag{31}$$

where the inequality comes from $g^* + 2\delta^* > \eta$. Then, the variance of firm size is:

$$d^{1*}(\tilde{m} - 1)^2 + d^{2*}(\tilde{m} - 2)^2 \equiv \sigma^2. \tag{32}$$

**Proposition 5** *Assume $1-\theta < \xi < \zeta$ and $\overline{h} = 2$. In equilibrium, the firm size distribution follows (30). The effects of a higher nominal interest, $i$, or inflation, $\pi$, on the ratio of large firms, $d^{2*}$, and the firm size dispersion, $\sigma^2$, are negative if the entry cost is large such that $\varpi \geq 1-\xi/\zeta$. Otherwise, the effects are positive, negative, or lead to an inverted U-shape, which occur if severity $\zeta$ of the CIA constraint for R&D is, respectively, high, low, or moderate.*

**Proof.** See the Appendix. ■

The mechanism behind Proposition 5 is as follows. In the case with a higher entry cost, a higher nominal interest, $i$, leads to increases in the entry and exit rates, $\varepsilon^*$ and $\delta^*$, which accelerates the dynamics of firms (Propositions 1 and 2). This results in a small-size bias in the firm size distribution because firms that newly enter the market are are all small firms.

If the entry cost is not large, a higher nominal interest, $i$, leads to a lower entry rate (Proposition 2), while leading to a higher exit rate. In this case, the effect is potentially

[26] For firms with two innovations, for instance, the probability at which either innovation becomes obsolete before creating an innovation is $[\delta^* dt(1 - \delta^* dt) + (1 - \delta^* dt)\delta^* dt]\ (1 - \eta dt)(1 - \eta dt) \approx 2\delta^* dt$ during short time interval $dt$. Term $\delta^* dt(1-\delta^* dt)$ or $(1-\delta^* dt)\delta^* dt$ represents the probability with which only one of the two innovations becomes obsolete. Term $(1 - \eta dt)(1 - \eta dt)$ is the probability with which an invention by accident does not occur in either production line. We can express other probabilities similarly.

[27] Note that this property is subject to assumption $\overline{h} = 2$. For example, if $\overline{h} = 3$, it can be a unimodal distribution.

ambiguous. Proposition 5 predicts that, when R&D firms face a more stringent CIA constraint ($\zeta$ is relatively small), the discouraging-entry effect tends to dominate, which increases the large-firm ratio, $d^{2*}$, and the firm size dispersion, $\sigma^2$.[28]

In addition to the entry cost, $\varpi$, our theory explains the role of financial frictions $\zeta$ for R&D under this prediction as follows. On the one hand, nominal interest $i$ negatively affects the entry incentives, generating a discouraging-entry effect that decreases $\psi^*$ in (20). This discouraging-entry effect is more dominant for more financial frictions at the entry stage, $\zeta$. On the other hand, it negatively affects the survival incentives to generate an encouraging-exit effect that increases $\delta^*$ (see (21)). This encouraging-exit effect is *less* dominant for more financial frictions at the entry stage, $\zeta$. As a result, when the CIA severity for R&D is high, the discouraging-entry effect of $i$ on $\varepsilon^*$ tends to dominate; further, fewer new entries imply a smaller share of young, small firms. This explains the role of CIA severity, $\zeta$, for R&D in determining the effect of $i$ on firm size distribution.

# 7 Financial Innovation: A Market Quality Perspective

Cash in advance is the only friction in our model. If the introduction of new financial technologies or institutions occurs (which we roughly call financial innovations), they could mitigate the severity of CIA constraints. We also consider an *exogenous* decrease in friction as financial innovation.

We consider two types of financial innovation. The first is an exogenous financial innovation that reduces financial friction in the entry/R&D stage, $\zeta$ to $q_{st}^{-1}\zeta$, where $q_{st} > 1$. According to the market quality theory terminology of Yano (2009), we refer to $q_{st}$ as "market infrastructure" for startups. An increase in $q_{st}$ relates to financial innovation in that it mainly facilitates payment by startup firms involved in R&D. We also refer to $q_{in}$ as "market infrastructure" for incumbents (including large firms). An increase in $q_{in}$ relates to financial innovation that facilitates payment by incumbent firms surviving on the market.[29]

Using (21) and (20), we can verify that startup-complementary financial innovation with a higher $q_{st}$ has different effects on social welfare $U^*$ and firm size disparity $\sigma^2$. If consumers are heterogeneous, in that different people own firms of different sizes, a higher firm-size disparity could increase income inequality from financial assets. Under market quality theory, macroeconomic performance is evaluated by how well the society achieves "healthy economic growth" (Yano, 2009). This literature stream measures the health of economic growth as a composite of two sub-measures, welfare and fairness, which are central concepts in economics. Therefore, if we presume that inequality negatively affects market fairness, an analysis in our model would suggest that a higher quality of a *single* financial market infrastructure is not always better for society's well-being; thus, society needs to find an "appropriate coordination of market infrastructure" between the two different financial markets for startups and incumbents. This implication supports one

[28] Note that the number of small firms is always larger than that of larger firms. Thus, an increase in small firms reduces the variance $\sigma^2$ of firm size.

[29] See also Yano (2019) for a recent contribution, who introduces a general theory of money that can explain commodity money, fiat systems of paper money, and virtual currencies in a single setup.

of the fundamental propositions in market quality theory, "appropriate coordination of market infrastructure is indispensable for a high-quality market" (Yano, 2009).

# 8 Concluding Remarks

We link two separately discussed issues, deflation and declining business dynamism, and explore their relationship in the framework of monetary R&D-based growth models a la Chu and Cozzi (2014). We do this by incorporating the view that firms are financially constrained at all stages of their life cycle—entry, exit, and survival. The essential departure from existing models is that (a) firms successfully innovating new goods need to further pay an entry cost; and after entry, (ii) incumbent firms invest in market survival, facing a CIA constraint, by which the effects of inflation on entry/exit rates are nontrivial in our model. Using this model, we find that if the entry cost is sufficiently large, deflation causes declining business dynamism, which is featured by lower entry and exit rates and a higher average firm age (or a larger share of older firms), which seems consistent with the date for the U.S. and Japan (Figure 1). Calibrating the model to the U.S. economy, we show that deflation causes declining business dynamism under realistic values of entry, exit, and growth rates, because the calibrated value of entry cost exceeds the threshold. Further, if the natural rate of firm exit is higher, deflation can also cause welfare loss.

Our model abstracts several important aspects of reality. First, the firm size distribution does not follow a Pareto distribution, which is inconsistent with some empirical evidence (e.g., Axtell, 2001) and related theories (e.g., Luttmer, 2007). Instead, the model is equipped to understand the impacts of inflation on the different stages of firms' dynamics—entry, exit, and survival. We achieve this by considering a relatively stylized equilibrium form of firm size distribution. Alternatively, the model can explain the mechanism behind an empirically observed form of the exponential age distribution. Second, consumers are homogeneous in the model. Therefore, there is no direct contribution to the vast literature on growth and income/wealth inequality (e.g., Acemoglu and Cao, 2015, and Jones and Kim, 2018).[30] Our model could be extended by assuming that different consumers own shares of different firms that vary in size and profit. Under such an extension, consumers are heterogeneous in their interest incomes. Third, in our model, firm growth is driven by an exogenous factor. This could be extended by introducing inhouse R&D (e.g., Peretto 1996). Forth, our results should depend on the setting where when firms get cash, they have no choice but to borrow from consumers. Introducing an alternative way to raise fund from, for example, a venture capital market would be interesting. We leave these possible extensions for future research.

Finally, we discuss the theoretical insights and policy implications of our study. Theoretically, we offer a novel explanation for the mechanism behind the relationship between deflation and declining business dynamism via the CIA channel, which is new to the literature. As in existing models, deflation yields a lower cost of money holding and, thereby, money borrowing because of the lower nominal interest, which encourages R&D investment for entry. However, in our model, firms are also financially constrained in the survival and exit stage in the equilibrium where incumbent firms borrow cash. As

[30] See, for example, Haruyama (2021) for a recent contribution to this literature, who develops a Schumpeterian growth model in which a double Pareto distribution of income emerges in equilibrium as a result of entrant (drastic) and incumbent innovations.

a result, deflation directly encourages survival investment and decreases firm exit, due to lower interest payment. Notably, it further has an indirect, general-equilibrium effect that emerges from a shift of labor resources between innovation and survival investment. The direction of this indirect effect is potentially ambiguous, and depends on the size of the entry cost. The reason is as follows. If the entry cost after successful R&D investment is very high; the net payoff of R&D investment is very small. In this case, R&D activity is originally very weak; the positive effect of deflation on entry is insignificant. Therefore, deflation causes a shift of resources from entry to survival, discouraging R&D for firm entry and encouraging survival to delay exit. This explains why deflation can be associated with a lower entry and exit rate and a higher average firm age (i.e., declining business dynamism).

These findings deliver the important policy implication that monetary policy controlling the interest rate, or targeting the inflation rate, can essentially affect the nature of business dynamics. For example, a low-interest-rate policy encourages both innovation and survival investment, thereby, enhancing long-run growth as a standard effect. However, as an additional effect, it can also cause declining business dynamism, depending on the size of an entry cost. If the latter effect dominates the former effect, deflation also causes welfare loss, which is the case when the natural rate of firm exit is higher. These theoretical findings suggest that the monetary authorities should consider not only standard macroeconomic variables, such as the growth rate, but also variables for firm demographics, in creating a desirable monetary policy.

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# Appendix

**Proof of Lemma 2.** Using Lemma 1, we combine (17)–(19) to obtain

$$\frac{\dot{e}_t}{e_t} = \kappa e_t + \left(\frac{1+\lambda^{\omega}(1-\theta)i}{1+\lambda^{\omega}\xi i} - \frac{1+\alpha}{\alpha}\right)\kappa\Pi + (k^* - \psi^* + \varpi) - \rho. \tag{A1}$$

Because $k_t^*$ and $\psi_t^*$ are independent of $t$, (A1) is an autonomous dynamic system for $e_t$. Applying the standard argument using the transversality condition, (A1) has a globally saddle-point stable steady state, $e^*$; at time 0, $e_t$ jumps to:

$$e^* \equiv \left[\rho + (\psi^* - k^* - \varpi) + \left(\frac{1+\alpha}{\alpha} - \frac{1+\lambda^{\omega}(1-\theta)i}{1+\lambda^{\omega}\xi i}\right)\kappa\Pi\right]\frac{1}{\kappa}. \tag{A2}$$

Along this unique balanced growth path, the growth rate for $c_t$ and $N_t$ are given as (21) from (17). Substituting (A2) into (19) yields (22) by using $\delta_t = \overline{\delta} - z_t$ and Lemma 1.

To ensure the positivity of $g^*$ and $\delta^*$, we first take $\kappa_0$ such that:

$$z^* = \frac{1+\lambda^{\omega}(1-\theta)i}{1+\lambda^{\omega}\xi i}\kappa\Pi - (\rho + \psi^*) > 0 \text{ for any } \kappa > \kappa_0. \tag{A3}$$

Then, we can consider some sufficiently high $\overline{\delta}$ that can ensure $\delta^* > 0$; we label the threshold as $\overline{\delta}_0$. Finally, we can also find some threshold value of $\eta$, $\eta_0$, such that $g^* > 0$ for $\eta > \eta_0$ because $z^*$ and $\delta^*$ are free from $\eta$, proving the positivity of $g^*$ and $\delta^*$.

An equilibrium where the existing firms borrow uniquely occurs if and only if $\frac{\xi z^*}{\kappa}\big|_{\omega=1} > (1-\theta)\Pi$ and $\frac{\xi z^*}{\kappa}\big|_{\omega=0} > (1-\theta)\Pi$, which is equivalent to

$$\frac{\xi-(1-\theta)}{\xi}\kappa\Pi > \max\left\{(1+\xi i)\left(\rho + \left(\frac{\gamma(1+\xi i)}{1+\zeta i}\right)^{\frac{\gamma}{1-\gamma}}\right), \left(\rho + \left(\frac{\gamma}{1+\zeta i}\right)^{\frac{\gamma}{1-\gamma}}\right)\right\} \tag{A4}$$

based on (A3). Further, (A4) holds for $\xi - (1-\theta) > 0$ and a sufficiently large $\kappa$.[31] ∎

**Proof of Proposition 1.** We differentiate (22) with respect to $i$, with $\omega = 1$, $d\delta^*/di > 0$ holds (implying $dz^*/di < 0$ by definition) if and only if:

$$\kappa\Pi > \frac{\gamma^{\frac{1}{1-\gamma}}}{1-\gamma}\frac{\zeta-\xi-\zeta\varpi}{\xi-(1-\theta)}\left(\frac{1-\varpi+\xi i}{1+\zeta i}\right)^{\frac{1}{1-\gamma}}\left(\frac{1+\xi i}{1-\varpi+\xi i}\right)^2 \equiv \vartheta(i) \tag{A5}$$

holds. Considering the definition of $\hat{\Pi}$, we can verify that (A5) holds for *any* $i \geq 0$ if $\varpi \geq 1 - \xi/\zeta$. Under the opposite inequality, $\varpi < 1 - \xi/\zeta$, $\vartheta' < 0$ holds. Therefore, (A5) holds for *any* $i \geq 0$ if and only if $\kappa\Pi > \vartheta(0)$ holds, or equivalently

$$\kappa\Pi > \frac{\gamma^{\frac{1}{1-\gamma}}}{1-\gamma}\frac{\zeta-\xi-\zeta\varpi}{\xi-(1-\theta)}(1-\varpi)^{\frac{1-2(1-\gamma)}{1-\gamma}}. \tag{A6}$$

Otherwise, if (A6) is violated, there are two cases: (i) if

$$\frac{\zeta-\xi-\zeta\varpi}{\xi-(1-\theta)}\left(\frac{\xi}{\zeta}\right)^{\frac{1}{1-\gamma}} < \hat{\Pi} < \frac{\zeta-\xi-\zeta\varpi}{\xi-(1-\theta)}(1-\varpi)^{\frac{1}{1-\gamma}-2}, \tag{A7}$$

[31] We implicitly exclude any extremely large $i$.

$d\delta^*/di > (<)0$ holds for higher (lower) $i$ (a U shape), and (ii) if

$$\hat{\Pi} < \frac{\zeta - \xi - \zeta\varpi}{\xi - (1-\theta)} \left(\frac{\xi}{\zeta}\right)^{\frac{1}{1-\gamma}}, \tag{A8}$$

$d\delta^*/di < 0$ holds for any $i \geq 0$ (monotonically negative). ■

**Proof of Proposition 3.** Differentiating (24) yields:

$$\frac{d}{di} f_t(a) > 0 \;\Leftrightarrow\; \left(-(\varepsilon^*)'\right)\left(1 - a(\varepsilon^* + \eta)\right) e^{-(\varepsilon^*+\eta)a} < 0. \tag{A9}$$

As shown above, under $1 - \theta < \xi$, $(\varepsilon^*)' < 0$ holds. In this case, there necessarily exists some cutoff value of $a$, $\tilde{a}$, such that $\left(-(\varepsilon^*)'\right)\left(1 - a(\varepsilon^* + \eta)\right) < 0$ for any $a > \tilde{a}$. Therefore, a higher $i$ decreases the density of older firms. ■

**Proof of Proposition 4.** From (A2), $c_0 = e^* N_0$ is:

$$c_0 = \frac{1}{\kappa}\left[\rho + \left(\left(\tfrac{\gamma(1-\varpi+\xi i)}{1+\zeta i}\right)^{\frac{\gamma}{1-\gamma}} - \left(\tfrac{\gamma(1-\varpi+\xi i)}{1+\zeta i}\right)^{\frac{1}{1-\gamma}}\right) + \left(\tfrac{1+\alpha}{\alpha} - \tfrac{1+(1-\theta)i}{1+\xi i}\right)\kappa\Pi\right], \tag{A10}$$

for which we take $N_0 = 1$. Using (21), (20), and (A2) with (25), $dU^*/di > 0$ if and only if:

$$\frac{d(\rho U^*)}{di} = \frac{1}{(1+\xi i)^2}\left(\tilde{c}_0(i) - \frac{\xi - (1-\theta)}{\rho}\kappa\Pi\right). \tag{A11}$$

where:

$$\tilde{c}_0(i) \equiv \frac{1}{\kappa c_0}\left[\left(\xi - (1-\theta)\right)\kappa\Pi + \tfrac{\gamma^{\frac{1}{1-\gamma}}(\xi-\zeta+\zeta\varpi)((\zeta-\xi)i+\varpi)}{(1-\gamma)(1+\zeta i)}\left(\tfrac{1+\xi i}{1-\varpi+\xi i}\right)^2\left(\tfrac{1-\varpi+\xi i}{1+\zeta i}\right)^{\frac{1}{1-\gamma}}\right]. \tag{A12}$$

Using (A11) and (A12):

$$\left.\frac{d(\rho U^*)}{di}\right|_{i=0} = \frac{\frac{\gamma^{1/(1-\gamma)}(\xi-\zeta+\zeta\varpi)\varpi}{(1-\gamma)}(1-\varpi)^{1/(1-\gamma)-2} - \frac{\xi-(1-\theta)}{\rho}\kappa\Pi\left[(\gamma(1-\varpi))^{\gamma/(1-\gamma)}(1-\gamma(1-\varpi))+\kappa\Pi/\alpha\right]}{\left[\rho+(\gamma(1-\varpi))^{\gamma/(1-\gamma)}(1-\gamma(1-\varpi))+\kappa\Pi/\alpha\right]}, \tag{A13}$$

which can be negative or positive. For example, taking sufficiently small (large) $\Pi$, it is positive (negative). ■

**Proof of Proposition 5.** Differentiating (32) with respect to $i$ yields:

$$\frac{d\sigma^2}{di} = \underbrace{-\eta\frac{g^* + 2\delta^* - \eta}{(g^* + 2\delta^* + \eta)^3}}_{<0}(g^* + 2\delta^*)'. \tag{A14}$$

Note, by definition, $g^* + 2\delta^* \equiv \delta^* + \varepsilon^* + \eta$. Using (20), (21), and (22):

$$(g^* + 2\delta^*)' = \frac{\gamma^{\frac{1}{1-\gamma}}\left(\xi - (1-\theta)\right)}{(1+\xi i)^2(1-\gamma)}\varrho(i) \tag{A15}$$

where

$$\varrho(i) \equiv \hat{\Pi} + 2\gamma\left(\frac{\xi - \zeta(1-\varpi)}{\xi - (1-\theta)}\right)\left(\frac{1-\varpi+\xi i}{1+\zeta i}\right)^{\frac{1}{1-\gamma}}\left(\frac{1+\xi i}{1-\varpi+\xi i}\right)^2. \tag{A16}$$

Note that $\varrho(i) > 0$ always holds if $\varpi \geq 1 - \xi/\zeta$. If $\varpi < 1 - \xi/\zeta$, $\varrho(i)$ is decreasing in $i$; thus, $\varrho(i) > 0$ for any $i \geq 0$ if $\zeta < \xi/(1-\varpi) + \hat{\Pi}\ (\xi - (1-\theta))\,/(2\,(1-\varpi)^{\gamma/(1-\gamma)})$ (lower $\zeta$) and $\varrho(i) < 0$ for any $i$ if $\zeta > (\xi + (\zeta/\xi)^{(1-2\gamma)/(1-\gamma)}\ \hat{\Pi}\,(\xi - (1-\theta))\,/2)/(1-\varpi)$ (higher $\zeta$). Otherwise, with an intermediate $\zeta$, $(g^* + 2\delta^*)' < (>) 0$ for lower $i$ (higher $i$). From (30) and (A14), the effects on $d^{2*}$ and $\sigma^2$ are opposite to those on $(g^* + 2\delta^*)$. Therefore, the effect on $\sigma^2$ is negative if $\varpi \geq 1 - \xi/\zeta$; otherwise, with higher, lower, or intermediate $\zeta$, the effect is positive, negative, or inverted U-shaped, respectively. ■