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Endogenous Business Cycles with Bubbles*

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Abstract
By using a simple macroeconomic model, this study demonstrates the possibility that a bubble exists whether the economy is in a boom or a recession. We use an overlapping-generations model with endogenous growth. The results demonstrate that in an economy where banks can lend to consumers, there exists a single equilibrium path in which the economic growth rate fluctuates with the bubble. On the contrary, in an economy where banks cannot lend to consumers, the equilibrium path does not exist.

Keywords: Bubbles, Endogenous growth, Business cycles
JEL classification: E32,O41,O42

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1 Introduction

It is well known that there are bubbles in a booming economy[1]. However, even for sluggish economies, there may be bubbles. In fact, in China, stock prices[2] kept rising sharply from 2014[3], whereas the GDP growth rate has been stagnating since 2006. Prior to 2006, the Chinese economy experienced an economic boom with rising stock prices[4]. When we consider these phenomena as a bubble economy, we find bubbles whether the economy is in a boom or recession. No studies examine these phenomena and explain them by using an equilibrium dynamic model.

The main purpose of this study is to show that a bubble exists along a single equilibrium path in which the economic growth rate fluctuates. The result implies that the bubble exists whether the economy is in boom or recession. In order to show the result, we provide an overlapping-generations model with a standard one-period monopoly production sector.

Regarding the existence of bubbles, whether the economy is in boom or recession, our study is the first to explain the situation of the Chinese economy, as no existing study explains this. Martin and Ventura (2012) show that the economic growth rate increases with the expansion of a bubble along a single equilibrium path[5]. Their result corresponds to the case of economic booms with bubbles. However, it does not correspond to the case of the Chinese economy to show that the bubble always exists along the single equilibrium path in which the economic growth rate fluctuates.

This study is also related to Grossman and Yanagawa (1993) and Olivier (2000). Grossman and Yanagawa (1993) explore the existence of a bubble in an endogenous growth model by using the framework of Samuelson (1958)

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[1] The Japanese economy was growing when asset prices were inflated from 1985 to 1989. The economic boom in the United States from 2000 to 2006 came with rising prices of real estate.

[2] See the SSE Composite Index at Macrotrends.net.

[3] At the same time, housing prices in major cities kept rising and liquidity expanded (see OECD, 2017).

[4] Real estate prices had boomed in major cities, and the GDP growth rate had increased by the mid-2000s (see OECD, 2010, 2017).

[5] Martin and Ventura (2012) introduce financial frictions and investor sentiment shocks into the framework of Samuelson (1958) and Tirole (1985). Moreover, Hirano and Yanagawa (2017) show that bubbles boost long-run growth with endogenous growth and financial frictions.

and Tirole (1985)[6]. They show that the growth rate decreases as the bubble expands. Conversely, Olivier (2000) shows that the bubble creates technologies (or firms) and positively affects economic growth. However, their results do not imply that there exist bubbles along a single equilibrium path, whether an economy is in a boom or slump.

Our model is based on Tirole (1985), Benhabib and Laroque (1988), and Kojima (2012a,b). Tirole (1985) extends Samuelson's (1958) research and provides the basic idea of a rational bubble in an overlapping-generation model. In Tirole's (1985) model, young consumers have an asset to save, which can form a bubble in the long run. Then, he demonstrates that a steady-state equilibrium with the bubble can exist if the interest rate is lower than the growth rate in the steady-state equilibrium without the bubble. In the bubble economy, the interest rate equals the growth rate. Thus, the bubble improves the problem of dynamic inefficiency. In the framework of Samuelson (1958) and Tirole (1985), banks cannot lend to consumers since consumers save their assets. Conversely, Benhabib and Laroque (1988) and Kojima (2012a,b)[7] set up overlapping-generations models wherein young consumers borrow an asset (or money), and old consumers repay it. Thus, in their studies, banks can lend to consumers[8]. Particularly, Kojima (2012a,b) asserts that a steady-state equilibrium with a bubble can exist if the interest rate exceeds the growth rate in the steady-state equilibrium without the bubble. This result implies that the bubble resolves the problem of inadequate capital accumulation[9]. Hereafter, in this study, the model of Samuelson (1958) and Tirole (1985) is called the Samuelson–Tirole case, and the model of Benhabib and Laroque (1988) and Kojima (2012a,b) is called the Benhabib–Laroque–Kojima case[10]. To achieve our research aims, we consider

[6]Saint-Paul (1992) and King and Ferguson (1993) demonstrate the same results as those of Grossman and Yanagawa (1993). Futagami and Shibata (2000) also show that the bubble negatively affects economic growth.

[7]Kojima's (2012a,b) framework is similar to that of Benhabib and Laroque's (1988) model.

[8]Benhabib and Laroque (1988) also consider an economy wherein banks cannot lend to consumers.

[9]In Kojima's (2012a,b) model, the expansion of a bubble raises capital accumulation and the output growth rate.

[10]Following Gale (1973), an economy is called the Samuelson case if the growth rate is higher than the interest rate and the classical case if the interest rate is higher than the growth rate. Therefore, we can classify bubbles into two types: Tirole's results, which correspond to the Samuelson case, and Kojima's results, which correspond to the classical

two cases.

The above studies provide important implications regarding the role of bubbles in a growing economy. However, they do not imply that bubbles exist independent of the economic growth rate.

Grossman and Yanagawa (1993) consider an economy where endogenous innovation sustains growth. Conversely, we consider a standard one-period monopoly model of innovation cycles (Judd, 1985; Deneckere and Judd, 1992; Matsuyama, 1999, 2001)[11]. In particular, we introduce two growth engines (capital accumulation and innovation), as described by Matsuyama (1999, 2001), into an overlapping-generations model. Matsuyama (1999, 2001) shows that there is upward and downward movement of the growth rate along a single equilibrium path. To show this, he introduces a temporary monopoly power when innovators produce new goods in the lab equipment model of Rivera–Batiz and Romer (1991). As a result, an economy achieves growth through cycles, moving back and forth between two phases. In one phase, called the high-growth regime in this study, there is no innovation, and the growth rates of investment and output are high. In the other phase, called the low-growth regime in this study, there is high innovation, and the growth rates of investment and output are low. Therefore, this framework is suitable for the analysis of endogenous business cycles.

This study obtains two major results. First, we derive necessary and sufficient conditions for the existence of the steady-state equilibrium with a bubble in both the low- and high-growth regimes and demonstrate the dynamic properties around the steady states. In particular, a bubble can exist if the growth rate is higher (lower) than the interest rate in the Samuelson–Tirole (Benhabib–Laroque–Kojima) case. This result is similar to that of Kojima (2012a). The steady state with the bubble is locally determinate in the Samuelson–Tirole case; it is either locally indeterminate or locally determinate in the Benhabib–Laroque–Kojima case. Second, and most importantly, there is a bubble along a single equilibrium path with a business cycle. Specifically, there is a single equilibrium path with upward and downward movement of the output growth rate in the Benhabib–Laroque–Kojima case. Along the single equilibrium path, there always exists a bubble. Moreover, the economy does not have the single equilibrium path in the Samuelson–

case.

[11] Judd (1985) is the first to show that innovation cycles occur by assuming that labor productivity grows at a constant rate through the accumulation of experiences. Deneckere and Judd (1992) assume that, at a constant rate, existing products become obsolete.

Tirole case.

The rest of this chapter is organized as follows. The next section presents an endogenous growth model. Section 3 analyzes the equilibrium dynamics. We show the conditions for the existence of the steady-state equilibrium with a bubble and discuss its stability. Furthermore, we show the conditions for the existence of the business cycle with a bubble. Section 4 concludes the paper. Finally, some proofs are shown in the Appendix.

2 The Model

This section develops an overlapping-generations model. It introduces two aspects to our analysis. First, we consider two cases of a bubble: banks can lend to consumers, as in Samuelson (1958) and Tirole (1985), and banks cannot lend to consumers, as in Benhabib and Laroque (1988) and Kojima (2012a,b). Second, we introduce the one-period monopoly production sector, as in Matsuyama (1999, 2001).

2.1 Households

We consider the standard overlapping-generations model in discrete time ($t = 0, 1, 2, ...$); additionally, L households live for two periods. Households born at time t supply one unit of labor when young and receive a wage income from a production sector, where w_t represents the real wage. Additionally, young consumers allocate income to the consumption of goods, c_t, savings, s_t, and an asset B_{t+1}. Let r_{t+1} be the return factor of the asset between time t and time $t + 1$[12]. The elderly consume goods d_{t+1} using savings and by selling the asset, or consume goods and repay the asset using savings. The utility function of the individual born at time t is given by $u(c_t, d_{t+1}) = \alpha \log c_t + \beta \log d_{t+1}$, where $\alpha, \beta > 0$. Savings by an individual born at time t are determined by the following maximization problem:

$$\begin{array}{rl} & \max u\left(c_t, d_{t+1}\right), \\ \text{s.t.} & c_t + s_t + \frac{B_{t+1}}{L} \leq w_t, \\ & d_{t+1} \leq r_{t+1}\left(s_t + \frac{B_{t+1}}{L}\right). \end{array} \tag{1}$$

[12] We assume a full capital depreciation within a period.

Following Tirole (1985), Benhabib and Laroque (1988), Grossman and Yanagawa (1993), and Kojima (2012a,b), we assume an asset whose fundamental value is zero. Let M and p_t be the asset's supply and price at time t. Then, $B_t := p_t M$ is the aggregate value of the asset at time t. Using the no-arbitrage condition between the asset and other assets, $p_{t+1}/p_t = r_t$, yields

$$B_{t+1} = r_t B_t. \tag{2}$$

We assume that M is positive or negative, as in Benhabib and Laroque (1988) and Kojima (2012a,b). In the case of $M > 0$, young consumers have an asset to save. Thus, banks cannot lend to consumers. This case corresponds to the framework of Samuelson (1958) and Tirole (1985). In the case of $M < 0$, consumers make a credit transaction wherein they borrow in an early period and repay in a later period. Thus, banks can lend to consumers. This case corresponds to the framework of Benhabib and Laroque (1988) and Kojima (2012a,b)[13]. We define the above as follows:

Definition 1 *Assume that $M > 0$; then, the economy is a Samuelson–Tirole case. Assume that $M < 0$; then, the economy is a Benhabib–Laroque–Kojima case.*

2.2 Production Sector

We consider Matsuyama's (1999,2001) production sector. The final goods Y_t are in a perfectly competitive market and produced by labor and intermediate goods. The production function of Y_t is

$$Y_t = D_0 \left(\int_0^{A_t} X_t(i)^{1-\frac{1}{\sigma}} di \right) L^{\frac{1}{\sigma}}, \tag{3}$$

where D_0 is total factor productivity, $X_t(i)$ represents the input of the i-th intermediate good at time t, $[0, A_t]$ is the range of variety, and $\sigma \in (1, +\infty)$. The first-order conditions can be expressed as

$$w_t = \frac{1}{\sigma} \frac{Y_t}{L}, \tag{4}$$

[13] Benhabib and Laroque (1988) assume both cases. However, their study is the first to consider the case of $M < 0$.

$$p_t(i) = \left(1 - \frac{1}{\sigma}\right) D_0 X_t(i)^{-\frac{1}{\sigma}} L^{\frac{1}{\sigma}}. \tag{5}$$

There are two types of intermediate goods sectors: the existing and new intermediate goods sectors. In the existing intermediate goods at time t, the economy developed all the intermediates in the range, $[0, A_{t-1}]$, with $A_0 > 0$. On the contrary, the new intermediate goods in the range, $[A_{t-1}, A_t]$, are produced if innovation occurs. In both sectors, capital is converted into intermediate products.

We consider the existing intermediate goods sector i, $i \in [0, A_{t-1}]$. Suppose that this sector is competitive. These existing intermediate goods are manufactured by converting a units of capital into one unit of the intermediate. The profit-maximization problem is $\pi_t^c(i) = \max p_t^c(i) X_t^c(i) - a r_t X_t^c(i)$. Hence, we obtain $p_t^c := p_t^c(i) = a r_t$ for all $i \in [0, A_{t-1}]$.

Conversely, we assume that the new intermediate goods sector i, $i \in [A_{t-1}, A_t]$, is monopolistic. In this sector, one unit of the intermediate is produced by a unit of capital and a fixed cost, F, to innovate new goods. Thus, the profit-maximization problem is $\pi_t^m(i) = \max p_t^m(i) X_t^m(i) - r_t \left(a X_t^m(i) + F\right)$. Then, we obtain $p_t^m := p_t^m(i) = a\sigma r_t/(\sigma - 1)$ for all $i \in [A_{t-1}, A_t]$. Since all intermediate goods are symmetrical, (5) yields

$$\frac{X_t^c}{X_t^m} = \left(\frac{p_t^c}{p_t^m}\right)^{-\sigma} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma}. \tag{6}$$

From $\pi_t^m < 0$, if and only if $a X_t^m < (\sigma - 1)F$, then the free-entry condition would ensure

$$a X_t^m \leq (\sigma - 1)F, A_t > A_{t-1}, \left(a X_t^m - (\sigma - 1)F\right)\left(A_t - A_{t-1}\right) = 0. \tag{7}$$

Let K_t be all capital in this economy. Then, the resource constraint at time t is

$$K_t = A_{t-1} a X_t^c + \left(A_t - A_{t-1}\right)\left(a X_t^m + F\right). \tag{8}$$

(6), (7), and (8) yield

$$a X_t^c = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} a X_t^m = \min\left\{\frac{K_t}{A_{t-1}}, \sigma\theta F\right\} \tag{9}$$

and

$$A_t = A_{t-1} + \max\left\{0, \frac{K_t}{\sigma F} - \theta A_{t-1}\right\}, \tag{10}$$

where $\theta := (1 - 1/\sigma)^{1-\sigma} \in (1, e), e = 2.71828....$

(7), (9), (10), and (3) can be rewritten as

$$Y_t = \begin{cases} D\left(\theta\sigma F A_{t-1}\right)^{\frac{1}{\sigma}} K_t^{1-\frac{1}{\sigma}} & K_t \leq \theta\sigma F A_{t-1} \\ DK_t & K_t \geq \theta\sigma F A_{t-1} \end{cases},$$

where

$$D := \frac{D_0}{a}\left(\frac{aL}{\theta\sigma F}\right)^{\frac{1}{\sigma}}.$$

Let $q_t := K_t/\theta\sigma F A_{t-1}$; thus, we obtain

$$\frac{Y_t}{K_t} = D\phi\left(q_t\right), \tag{11}$$

$$\frac{Y_t}{\theta\sigma F A_{t-1}} = Dq_t\phi\left(q_t\right), \tag{12}$$

$$\frac{A_t}{A_{t-1}} = \psi\left(q_t\right), \tag{13}$$

where

$$\phi\left(q_t\right) = \begin{cases} q_t^{-\frac{1}{\sigma}} & q_t \leq 1 \\ 1 & q_t \geq 1 \end{cases},$$

and

$$\psi\left(q_t\right) = \begin{cases} 1 & q_t \leq 1 \\ 1 + \left(q_t - 1\right)\theta & q_t \geq 1 \end{cases}.$$

The economy is a high-growth regime if $q_t \leq 1$ and a low-growth regime if $q_t \geq 1$, for reasons that will become clear in Proposition 2. There is no innovation and all intermediate goods are competitively supplied in the high-growth regime. On the contrary, in the low-growth regime, final goods and existing intermediate goods are competitive, but innovated intermediate goods are a monopoly.

3 Equilibrium Dynamics

This section presents the study results. First, we derive the equilibrium dynamics of our model presented in Section 2. Then, we provide a necessary and sufficient condition for the existence of the steady-state equilibrium with a bubble. Moreover, we study the dynamic property of our model around the steady-state equilibrium. Second, we show the existence of period-2 cycles with a bubble. Thus, we obtain the main result that there is the bubble along a single equilibrium path with a business cycle.

3.1 Steady-State Equilibrium

By the optimization condition of (1), the resulting savings are written as

$$s_t + \frac{B_{t+1}}{L} = \frac{\beta}{\alpha+\beta} w_t. \tag{14}$$

Thus, the budget constraint for the young and elderly holds in the equation.

The capital market-clearing condition is

$$Y_t = Lc_t + Ld_t + K_{t+1.}$$

All new savings by the young are invested in capital,

$$Ls_t = K_{t+1}. \tag{15}$$

Thus, in equilibrium, $Y_t = r_t K_t + w_t L$ and

$$r_t = \left(1 - \frac{1}{\sigma}\right) \frac{Y_t}{K_t} \tag{16}$$

hold.

We denote $b_t := B_t/\theta\sigma F A_{t-1}$. Using (4), (11), (12), (13), (15), and (16), (14) and (2) can be written as follows:

$$q_{t+1} = D \frac{\phi(q_t)}{\psi(q_t)} \left(\frac{\beta}{\alpha+\beta} \frac{1}{\sigma} q_t - \left(1 - \frac{1}{\sigma}\right) b_t \right) \tag{17}$$

and

$$b_{t+1} = D \left(1 - \frac{1}{\sigma}\right) b_t \frac{\phi(q_t)}{\psi(q_t)}. \tag{18}$$

Therefore, (17) and (18) are considered as a complete dynamic system with respect to q_t and b_t in this economy.

We consider the existence of the conditions of steady states in the economy. Proposition 1 shows the existence of steady states with and without a bubble in each regime in both cases. The value of the bubble is zero in the steady state without the bubble. Conversely, in the steady state with the bubble, the value of the bubble is a positive constant in the Samuelson–Tirole case or a negative constant in the Benhabib–Laroque–Kojima case. Moreover, Proposition 1 shows the local dynamics of (17) and (18). B_t is not predetermined, while K_t and A_t are predetermined. Thus, the economy is locally determinate if the steady state is a saddle point. If the steady state is a sink point, then it is stable, and the economy is locally indeterminate. If the steady state is a source point, then it is unstable. In Proposition 1, ST case and BLK case indicate the Samuelson–Tirole case and the Benhabib–Laroque–Kojima case, respectively.

Proposition 1 *1. Consider the Samuelson–Tirole case.*

(a) There is always a unique steady state without a bubble.

i. $D\beta/((\alpha+\beta)\sigma) > 1$ if and only if there exists a unique steady state without the bubble, (q^{+}, b^{+}), in the low-growth regime, given by (19). This steady state is a bubbleless balanced growth path, given by (23). Moreover, the local stability of the steady state is summarized in Table 2-1.

ii. $D\beta/((\alpha+\beta)\sigma) < 1$ if and only if there exists a unique steady state without the bubble, (q^{++}, b^{++}), in the high-growth regime, given by (21). The economy does not grow in this steady state, which is a neoclassical stationary path. We summarize the local stability of the steady state in Table 2-2.

(b) Assume that $(1 - 1/\sigma)\, D < D\beta/((\alpha+\beta)\sigma)$.

i. $(1 - 1/\sigma)\, D > 1$ if and only if there exists a unique steady state with the bubble, (q^{*}, b^{*}), in the low-growth regime, given by (20). This steady state is a bubbly balanced growth path, given by (24). We summarize the local stability of the steady state in Table 2-3.

ii. $(1 - 1/\sigma)\, D < 1$ if and only if there exists a unique steady state with the bubble, (q^{**}, b^{**}), in the high-growth regime, *given by (22). The economy does not grow in this steady state, which is a neoclassical stationary path. We summarize the local stability of the steady state in Table 2-4.*

2. *Consider the Benhabib–Laroque–Kojima case.*

   (a) *There is always a unique steady state without a bubble.*

      i. $D\beta/((\alpha+\beta)\sigma) > 1$ *if and only if there exists a unique steady state without the bubble,* (q^+, b^+)*, in the low-growth regime, given by (19). This steady state is a bubbleless balanced growth path, given by (23). Moreover, the local stability of the steady state is summarized in Table 2-1.*

      ii. $D\beta/((\alpha+\beta)\sigma) < 1$ *if and only if there exists a unique steady state without the bubble,* (q^{++}, b^{++})*, in the high-growth regime, given by (21). The economy does not grow in this steady state, which is a neoclassical stationary path. Moreover, the local stability of the steady state is summarized in Table 2-2.*

   (b) *Assume that* $(1 - 1/\sigma)\, D > D\beta/((\alpha+\beta)\sigma)$*.*

      i. $(1 - 1/\sigma)\, D > 1$ *if and only if there exists a unique steady state with the bubble,* (q^*, b^*)*, in the low-growth regime, given by (20). This steady state is a bubbly balanced growth path, given by (24). Moreover, the local stability of the steady state is summarized in Table 2-3.*

      ii. $(1 - 1/\sigma)\, D < 1$ *if and only if there exists a unique steady state with the bubble,* (q^{**}, b^{**})*, in the high-growth regime, given by (22). The economy does not grow in this steady state, which is a neoclassical stationary path. Moreover, the local stability of the steady state is summarized in Table 2-4.*

$$\left(q^+, b^+\right) = \left(\frac{1}{\theta}\left(D\frac{\beta}{(\alpha+\beta)\sigma} - 1\right) + 1, 0\right) \tag{19}$$

$$(q^*, b^*) = \left(\frac{1}{\theta}\left(\left(1 - \frac{1}{\sigma}\right) D - 1\right) + 1, q^*\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right)\right) \tag{20}$$

$$\left(q^{++}, b^{++}\right) = \left(\left(D\frac{\beta}{(\alpha+\beta)\sigma}\right)^{\sigma}, 0\right) \tag{21}$$

$$(q^{**}, b^{**}) = \left(\left(\left(1 - \frac{1}{\sigma} \right) D \right)^{\sigma}, q^{**} \left(\frac{\beta}{\alpha+\beta} \frac{1}{\sigma - 1} - 1 \right) \right) \tag{22}$$

$$\frac{Y_{t+1}}{Y_t} = \frac{A_t}{A_{t-1}} = \frac{K_{t+1}}{K_t} = D \frac{\beta}{\sigma(\alpha+\beta)}, B_t = 0 \textit{ for all } t \tag{23}$$

$$\frac{Y_{t+1}}{Y_t} = \frac{A_t}{A_{t-1}} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \left(1 - \frac{1}{\sigma} \right) D \tag{24}$$

Table 1-1: A steady state without the bubble in the low-growth regime (both cases)

Low-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$
$D\frac{\beta}{(\alpha+\beta)\sigma} > \theta - 1$	*Sink point*	*Saddle point*
$D\frac{\beta}{(\alpha+\beta)\sigma} < \theta - 1$	*Non-existent*	*Source point*

Table 1-2: A steady state without the bubble in the high-growth regime (both cases)

High-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$
	Sink point	*Saddle point*

Table 1-3: A steady state with the bubble in the low-growth regime

Low-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$ *(ST case)*	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$ *(BLK case)*
$D\left(1 - \frac{1}{\sigma}\right) > \theta - 1$	*Saddle point*	*Sink point*
$D\left(1 - \frac{1}{\sigma}\right) < \theta - 1$	*Non-existent*	*Saddle point*

Table 1-4: A steady state with the bubble in the high-growth regime

High-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$ *(ST case)*	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$ *(BLK case)*
	Saddle point	*Sink point*

Proof. See Appendix. ■

Let $\hat{g}^{+}$ and $\hat{g}^{++}$ be the capital growth rate in the steady state without the bubble in the low-growth regime and the high-growth regime. From Proposition 1, we obtain

$$\hat{g}^{+} = D \frac{\beta}{\sigma(\alpha+\beta)}, \hat{g}^{++} = 1.$$

$\hat{r}^{+}$ and $\hat{r}^{++}$ denote the interest rate in the steady state without the bubble in the low- and the high-growth regimes, respectively. Then,

$$\hat{r}^{+} = D\left(1 - \frac{1}{\sigma}\right), \hat{r}^{++} = \frac{(\alpha+\beta)(\sigma-1)}{\beta}.$$

Therefore, we obtain the following relation:

$$\begin{aligned} \hat{g}^{+} &\lesseqqgtr \hat{r}^{+} \Longleftrightarrow \frac{\beta}{\alpha+\beta} \lesseqqgtr \sigma - 1, \\ \hat{g}^{++} &\lesseqqgtr \hat{r}^{++} \Longleftrightarrow \frac{\beta}{\alpha+\beta} \lesseqqgtr \sigma - 1. \end{aligned}$$

By combining the above relations and Proposition 1, the bubble can exist if the growth rate is higher (lower) than the interest rate in the Samuelson–Tirole (Benhabib–Laroque–Kojima) case. This result is similar to that of Kojima (2012a). In the low-growth regime, moreover, the growth rate with a bubble is lower (higher) than the growth rate without the bubble in the Samuelson–Tirole (Benhabib–Laroque–Kojima) case.

3.2 Cycles with Bubbles

Proposition 2 shows period-2 cycles in the Benhabib–Laroque–Kojima case. Propositions 1 and 2 establish that the bubble economy is locally determinate and globally indeterminate in the Benhabib–Laroque–Kojima case.

Proposition 2 *Suppose that* $(1-1/\sigma)D > 1$. $(1-1/\sigma)D < \theta - 1$ *and* $(1-1/\sigma)\,D > D\beta/((\alpha+\beta)\sigma)$ *if and only if there are unique period-2 cycles with a bubble, such as* $q^L < 1 < q^H$ *and* $\left(q^L, b^L\right) \neq \left(q^H, b^H\right)$ *in the Benhabib–Laroque–Kojima case, thereby satisfying the following equations:*

$$\zeta\left(q^H\right) := q^L = \left(\frac{\left(1-\frac{1}{\sigma}\right)^2 D^2}{\psi\left(q^H\right)}\right)^{\sigma}, \tag{25}$$

$$\left(1-\frac{1}{\sigma}\right) D q^H = \psi\left(q^H\right)\zeta\left(q^H\right), \tag{26}$$

$$b^L = \left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right)\zeta\left(q^H\right), \tag{27}$$

$$b^H = \left(\frac{\beta}{\alpha + \beta} \frac{1}{\sigma - 1} - 1 \right) q^H. \tag{28}$$

Moreover, period-2 cycles never emerge in the Samuelson–Tirole case.

Proof. See Appendix. ■

Next, we discuss the properties of Proposition 2. g_X denotes the growth rate of X. Similar to Proposition 2 (Matsuyama, 1999), we obtain $g_A = 1 < D(1 - 1/\sigma) < D(1 - 1/\sigma)\phi\left(q^L\right) = g_K = g_Y = g_B$ in the high-growth regime and $g_A = \psi\left(q^H\right) > D(1 - 1/\sigma) = g_K = g_Y = g_B$ in the low-growth regime. Since the result that the output growth rate in the high-growth regime is higher than that in the low-growth regime, the output growth rate moves up and down over the cycle. This result implies that there is a single equilibrium path with an endogenous business cycle. Moreover, along the path, there is always a bubble. Therefore, there is the possibility that the bubble exists whether the economy is in boom or recession.

4 Conclusion

Using the framework of an overlapping-generations model with Matsuyama's (1999, 2001) production sector, we explore the existence of bubbles regardless of whether an economy is in a boom or recession. Specifically, we show that there is upward and downward movement of the growth rate along a single equilibrium path. Along the path, a bubble always exists, independent of the growth rate. Moreover, we show that this movement emerges in the Benhabib–Laroque–Kojima case, but not the Tirole–Samuelson case. Furthermore, we provide the necessary and sufficient conditions for the existence of an equilibrium steady state with a bubble in both the Samuelson–Tirole case and Benhabib–Laroque–Kojima case.

We use Matsuyama's (1999, 2001) framework to generate the cycle. Since his framework is applicable to the production sector, the existence of cycles does not depend on asset markets. Therefore, building a framework for an asset market to generate cycles is an interesting direction for future research.

Appendix

Lemma

Before providing the proofs of Proposition 1 and 2, we present two lemmas of the parameters.

Lemma 1 *We assume that* $D\beta/\left((\alpha+\beta)\,\sigma\right) > 1$*. Then,*

$$\sigma - 1 \leq \frac{\beta}{\alpha+\beta} \Longrightarrow D\frac{\beta}{(\alpha+\beta)\sigma} \geq \theta - 1.$$

Proof. We assume that $D\beta/\left((\alpha+\beta)\,\sigma\right) < \theta - 1$. Then, $1 < D\beta/\left((\alpha+\beta)\,\sigma\right) < \theta - 1$, which yields $2 < \theta < e$. However, using $0 < \beta/\left(\alpha+\beta\right) < 1$ and $\sigma - 1 \leq \beta/\left(\alpha+\beta\right)$, σ is in $(1,2)$. Since θ is an increasing function with respect to σ, $\theta \in (1,2)$ as $\sigma \in (1,2)$. This result contradicts $2 < \theta < e$. ■

Lemma 2 *We assume that* $\left(1 - 1/\sigma\right) D > 1$*. Then,*

$$\sigma - 1 \leq \frac{\beta}{\alpha+\beta} \Longrightarrow \left(1 - \frac{1}{\sigma}\right) D \geq \theta - 1.$$

Proof. We assume that $(1-1/\sigma)D < \theta - 1$. Then, $1 < (1-1/\sigma)D < \theta - 1$, which yields $2 < \theta < e$. Conversely, using $0 < \beta/\left(\alpha+\beta\right) < 1$ and $\sigma - 1 \leq \beta/\left(\alpha+\beta\right)$, σ is in $(1,2)$. Since θ is an increasing function with respect to σ, $\theta \in (1,2)$ as $\sigma \in (1,2)$. This result contradicts $2 < \theta < e$. ■

Proof of Proposition 1

First, we show the existence of steady states. Next, we consider local stabilities around steady states.

Conditions for the existence of steady states

Let (q^+, b^+) and (q^*, b^*) be the value of the steady state without a bubble and the value of the steady state with a bubble in the low-growth regime, respectively. Then, $q^+ > 1$ and $q^* > 1$ must hold. b^* is a positive constant in the

Samuelson–Tirole case and a negative constant in the Benhabib–Laroque–Kojima case. In the steady state without the bubble, b^+ equals zero. From (17), we obtain

$$q^+ = \frac{1}{\theta}\left(D\frac{\beta}{(\alpha+\beta)\sigma} - 1\right) + 1.$$

Thus, $D\beta/((\alpha+\beta)\sigma) > 1$ if and only if $q^+ > 1$. In the steady state with the bubble, $b^* \neq 0$. Using (18), we obtain

$$q^* = \frac{1}{\theta}\left(\left(1 - \frac{1}{\sigma}\right)D - 1\right) + 1.$$

Thus, $(1 - 1/\sigma)\,D > 1$ if and only if $q^* > 1$. From (17), we obtain

$$b^* = q^*\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right).$$

Then, we need the condition given by $\sigma - 1 \lesseqgtr \beta/\left(\alpha+\beta\right)$ to guarantee that $b^* \gtreqless 0$.

Let (q^{++}, b^{++}) and (q^{**}, b^{**}) be the value of the steady state without a bubble and the value of the steady state with a bubble in the high-growth regime, respectively. Then, $q^{++} < 1$ and $q^{**} < 1$ must hold. b^{**} is a positive constant in the Samuelson–Tirole case and a negative constant in the Benhabib–Laroque–Kojima case. In the steady state without the bubble, b^{++} equals zero. From (17), we obtain

$$q^{++} = \left(D\frac{\beta}{(\alpha+\beta)\sigma}\right)^{\sigma}.$$

Then, $D\beta/((\alpha+\beta)\sigma) < 1$ if and only if $q^{++} < 1$. In the steady state with the bubble, $b^{**} \neq 0$. Using (18), we obtain

$$q^{**} = \left(\left(1 - \frac{1}{\sigma}\right)D\right)^{\sigma}.$$

Thus, $(1 - 1/\sigma)\,D < 1$ if and only if $q^{**} < 1$. From (17), we obtain

$$b^{**} = q^{**}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right).$$

Then, we need the condition given by $\sigma - 1 \lesseqgtr \beta/\left(\alpha+\beta\right)$ to guarantee that $b^{**} \gtreqless 0$.

Local stabilities around steady states

As a first step, we compute the partial derivatives of (17) and (18) with respect to q_t and b_t. We obtain

$$\frac{\partial q_{t+1}}{\partial q_t} = \begin{cases} \frac{D}{\sigma}\frac{1}{\psi(p_t)^2}\left(\theta\left(\sigma-1\right)b_t - \left(\theta-1\right)\frac{\beta}{\alpha+\beta}\right) & \text{low-growth regime} \\ \left(1-\frac{1}{\sigma}\right)D\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}\phi\left(q_t\right) - b_t\phi'\left(q_t\right)\right) & \text{high-growth regime} \end{cases},$$

$$\frac{\partial q_{t+1}}{\partial b_t} = \begin{cases} -\left(1-\frac{1}{\sigma}\right)D\frac{1}{\psi(q_t)} & \text{low-growth regime} \\ -\left(1-\frac{1}{\sigma}\right)D\phi\left(q_t\right) & \text{high-growth regime} \end{cases},$$

$$\frac{\partial b_{t+1}}{\partial q_t} = \begin{cases} -\left(1-\frac{1}{\sigma}\right)D\theta b_t\frac{1}{\psi(q_t)^2} & \text{low-growth regime} \\ \left(1-\frac{1}{\sigma}\right)Db_t\phi'\left(q_t\right) & \text{high-growth regime} \end{cases},$$

$$\frac{\partial b_{t+1}}{\partial b_t} = \begin{cases} \left(1-\frac{1}{\sigma}\right)D\frac{1}{\psi(q_t)} & \text{low-growth regime} \\ \left(1-\frac{1}{\sigma}\right)D\phi\left(q_t\right) & \text{high-growth regime} \end{cases}.$$

Next, we analyze the Jacobian matrix. We now demonstrate the stability around the steady state with and without a bubble.

Around the steady state without the bubble We consider bubble-less steady states. In each steady state, we have

$$J_{11} := \left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{\text{steady-state}} = \begin{cases} -\frac{(\theta-1)\sigma}{D\left(\frac{\beta}{\alpha+\beta}\right)} & \text{low-growth regime} \\ 1-\frac{1}{\sigma} & \text{high-growth regime} \end{cases},$$

$$J_{12} := \left.\frac{\partial q_{t+1}}{\partial b_t}\right|_{\text{steady-state}} = \begin{cases} -\frac{(\alpha+\beta)(\sigma-1)}{\beta} & \text{low-growth regime} \\ -\frac{(\alpha+\beta)(\sigma-1)}{\beta} & \text{high-growth regime} \end{cases},$$

$$J_{21} := \left.\frac{\partial b_{t+1}}{\partial q_t}\right|_{\text{steady-state}} = \begin{cases} 0 & \text{low-growth regime} \\ 0 & \text{high-growth regime} \end{cases},$$

$$J_{22} := \left.\frac{\partial b_{t+1}}{\partial b_t}\right|_{\text{steady-state}} = \begin{cases} \frac{(\alpha+\beta)(\sigma-1)}{\beta} & \text{low-growth regime} \\ \frac{(\alpha+\beta)(\sigma-1)}{\beta} & \text{high-growth regime} \end{cases}.$$

We can write the Jacobian matrix evaluated at each steady state as

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

The eigenvalues of J are the solutions of the following function

$$
\begin{aligned}
0 &= \lambda^2 - (J_{11}+1)\lambda + J_{11} + J_{21} \\
&= \begin{cases} \left(\lambda - \frac{(\alpha+\beta)(\sigma-1)}{\beta}\right)\left(\lambda + \frac{(\theta-1)\sigma}{D\frac{\beta}{\alpha+\beta}}\right) & \text{low-growth regime} \\ \left(\lambda - \frac{(\alpha+\beta)(\sigma-1)}{\beta}\right)\left(\lambda - \left(1-\frac{1}{\sigma}\right)\right) & \text{high-growth regime} \end{cases}.
\end{aligned}
$$

From Lemma 1 and $\sigma \in (1, +\infty)$, we obtain the following tables.

Table 1-1: A steady state without the bubble in the low-growth regime

Low-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$
$D\frac{\beta}{(\alpha+\beta)\sigma} > \theta - 1$	Sink point	Saddle point
$D\frac{\beta}{(\alpha+\beta)\sigma} < \theta - 1$	Non-existent	Source point

Table 1-2: A steady state without the bubble in the high-growth regime

High-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$
	Sink point	Saddle point

Around the steady state with the bubble We consider steady states with the bubble. In each steady state, we have

$$
J_{11} := \left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{\text{steady-state}} = \begin{cases} \frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta} - 1 - \frac{1}{D}\frac{\sigma}{\sigma-1}(\theta-1) & \text{low-growth regime} \\ \frac{1}{\sigma}\left(\frac{\beta}{\alpha+\beta}\frac{\sigma}{\sigma-1} - 1\right) & \text{high-growth regime} \end{cases},
$$

$$
J_{12} := \left.\frac{\partial q_{t+1}}{\partial b_t}\right|_{\text{steady-state}} = \begin{cases} -1 & \text{low-growth regime} \\ -1 & \text{high-growth regime} \end{cases},
$$

$$
\begin{aligned}
J_{21} &:= \left.\frac{\partial b_{t+1}}{\partial q_t}\right|_{\text{steady-state}} \\
&= \begin{cases} -\frac{\sigma}{\sigma-1}\frac{1}{D}\left(\left(1-\frac{1}{\sigma}\right)D + \theta - 1\right)\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right) & \text{low-growth regime} \\ -\frac{1}{\sigma}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma-1} - 1\right) & \text{high-growth regime} \end{cases},
\end{aligned}
$$

$$
J_{22} := \left.\frac{\partial b_{t+1}}{\partial b_t}\right|_{\text{steady-state}} = \begin{cases} 1 & \text{low-growth regime} \\ 1 & \text{high-growth regime} \end{cases}.
$$

We can write the Jacobian matrix evaluated at each steady state as

$$
J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.
$$

The eigenvalues of J are the solutions of the following function

$$\begin{aligned} 0 &= \lambda^2 - (J_{11}+1)\lambda + J_{11} + J_{21} \\ &= \begin{cases} \left(\lambda - \frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta}\right)\left(\lambda + \frac{\sigma}{\sigma-1}\frac{\theta-1}{D}\right) & \text{low-growth regime} \\ \left(\lambda - \frac{1}{\sigma-1}\frac{\beta}{\alpha+\beta}\right)\left(\lambda - \left(1-\frac{1}{\sigma}\right)\right) & \text{high-growth regime} \end{cases}. \end{aligned}$$

From Lemma 2, $\sigma \in (1, +\infty)$, and the existence conditions of the steady states, we obtain the following tables[14].

Table 1-3: A steady state with the bubble in the low-growth regime

Low-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$ (ST case)	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$ (BLK case)
$D\left(1-\frac{1}{\sigma}\right) > \theta - 1$	Saddle point	Sink point
$D\left(1-\frac{1}{\sigma}\right) < \theta - 1$	Non-existent	Saddle point

Table 1-4: A steady state with the bubble in the high-growth regime

High-growth regime	$\sigma - 1 < \frac{\beta}{\alpha+\beta}$ (ST case)	$\sigma - 1 > \frac{\beta}{\alpha+\beta}$ (BLK case)
	Saddle point	Sink point

Proof of Proposition 2

Substituting $((q_t, b_t), (q_{t+1}, b_{t+1})) = \left(\left(q^L, b^L\right), \left(q^H, b^H\right)\right)$ for (17) and (18), we obtain

$$q^H = D\left(q^L\right)^{-\frac{1}{\sigma}}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}q^L - \left(1-\frac{1}{\sigma}\right)b^L\right), \tag{29}$$

$$b^H = D\left(1-\frac{1}{\sigma}\right)b^L\left(q^L\right)^{-\frac{1}{\sigma}}. \tag{30}$$

Substituting $((q_t, b_t), (q_{t+1}, b_{t+1})) = \left(\left(q^H, b^H\right), \left(q^L, b^L\right)\right)$ for (17) and (18), we obtain

$$q^L = D\frac{1}{1+\left(q^H-1\right)\theta}\left(\frac{\beta}{\alpha+\beta}\frac{1}{\sigma}q^H - \left(1-\frac{1}{\sigma}\right)b^H\right), \tag{31}$$

$$b^L = D\left(1-\frac{1}{\sigma}\right)b^H\frac{1}{1+\left(q^H-1\right)\theta}. \tag{32}$$

[14]ST case and BLK case indicate the Samuelson–Tirole case and the Benhabib–Laroque–Kojima case, respectively.

Combining (29), (30), (31), and (32), we obtain (25), (26), (27), and (28).

Next, we show that q^L and q^H satisfy $0 < q^L < 1 < q^H$. By (25),

$$0 < q^L < 1 \Longleftrightarrow \underline{q} := \frac{1}{\theta}\left(x^2 - 1\right) + 1 < q^H, \tag{33}$$

where $x := (1 - 1/\sigma)D$, and $\underline{q} > 1$ by $(1 - 1/\sigma)D > 1$. Let (26) be defined as

$$f\left(q^H\right) := \psi\left(q^H\right)\zeta\left(q^H\right) - \left(1 - \frac{1}{\sigma}\right)Dq^H.$$

Since

$$f'\left(q^H\right) < 0, \tag{34}$$

$$\lim_{q^H \to \infty} f\left(q^H\right) = -\infty, \tag{35}$$

and

$$f\left(\underline{q}\right) > 0 \Longleftrightarrow \left(1 - \frac{1}{\sigma}\right)D < \theta - 1, \tag{36}$$

there is a solution, q^H, satisfying $1 < \underline{q} < q^H$. Moreover, we obtain

$$b^L < 0, b^H < 0 \Longleftrightarrow \frac{\beta}{\alpha + \beta} < \sigma - 1. \tag{37}$$

Therefore, $\left(\left(q^L, b^L\right), \left(q^H, b^H\right)\right)$ comprises the period-2 cycles with a bubble in the Benhabib–Laroque–Kojima case.

Conversely, we consider that (25), (26), (27), and (28) are given, and $\left(\left(q^L, b^L\right), \left(q^H, b^H\right)\right)$ is in the Benhabib–Laroque–Kojima case. Similarly, from (33), (34), (35), (36), and (37), we obtain $(1 - 1/\sigma)D < \theta - 1$ and $\beta/(\alpha + \beta) < \sigma - 1$.

Finally, we check whether period-2 cycles never emerge in the Samuelson–Tirole case. If (17) and (18) have period-2 cycles in the Samuelson–Tirole case, we obtain equations (25), (26), (27), and (28). By the same argument, $(1 - 1/\sigma)D < \theta - 1$ and $b^L > 0, b^H > 0 \Longleftrightarrow \beta/(\alpha + \beta) > \sigma - 1$ must hold. These relations contradict Lemma 2.

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