RIETI Discussion Paper Series 19-E-008

Two-dimensional Constrained Chaos and Time in Innovation: An analysis of industrial revolution cycles

YANO Makoto
RIETI

FURUKAWA Yuichi
RIETI

Research Institute of Economy, Trade & Industry, IAA
The Research Institute of Economy, Trade and Industry
https://www.rieti.go.jp/en/

RIETI Discussion Paper Series 19-E-008

February 2019

Two-dimensional Constrained Chaos and Time in Innovation: An analysis of industrial revolution cycles[1]

YANO Makoto[2]

Research Institute of Economy, Trade and Industry

FURUKAWA Yuichi

Chukyo University and Research Institute of Economy, Trade and Industry

Abstract

Many issues in cycles and fluctuations cannot be explained without multiple state variables. Time needed for innovation is one such issue. This study uncovers, and provides a new characterization for, ergodic chaos with two state variables and builds a model of innovation highlighting time in innovation and intellectual property protection. We demonstrate that the explosive industrial takeoffs and Kondratieff's long waves over the past two centuries and a half may be explained as a single equilibrium phenomenon. Time in innovation tends to stabilize chaotic innovation dynamics. Intellectual property protection is a necessary condition for chaotic takeoffs from the no-innovation phase.

Keywords: Industrial revolutions, Chaotic cycles, Intellectual properties, Market quality dynamics.

JEL Classification Codes: C62; E32; O41

[1]This study is conducted as part of a research project, Evidence-based Policy Study on the Law and Economics of Market Quality, undertaken by the Research Institute of Economy, Trade and Industry (RIETI). We are grateful to Paulo Brito, Ken Chan, Angus Chu, Guido Cozzi, Steve Easton, Jean-Michel Grandmont, Richard Harris, Jun-ichi Itaya, Christoph Luelfesmann, Tapan Mitra, Felipe Saffie, Nic Schmitt, Hitoshi Shigeoka, and seminar/conference participants at Chukyo University, Durham University, RIETI (Discussion Paper Seminar), Simon Fraser University, University of Southern California, the Conference in honor of Jean-Michel Grandmont held at Aix-Marseille, June, 2013, and the 2013 European Meeting of the Econometric Society for useful conversations and comments. Partial financial support from the Keio-Kyoto Global COE Program, the Grant-in-Aid for Specially Promoted Research #23000001, the Grant-in-Aid for Scientific Research (A) #16H02015, the Grant-in-Aid for Scientific Research (C) #18K01522, the Grant-in-Aid for Young Scientists (B) #23730198, and a Research Grant Program of the Toyota Foundation are gratefully acknowledged. Yuichi Furukawa acknowledges the hospitality and support of Simon Fraser University, at which part of the work was conducted.

2 Corresponding author.

1 Introduction

Chaos theory has greatly contributed to our understanding of economic cycles and fluctuations. Despite this, it has not touched the issues involving multiple state variables. This is partly because there are few structural characterizations for two-dimensional chaos that render economic dynamics tractable and are as powerful as those on single dimension. One of those issues is the role of time needed before R&D investments materialize. This study uncovers, and provides a characterization for, a new two-dimensional chaos and analyze the role of R&D investments in innovation cycles.

The development of our method is prompted by the puzzle that originates from Kondratieff (1925). That is, why have explosive technological progresses come in very long waves? It is often said that we are currently in the midst of the third industrial revolution; some even argue that we are facing the start of the fourth industrial revolution. While the first through the third industrial revulsions are more than one hundred years apart, Kondratieff (1925, 1935) focuses on a bit "shorter" innovation waves, spanning fifty to sixty years. Kondratieff (1925, 1935) argues that those long waves are driven by an economic mechanism rather than a sequence of random events; he writes,

> "In asserting the existence of long waves and in denying that they arise out of random causes, we are also of the opinion that the long waves arise out of causes which are inherent in the essence of the capitalistic economy" (see Kondratieff (1935, p. 115)).

Chaotic dynamics is a perfect tool for explaining the mechanism behind those waves, some of which are explosive and exceptionally long, and the others not as extreme. We incorporate time in innovation into a standard one-period monopoly model of innovation cycles (Judd (1985), Deneckere and Judd (1992), and Matsuyama (1999, 2001)), in which innovation, production, and consumption are assumed to complete within a single period (atemporal-innovation model).[1] We demonstrate that the phase of active innovation and that of no innovation alternate in a chaotic manner. Time in innovation opens

[1] As Judd (1985) and Deneckere and Judd (1992) show, innovation cycles can be modelled with a single state variable under the assumption of atemporal innovation. By adding capital accumulation to their model, Matsuyama (1999, 2001) builds a model with two-state variables that is simple enough to be transformed into a single state variable model. See also Shleifer (1986) and Gale (1996), who study a different mechanism for innovation cycles that emerge due to a strategic delay in implementing inventions.

the channel through which the adjustment in market structure stabilizes, but not wipe out completely, the chaotic dynamics that exists in the atemporal-innovation version of our model. Two distinct types of takeoffs from the no-innovation phase to the innovation phase emerge in this process: They may be referred to as large and small.[2]

The two dimensional chaos that this study uncovers is an ergodic chaos (Grandmont (1985, 1986) and Bhattacharya and Majumdar (2007)), which permits a probabilistic characterization for a deterministic system (Birkov (1931), von Neumann (1932), and Lasota and Yorke (1973)). Relying on the ergodic theorem, we find reasonable parameter values with which a large takeoff on average emerges along an equilibrium path once in more than one hundred years whereas a takeoff in general once in fifty to sixty years along the same equilibrium path. These results open a way to explain the coexistence of industrial revolution cycles and Kondratieff's long waves in a single dynamic general equilibrium. Judd (1985) and Deneckere and Judd (1992), in contrast, build models of atemporal-innovation in order to capture similar but shorter chaotic innovation cycles of "40, 20 and 13 years" (Judd (1985, p. 580)). Our results show that longer chaotic waves can be captured in their atemporal-innovation models as well.

In addition to time in innovation, this study highlights the role of intellectual property protection in innovation cycles. We follow Helpman (1993) in incorporating the level of intellectual property protection and demonstrates that a sufficient protection is a necessary condition for our chaotic industrial innovation cycles. Intellectual property protection has been regarded as an important key to start the first industrial revolution (North (1981, 1990)).[3] Since then, a large volume of studies has been concerned with economic growth in relation to various institutional factors.[4] In contrast, this study is new in highlighting the role of such an institutional factor in innovation cycles.

In the one-period monopoly model, innovation cycles are attributable to a rise in the relative importance of innovation during the period in which no innovation is made. Following Judd (1985), the present study incorporates this feature by assuming that labor productivity grows at a constant rate through the accumulation of experiences. In contrast, Deneckere and Judd

[2]While this study focuses on cycles and fluctuations drived by innovation, there is a large volume of literature that focuses on innoavtion driven growth (Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992)).

[3]This view is widely supported in the literature, including those who do not regard intellectual property protection as the essential driving force of the first industrial revolution (Mokyr, 2009).

[4]See Acemoglu, Johnson, and Robinson (2005) and Helpman (2008).

(1992) assume that the existing products become obsolete at a constant rate while Matsuyama (1999, 2001) assumes that physical capital accumulates.

Our two-dimensional chaos may be thought of as an extension of what may be called constrained chaos. Nishimura and Yano (1994, 1995a, 1995b, 1996) show that chaos may emerge in an optimal growth model with a single state variable if an unstable and oscillatory dynamical system is restricted to a bounded feasible region. This study reveals that if the domain of a system of difference equations with two state variables is restricted by an externally imposed constraint, the "double-period dynamics" relating the state variable at the beginning of a period to that at the end of the subsequent period may be described by a first order dynamical system with a single state variable. We demonstrate that this first order system can be expansive and unimodal, i.e., Lasota-Yorke's ergodic chaos (see Lasota and Yorke (1973). Even if so, the original single-period dynamics, relating the two state variables at the beginning of a period to those at the end of the same period, cannot be transformed into a first order system with a single state variable.

The economic literature on chaotic dynamics can go back long way to Benhabib and Day (1980) and Grandmont (1985). Since then, an important, and the most difficult, issue has been whether or not chaotic dynamics is consistent with the optimization of infinitely-lived consumers with perfect foresight and/or rational expectations. For the single dimensional case, this question is first addressed by the bifurcation method in the early literature (Benhabib and Nishimura (1979, 1985)). Whether or not the infinite time horizon optimization may result in chaotic dynamics has been solved by two different approaches, model seeking and structural. The model seeking approach starts with a chaotic system and looks for an intertemporal optimization model the solution of which coincides with that chaotic system. Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) first solve this question, which is extended by Nishimura, Sorger and Yano (1994), Nishimura and Yano (1996), Mitra (1996), and Mitra and Sorger (1999). In contrast, Nishimura and Yano (1994, 1995a, 1995b) and Baierl, Nishimura and Yano (1998), and Khan and Mitra (2005, 2012) show that the feasibility constraints in an economic model may restrict interior dynamics (or the solutions to an Euler equation) in such a way that the model may generate optimal chaotic dynamics. The present study follows this line of research in dealing with the infinite-time horizon optimization of a consumer.

After explaining Lasota-Yorke's theorem on ergodic chaos, in Section 2, we will give the mathematical characterization of our two-dimensional constrained chaos. In Section 3, we will build our model with time in innovation and intellectual property protection. We will reveal the way in which time in innovation brings a market structure in innovation dynamics in the atemporal-innovation version of our model and demonstrate that intellectual property protection together with exogenous growth will eventually lead an economy without innovation to take-off. In Section 4, we will provide a complete characterization for chaotic innovation dynamics for the case in which the elasticity of substitution between differentiated products. In Section 5, we will extend this result to the case of general elasticity of substitution and show the role of time in innovation in partially stabilizing chaotic innovation dynamics. Section 6 is for concluding remarks.

1.1 Preliminary Facts

In this study, we will build an equilibrium model in which dynamics is governed by ergodic chaos. Ergodic chaos implies that solutions to a dynamical system behave as if they were generated by a random process. For the sake of explanation, let I be a closed interval in $\mathbb{R}$. Adopt the convention $f^t = f \circ f^{t-1}$, $t = 1, 2, \ldots$, $f^0(x) = x$ and $f^1(x) = f(x)$.

Denote as m the Lebesgue measure. A measure μ on I is absolutely continuous (with respect to the Lebesgue measure) if $m(A) = 0$ implies $\mu(A) = 0$. Moreover, a system, f, is said to be invariant if for any μ-measurable $A \subset I$, $f^{-1}(A) = A$ implies $\mu(A) = \mu(f^{-1}(A))$. It is ergodic with respect to a measure μ (or μ-ergodic) if $\mu(A) = \mu(f^{-1}(A))$ implies $\mu(A) = 0, 1$.

Birkhoff (1931) and von Neumann (1932) show the existence of a natural coupling between a deterministic system, $f : I \to I$, and a probability measure on I, μ, such that the action average can be related to the space average, i.e., for any measurable $A \subset I$, it holds that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \chi_A(f^t(x_0)) = \mu(A), \tag{1}$$

where χ is the characteristic function defined by $\chi_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$. In particular, Birkhoff (1931) proves that if the system is ergodic with respect to the invariant measure, μ, equation (1) holds for almost every A with respect to the invariant measure, μ.

This prompts the standard definition of ergodic chaos in the standard literature. An ergodic chaos is a system $f : I \to I$ that can be associated with a Lebesgue-absolutely continuous, invariant probability measure with respect to which the system is ergodic, μ (Grandmont (1986, 2008) and Bhattacharya and Majumdar (2007)).

Our study is based on the characterization of an ergodic chaos obtained by Lasota and Yorke (1973), who show that an expansive and unimodal system is an ergodic chaos. A dynamical system $f : I \to I$ is said to be expansive if it is piecewise twice continuously differentiable and if $\inf |f'(x)| > 1$ over the set of x at which $f'(x)$ is well defined. Moreover, a system, $f : I \to I$ is unimodal if it is continuous and if there is $c \in I$ either such that $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ or such that $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, whenever f' is defined. Lasota and Yorke (1973) shows the following:

Theorem 1 (Lasota and Yorke's Chaos) *An expansive and unimodal dynamical system on an closed interval I is an ergodic chaos.*

2 Chaos in Two Dimension

In this section, we introduce a new chaotic system with two state variables, which is based on a constrained domain. In order to motivate our method, it may be useful to start out with our equilibrium dynamical system, which will be developed in the subsequent part of this study.

$$\begin{cases} x_{t+1} &= y_t \\ y_{t+1} &= \max\left\{ \frac{1}{\alpha} y_t, \frac{1}{\alpha} y_t + \frac{1}{\alpha} \frac{\psi \frac{\delta(\alpha y_t - x_t) + x_t}{\gamma(\alpha y_t - x_t) + x_t} - y_t}{\chi \frac{\upsilon(\alpha y_t - x_t) + x_t}{\upsilon(\alpha y_t - x_t) + x_t} + \delta} \right\}. \end{cases} \tag{2}$$

For economists, this system might be of little interest unless the underlying economic structure is explained. For the moment, however, we focus purely on the mathematical features of solutions to system (2).

Not to mention, system (2) is a highly complicated two-dimensional dynamical system. It is more manageable than the standard two dimensional system in the form of $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$ in that it is assumed that $x_{t+1} = y_t$. This assumption is, however, innocuous because in most cases, the general system, $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$, can be transformed into a system in the form of $(x_{t+1}, y_{t+1}) = (y_t, f(x_t, y_t))$.

What is surprising is that the plots of solutions to system (2) has a very simple structure, which is shown in *Figure* 1. The plots depict the double-period solution orbit, $(y_{2\tau}, y_{2(\tau+1)})$, $\tau = 0, 1, 2, ...$, that are taken from a solution, (x_t, y_t), to system (2) from $(x_0, y_0) = (2.263, 2.368)$. For the values of parameters, we adopt

$$(\alpha, \psi, \delta, \upsilon, \chi) = (1.17, 2.264, 0.377, 0.358, 0.01). \tag{3}$$

These plots forms a unimodal and expansive graph, which suggests that the original system, (2), might be obeying Lasota-Yorke's chaos (see Theorem 1) in double periods.

This observation motivates our characterization of two-dimensional chaos. In what follows, we will explain the underlying mechanisms and demonstrate that the double period dynamics of the original two-dimensional system, relating $y_{2\tau}$ to $y_{2(\tau+1)}$, obeys a single dimensional chaos.

For the sake of explanation, think of a simple two dimensional system

$$(x_{t+1}, y_{t+1}) = F(x_t, y_t) = (y_t, F_2(x_t, y_t)). \tag{4}$$

If, for example, the fixed point of F is a saddle point with with negatively sloped stable and unstable manifold, M^s and M^u, a solution to the system follows a pair of orbits. *Figure* 2 illustrates such a solution; if the initial point is at the blank circle, the solution can be illustrated by the arrows form the circle.

Now introduce an exogenous constraint that restricts the state variable vector, (x_t, y_t), the region above and on $y_t = A(x_t)$. That is to say, if $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$ satisfies $y_{t+1} \geq A(x_{t+1})$, dynamics follows (4), i.e., $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$. If it does not satisfy the constraint, the state-variable vector falls down to the boundary; i.e., if $y_t < A(x_t)$, $y_{t+1} = A(x_{t+1}) = A(y_t)$. This system may be written as

$$(x_{t+1}, y_{t+1}) = (y_t, \max\{F_2(x_t, y_t), A(y_t)\}) = f(x_t, y_t). \tag{5}$$

For the sake of analysis below, it is convenient to write

$$B(x_t, y_t) = F_2(x_t, y_t) - A(y_t). \tag{6}$$

With this function, B, we may define the "core" of feasible state-variable vectors as

$$C = \{(x_t, y_t) : B(x_t, y_t) \geq 0 \text{ and } y_t \geq A(x_t)\}. \tag{7}$$

If $(x_t, y_t) \notin C$, then (5) implies

$$(x_{t+1}, y_{t+1}) = (y_t, A(y_t)). \tag{8}$$

If, instead, $(x_t, y_t) \in C$, then

$$(x_{t+1}, y_{t+1}) = (y_t, B(x_t, y_t) + A(y_t)). \tag{9}$$

For the sake of simplicity, assume the following:

Assumption 1 Function A is continuously differentiable; $y > A(y)$ and $A' > 0$.

Assumption 2 Function B is continuously differentiable. Equation $B(x, y) = 0$ can be solved for y. The solution $y = \bar{B}(x)$ satisfies $\bar{B}' < 0$ and that $y \leq \bar{B}(x)$ if and only if $B(x, y) \geq 0$.

See *Figure* 3, which incorporates constraint $y \geq A(x)$ to the unconstrained system in *Figure* 2. Assume that $B(x, y) \geq 0$ if and only if (x, y) lies below or on curve $\bar{B}$ in *Figure* 3; that is, curve $\bar{B}$, capturing $B(x, y) = 0$, is the upper boundary of core C. Moreover, the lower boundary of C is illustrated by line A in *Figure* 3; in system (2), function A is $y_{t+1} = \frac{1}{\alpha} y_t$.

Suppose that, again, the initial point lies at the blank circle (point 0) which lies above curve $\bar{B}$. Since this implies that $B(x_0, y_0) < 0$, by (8), it holds that $(x_1, y_1) = (y_0, A(y_0))$. This point is at the intersection of the first downward dotted arrow from the blank circle and line A, which is indicated by the solid circle (point 1). Another orbit goes through this solid circle, as is shown by dotted curve in *Figure* 3. This orbit is paired with an orbit in the opposite "quadrant," which is indicated by another dotted curve. The state variable vector, therefore, moves from the first solid circle (point 1) to the second (point 2). It will then move to the third (point 3) on line A. As this shows, the constrained dynamics may exhibit stark nonlinearity, which is captured by the plots in *Figure* 1.

Next, by using this idea, we will provide a sufficient condition under which system f is in fact chaotic. Towards this end, note that, under Assumptions 1 and 2, $y = \bar{B}(x)$ and $y = A(x)$ have a unique intersection. Define y_H as satisfying $\bar{B}(y_H) = A(y_H)$. Moreover, define the following functions:

$$R(y) = B(y, A(y)) + A(A(y)); \tag{10}$$

$$G(y) = B(A(R^{-1}(y)), y) + A(y); \tag{11}$$

$$L(y) = B(y, G(y)) + A(G(y)). \tag{12}$$

The next lemma gives a sufficient condition under which an equilibrium path that reaches (x_t, y_t) such that $\bar{B}(x_t) < y_t < y_H$ will follow $y_{t+2} = R(y_t)$ and $y_{t+4} = L(y_{t+2})$.

Lemma 1 *Suppose that an equilibrium path (x_t, y_t), solving system* (53)*, satisfies the following conditions:*

$$\bar{B}(x_t) \leq y_t < y_H; \tag{13}$$

$$y_{t+2} < \bar{B}(x_{t+2}) < y_H; \tag{14}$$

$$y_{t+3} < \bar{B}(x_{t+3}) < y_H. \tag{15}$$

If that R^{-1} exists, the equilibrium path satisfies $(x_{t+2}, y_{t+2}) = (A(y_t), R(y_t))$ and $(x_{t+4}, y_{t+4}) = (G(y_{t+2}), L(y_{t+2}))$.

Proof. Let $y_t \geq \bar{B}(x_t)$ and $y_t < y_H$. Then, $B(x_t, y_t) \leq 0$. By (53),

$$y_{t+1} = A(y_t) \text{ and } x_{t+1} = y_t.$$

Since B and A are, respectively, decreasing and increasing, $y_t < y_H$ implies $y_{t+1} < \bar{B}(x_{t+1})$. Since this implies $B(x_{t+1}, y_{t+1}) > 0$, by $y_{t+1} = A(y_t)$ and (53), we have

$$y_{t+2} = B(y_t, A(y_t)) + A(A(y_t)) \text{ and } x_{t+2} = A(y_t),$$

which implies $y_{t+2} = R(y_t)$. Since $y_{t+2} < \bar{B}(x_{t+2}) < y_H$ by (14), similarly, we have

$$y_{t+3} = B(y_{t+1}, y_{t+2}) + A(y_{t+2}) \text{ and } x_{t+3} = y_{t+2},$$

which implies $x_{t+4} = y_{t+3} = G(y_{t+2})$. Moreover, since $y_{t+3} < \bar{B}(x_{t+3}) < y_H$ by (15),

$$y_{t+4} = B(y_{t+2}, y_{t+3}) + A(y_{t+3}) \text{ and } x_{t+4} = y_{t+3},$$

which implies $y_{t+4} = L(y_{t+2})$. ■

Now, assume the following:

Assumption 3 $L'(x) > 1$ and $R'(x) < -1$. There is x such that $x < L(x)$ and $x < R(x)$.

Under this assumption, the function below is well defined and expansive and unimodal:

$$T(x) = \min_y \{L(x), R(x)\}. \tag{16}$$

Moreover, define

$$S(x) = \begin{cases} G(x) & \text{if } L(x) \leq R(x) \\ A(x) & \text{if } L(x) \geq R(x) \end{cases}. \tag{17}$$

The next theorem provides a sufficient condition under which an equilibrium path follows an ergodic chaos.

Theorem 2 *Suppose that Assumptions 1, 2 and 3 are satisfied. Let* $y_C = \arg\max_x T(x)$, $y_{\max} = \max_x T(x)$, *and* $y_L = R^{-1}(y_{\max})$. *Suppose that* $y_L < L(y_L)$ *and that if* $\bar{B}(x) \leq y < y_H$ *and* $R(y) < \bar{B}(A(y))$, *then the following holds:*

$$G(R(y)) < \bar{B}(R(y)); \tag{18}$$

$$L(R(y)) \geq \bar{B}(G(R(y))). \tag{19}$$

Then, a solution (x_t, y_t), $t = 0, 1, \ldots$, *to the original dynamical system,* (5), *from* (x_0, y_0), $\bar{B}(x_0) < y_0 < y_H$, *follows the ergodically chaotic system on* $[y_L, y_{\max}]^2$ *as follows:*

$$\left(x_{2(\tau+1)}, y_{2(\tau+1)}\right) = \left(S(y_{2\tau}), T(y_{2\tau})\right). \tag{20}$$

Proof. Note that $y_L < L(y_L)$ implies that $T(x)$ on $[y_L, y_{\max}]$ is a function into itself. Thus, by Lasota and Yorke's theorem (Theorem 1), T is an ergodically chaotic dynamical system on $[y_L, y_{\max}]$.

Suppose $y_C \leq \bar{B}(x_t) \leq y_t < y_H$ for an arbitrary t. As is shown in the proof of Lemma 1, this implies $(x_{t+2}, y_{t+2}) = (A(y_t), T(y_t))$. Since $y_{\max}$ is achieved at y_C and since $y_C \leq y_t < y_H$,

$$y_{t+2} = R(y_t) = T(y_t) \leq y_{\max} < y_H.$$

The resulting state variable vector, (x_{t+2}, y_{t+2}), may or may not lie in the core region, C. If it does not, then $y_{t+2} \geq \bar{B}(x_{t+2})$. In this case, by (18),

$$y_{t+4} = L(y_{t+2}) \text{ and } x_{t+4} = A(y_{t+2}).$$

Suppose, instead, that $y_{t+2} < \bar{B}(x_{t+2})$. Then, $R(y_t) < \bar{B}(A(y_t))$. This implies that, by (18) and (19), $y_{t+3} < \bar{B}(x_{t+3})$ and $y_{t+4} < \bar{B}(x_{t+4})$. Thus, by Lemma 1, $(x_{t+4}, y_{t+4}) = (A(y_{t+2}), T(y_{t+2}))$.

Finally since $y_C \leq \bar{B}(x_{t+4}) \leq y_{t+4} < y_H$ by (19), the above process repeats. Thus, $(x_{t+2\tau}, y_{t+2\tau}) = (S^\tau(y_t), T^\tau(y_t))$ for all τ if $(x_0, y_0) = (x, y)$ and $y_C \leq \bar{B}(x) \leq y < y_H$. ■

System (2) with parameter values (3) satisfies the conditions obtained in Theorem 2 and, thus, obeys the double-period chaotic system, (20). In the example, specific functional forms can be obtained for L, R, G, and $\bar{B}$ along with values for y_L, y_C, $y_{\max}$ and y_H; their characterizations are given in the subsequent part of this study.

By using those characterizations, it is possible to draw the graphs of those functions. See *Figure* 4. As (66) shows, function R is linear and negatively sloped. In contrast, as (68) shows, function L is non-linear and positively sloped, although it is very close to a linear function. On interval $[y_L, y_{\max}]$, L' and R' are given as follows:

$$1.60411 < L' < 1.60463 \text{ and } R' \approx 1.14899, \tag{21}$$

which shows that L has a practically linear graph. Moreover,

$$(y_L, y_C, y_{\max}, y_H) \approx (2.263, 2.276, 2.364, 2.649). \tag{22}$$

In *Figure* 4, functions L and R are illustrated by curve L and line R. Moreover, the graphs of functions $y = G(x)$, $y = \bar{B}(G(x))$ and $y = \bar{B}(A(x))$ are illustrated by curves G, $B \circ G$, and $B \circ A$. The graph of $y = \bar{B}(x)$ lies (slightly) below curve $B \circ A$. As is shown below (see the proof of Theorem 5), y_C is at the vertical coordinate of the intersection between curve $B \circ A$ and line R. Thus, if $R(y) < \bar{B}(A(y))$, it holds that $y_L \leq R(y) < y_C$. Since, as *Figure* 4 shows, curve G lies below curve $\bar{B} \circ A$, (18) is satisfied. Moreover, since curve L lies above curve $\bar{B} \circ G$, condition (19) is also satisfied. Thus, any solution to (2), (x_t, y_t), satisfies $y_{2(\tau+1)} = T(y_{2\tau})$, which is ergodically chaotic. This result may be summarized as follows:

Corollary 1 *The double-period dynamics of system (2) is Lasota-Yorke's chaos.*

Many economic applications exist for our two-dimensional chaotic system. There are many economic issues that cannot be analyzed without assuming more than one state variable. The main part of this study deals with such an issue. That is to investigate chaotic dynamics arising from the endogenous interaction between market structure and innovation.

3 Dynamic Model of Innovation

In this section, we introduce time in innovation and intellectual property protection into the standard model of atemporal innovation (Judd (1985), Deneckere and Judd (1992), and Matsuyama (1999, 2001)). We demonstrate that time in innovation opens the channel through which a market structure affects equilibrium dynamics.

3.1 Basic Model

Following Judd (1985), we assume that labor productivity increases due to Harrod-neutral exogenous technological progress. That is, the effective productivity of labor $\bar{E}$ grows at the rate of α (trend growth rate); i.e., the effective amount of labor in period t (or, more precisely, the period between time $t-1$ and time t) is

$$E_t = \alpha^t \bar{E} > 0. \tag{23}$$

With this assumption, we think of the case in which not physical labor but the quality of labor increases at the constant rate through accumulating experiences.

We assume the length of a single period to be just long enough for the monopolistic owner of a newly invented product to receive a monopolistic profit from that product. Newly invented products are protected by patents. They are in competition with the existing products. In the real world, however, many newly invented products become obsolete before patents expire. The length of a single period in a model must be thought of as shorter than 20 years.

As is noted above, we assume that there are two state variables, determining the market structure endogenously. The first state variable is the number of new inventions, Z_t, which represents the size of the monopolistic sector. In order to make invention, it is necessary to invest in research. If research activities are made in the period between time $t-1$ and time t, new inventions become available at time t. Each invention is associated with a differentiated product; thus, Z_t may be thought of as the number of differentiate products that are newly invented for the production in the period between t and $t+1$.

The other state variable is the number of differentiated products invented in the past and sold by the competitive sector, N_t. This variable represents the size of the competitive sector. These state variables, Z_t and N_t, obey the following dynamics:

$$N_t = N_{t-1} + Z_{t-1}. \tag{24}$$

Assume that at the beginning of period 1 (time 0), a positive number of differentiated products invented before the initial period $N_0 = \overline{N} > 0$ exists. Denote as $Z_0 = \overline{Z} \geq 0$ the number of newly invented differentiated product at time 0, which can be zero. Technological progress is irreversible with respect to both types of technologies. In the case of endogenous technological progress, therefore, it must hold

$$Z_{t-1} \geq 0. \tag{25}$$

In the system captured by (23) - (25), growth is driven both by an endogenous factor (represented by $Z_{t-1} > 0$) and an exogenous factor (represented by $E_t = \alpha^t \bar{E}$, $\alpha > 1$). Parameter α determines the trend growth rate of the economy. The actual growth rate is, in general, different from the trend rate because innovation level Z_t fluctuates over time.

Assume that, in each period, each R&D firm decides whether or not to invest in inventing a new technology for producing a differentiated middle product. Inventions are produced by using only labor. Let κ be the labor input needed for making a new invention. By using this technology, each manufacturer produce a differentiated middle output; let λ be the labor input needed to make one unit of a middle product. The retail sector transforms the middle products into a single final consumption good.

In order to examine the role of intellectual property protection in recurrent industrial takeoffs, we assume that in each period, only 100ϕ percent of the new inventions are actually protected. This assumption follows Helpman (1993) and is adopted to demonstrate in the simplest fashion that a sufficient level of intellectual property protection is a necessary condition for recurrent industrial takeoffs. As is discussed in the Introduction, parameter ϕ may be interpreted as capturing the enforcement level of the patent rule. It can be interpreted also as the standard that the patent authority applies to patent applications.[5] Under this assumption, the numbers of monopolistic and competitive markets in period t are, respectively, $N_t^M = \phi Z_{t-1}$ and $N_t^C = (1-\phi)Z_{t-1} + N_{t-1}$.

There are four types of firms in the economy. They are perfectly competitive retail firms, perfectly competitive manufacturers, monopolistic manufacturers, and innovation firms. Consumers, represented by a single agent, consume the final consumption good, sold by the retail sector. The representative consumer chooses a sequence of consumption, X_t, so as to maximize the following intertemporal utility,

$$U = \sum_{t=1}^{\infty} \beta^{t-1} \ln X_t, \tag{26}$$

where $0 < \beta < 1$. By normalizing the current value of consumption goods in each period to 1 and denoting as r_j the real interest rate in period t, the consumer's wealth constraint is expressed as

$$\sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) X_t \leq W_0 \tag{27}$$

[5] Under this interpretation, it is implicitly assumed that patentabilities differ across inventions, although the differentiated products, produced from invented technologies, are assumed to be symmetric.

with the wealth, W_0, consisting of the sum of discounted values of labor and his initial asset or, in other words, equal to

$$W_0 = \sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) w_t \alpha^t \bar{E} + A_0, \tag{28}$$

where A_0 is the value of initial asset at the beginning of period 1.

The retail sector, in period t, transforms the existing differentiated products (i.e., those in closed interval $[0, N_t]$) into the final consumption good in that period. Denote as X_t the amount of the final consumption good produced in period t and as v_{tj} the amount of good j employed to produce X_t. For the retail sector's production function, we adopt the standard CES function,

$$X_t = \left(\int_0^{N_t} {v_{tj}}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}, \quad 0 < \theta < 1; \tag{29}$$

see Dixit and Stiglitz (1977) and Ethier (1982). Note that θ is the inverse of the elasticity of substitution between any two differentiated products. The retail sector's optimization problem can be expressed as, for each period,

$$\max_{v_{tj}} \left[\left(\int_0^{N_t} {v_{tj}}^{1-\theta} dj \right)^{\frac{1}{1-\theta}} - \int_0^{N_t} p_{tj} v_{tj} dj \right], \tag{30}$$

where p_{tj} is the price of product j in period t.

Next, we will describe the market for patent licenses. In each period, an innovation firm can invent one technology to produce a new differentiated product by using κ units of labor. Denote as (t, i) the ith invention that is invented in period t. This invention can be utilized to produce a product in period $t+1$, which we call (t, i) as well. The innovation firm to invent technology (t, i) can sell the licence for its invention at price P_t if the invention is not to be free ridden and at 0 if it is to be free ridden. Denote as $\tilde{P}_t$ the distribution of this price.

Each innovation firm decides whether or not to employ workers for an invention. If, in period t, an innovation firm makes input for (t, i), denote its choice as $\delta^I_{(t,i)} = 1$. If it does not, denote the choice as $\delta^I_{(t,i)} = 0$. Thus, the profit maximization problem of the innovation firm to invent technology $(t-1, i)$ can be written as

$$\max_{\delta^I_{(t-1,i)} \in \{0,1\}} \mathcal{E} \left(\frac{\tilde{P}_t}{1+r_{t-1}} - w_t \kappa \right) \delta^I_{(t-1,i)}, \tag{31}$$

where $\mathcal{E}$ is the operator taking the expected value of a random variable.

Each individual manufacturer in the market for new inventions decides whether or not to invest in a licence for an invention ($\delta_{tj}^M = 1$ or 0) and how many units of the product it will produce by using the invention (v_{tj}^M), where $j = (t-1, i)$. Assume that, in doing so, it takes the manufacturing sector's aggregate output, X_t, as given. The manufacturer must pay P_{tj} for the license in purchasing the license of invention $j = (t-1, i)$. In addition, it must pay $w_t \lambda v_{tj}^M$ for labor to produce output since, by assumption, λ units of labor per unit of output are needed. Thus, the optimization problem of a manufacturer, acquiring $j = (t-1, i)$ invention, is

$$\max_{\delta_{tj}^M \in \{0,1\}, v_{tj}^M \geq 0} (p(v_{tj}^M; X_t) v_{tj}^M - w_t \lambda v_{tj}^M - P_{tj}) \delta_{tj}^M. \tag{32}$$

In an equilibrium in the market for licenses for patent, each new invention that is not free ridden must be purchased by a manufacturer. This implies that in an equilibrium in the period-t market for licenses of inventions, it holds that

$$\delta_{tj}^M = \delta_{(t-1,i)}^I \text{ if invention } j = (t-1, i) \text{ is not free ridden.} \tag{33}$$

We assume that free entry is guaranteed on both sides of the market for intellectual properties. On the demand side, this implies that the profit each monopolistic manufacturer can acquire by using the exclusive license for a technology must not exceed the price of a license. By (32), this can be written as

$$(p(v_{tj}^M; X_t) - w_t \lambda) v_{tj}^M \leq P_{tj}, \tag{34}$$

where strict equality holds if $\delta_{tj}^M > 0$. On the supply side, the expected present value of a price of a licence must not exceed the opportunity cost of an invention for an innovation firm. Since this opportunity cost is $w_{t-1}\kappa$, this condition can be expressed as

$$\frac{\phi P_{tj}}{1 + r_{t-1}} \leq w_{t-1}\kappa, \tag{35}$$

where strict equality holds if $\delta_{(t-1,i)}^I > 0$. In summary, the market for intellectual properties is described by (31)-(35).

The competitive manufacturing sector in period t consists of firms that use the technologies invented before period $t-2$ and those that are invented in period $t-1$ but failed to receive the patent protection. The optimization problem in a perfectly competitive manufacturing sector can be written as

$$\max_{v_{tj}^C} (p_{tj}^C v_{tj}^C - w_t \lambda v_{tj}^C), \tag{36}$$

where v^C_{tj} and p^C_{tj}, respectively, denote the amount and price of output produced by the competitive sector using publicly known technology j.

In the labor market, labor is employed by the manufacturing sector (monopolistic and competitive) and the R&D sector. Thus, the market clearing condition is

$$\int_0^{N_t} \lambda v_{tj} dj + \kappa Z_t = \alpha^t \bar{E}, \tag{37}$$

which implies that the sum of the amount of effective labor forces employed by the manufacturing sector and that employed in the invention sector must be equal to the existing amount of effective labor forces. This completes the description of our model.

3.2 Determination of an Equilibrium

The model above has two state variables, Z_t and N_t. One important feature of the model is that, even though it is based on the intertemporal optimization of an infinitely-lived consumer, unlike an optimal growth model, the equilibrium system can be written as a dynamical system of the two state variables independently of the co-state variables (prices); that system automatically satisfies the transversality condition for intertemporal optimization.

This dynamical system captures the two driving forces of cyclical innovation dynamics: (i) Harrod-neutral exogenous growth, which constantly pushes up the demand for inventions, and (ii) endogenous inventions, which create new manufacturing industries, spread out the demand for inventions as a whole, thereby reducing the demand for each individual invention.

Due to symmetricity, $v^M_{tj} = v^M_t$ and $p^M_{tj} = p^M_t$ for all monopolistically supplied goods tj. Define

$$\omega_t = \frac{\phi(p^M_{t+1} - \lambda w_{t+1})v^M_{t+1}}{(1 + r_t)w_t}, \tag{38}$$

which is the present-value expected profit, evaluated by physical labor, from an invention in period t. By (34) and (35), it must hold that

$$\kappa \geq \omega_t \tag{39}$$

where equality holds if $Z_t > 0$. This condition implies that ω_t must be exceeded by the marginal (labor) cost of innovation, κ, in equilibrium. This shows that ω_t may be thought of as a manufacturer's "derived" willingness to pay for an invention in period t, given Z_t.

For this reason, (38) can be transformed into a derived inverse demand function for Z_t. In order to make this transformation, by solving (30), obtain the inverse demand function of the retail sector for a differentiated manufactured good tj,

$$p_{tj} = p(v_{tj}; X_t) = (X_t/v_{tj})^{\theta}. \tag{40}$$

Facing this demand, the monopolistic manufacturer of tj maximizes its profit. The profit maximizing price is as follows:

$$p^M_{t+1} = \frac{1}{1-\theta}\lambda w_{t+1}. \tag{41}$$

Note that $v^M_{t+1} = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_{t+1}$ for a monopolistically manufactured good. By the first order condition for consumers, we have $X_{t+1}/X_t = \beta(1+r_t)$. By using these facts, $p^M_{t+1} - \lambda w_{t+1}$, v^M_{t+1}, and X_{t+1} can be eliminated from (38), which results in

$$\omega_t = \phi\beta\theta \frac{\left(\frac{\lambda w_{t+1}}{1-\theta}\right)^{1-1/\theta} X_t}{w_t}. \tag{42}$$

As (41) shows, parameter θ determines the mark-up rate of a monopolistic manufacturer. Expression (42) implies that the monopolistic manufacturer's willingness to pay for an invention is proportionate to this mark-up rate, θ.

By symmetricity, $v^C_{tj} = v^C_t$ and $p^C_{tj} = p^C_t$ hold for all competitively supplied goods tj. Since $p^C_t = \lambda w_t$ by profit maximization, and since $p_j(v_{ij}; X_t) = v_{tj}^{-\theta} X_t^{\theta}$, it holds that $v^C_t = (\lambda w_t)^{-1/\theta} X_t$. By plugging this and $v^M_t = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_t$ into the production function of the retail sector, (29), we have

$$w_t = \frac{1}{\lambda}\left(N_{t-1} + \left(1 - \phi\left(1 - \left(\tfrac{1}{1-\theta}\right)^{1-1/\theta}\right)\right) Z_{t-1}\right)^{\frac{\theta}{1-\theta}}. \tag{43}$$

Moreover, by plugging $v_{tj} = v^C_t = (\lambda w_t)^{-1/\theta} X_t$ for any competitively supplied tj and $v_{tj} = v^M_t = (\frac{\lambda w_t}{1-\theta})^{-1/\theta} X_t$ for any monopolistically supplied tj into the labor market clearing condition, (37), we obtain

$$X_t = \frac{\bar{E}\alpha^t - \kappa Z_t}{\lambda^{1-1/\theta}\left(N_{t-1} + (1 - \phi(1 - \left(\frac{1}{1-\theta}\right)^{-1/\theta}))Z_{t-1}\right) w_t^{-1/\theta}}. \tag{44}$$

Thus, by (43) and (44), (42) can be transformed into the (inverse) derived demand function as follows:

$$\omega_t(Z_t; N_{t-1}, Z_{t-1}) = \phi\beta\theta\eta \frac{\xi Z_{t-1} + N_{t-1}}{\zeta Z_{t-1} + N_{t-1}} \frac{\bar{E} - \kappa Z_t/\alpha^t}{N_t/\alpha^t + \xi Z_t/\alpha^t}, \tag{45}$$

where

$$\eta = (1-\theta)^{1/\theta - 1}, \tag{46}$$

$$\xi = 1 - \phi\left(1 - (1-\theta)^{1/\theta-1}\right) \tag{47}$$

and

$$\zeta = 1 - \phi \left(1 - (1 - \theta)^{1/\theta}\right). \tag{48}$$

With (45), (39) can be expressed as a non-autonomous system

$$\omega_t(Z_t; N_{t-1}, Z_{t-1}) \leq \kappa, \tag{49}$$

where $Z_t > 0$ only if (49) holds with equality. In summary, our equilibrium stem can be transformed into a two-state-variable dynamical system (24) and (49) together with (25).

Remark 1 *Our model is a specific form of the dynamic general equilibrium model with infinitely-lived consumers (Yano, 1998). In the standard model, an equilibrium is characterized by a set of Euler equations, or an Euler system, consisting of the first order conditions of optimization. In a system with two state variables, the Euler system is a dynamical system of two state variables and two co-state variables; in our model, N_t and Z_t may be thought of as state variables whereas p_t and q_t as a co-state variable. In the standard model, the sequences of state and co-state variables must be determined by simultaneously solving a system of infinitely many market clearing conditions which render the dynamic general equilibrium intractable. Once that system is solved, the equilibrium dynamics on state and co-state variables is described by the Euler system with the exogenously given initial values of state variables and the endogenously determined initial values of co-state variables (see Yano, 1998, for a more detailed explanation). It is a distinctive feature of our model that the equilibrium values of the co-state variables, p_t and q_t, can be determined without solving the entire infinite-dimensional equilibrium system.*

The basic working of this system can be illustrated by demand and supply curves in the intellectual property market. Note that (45) may be thought of the inverse demand function for new technologies; in that function, Z_t/α^t may be thought of as the effective demand for new technologies whereas ω_t is the technology firms' marginal willingness to pay for a new technology. See *Figure* 5. Curve D_t illustrates the graph of (45), relating ω_t to Z_t/α^t. The supply curve is the horizontal line through κ, S, which may be thought of the marginal cost of a new technology. The equilibrium effective number of new technologies inventions, Z_t/α^t, is determined at the intersection between demand curve D_t and supply curve S. The cumulative number of differentiated products, N_{t+1}/α^{t+1}, is also determined by (24) as the sum of Z_t/α^t and N_t/α^t.

Generally speaking, a "market structure" refers to the way in which competitive and non-competitive sectors are situated in an economy. In the context of innovation, firms sell new products monopolistically. As their monopolistic controls weaken, those products will eventually be sold competitively. In our model, N_{t-1} and Z_{t-1}, respectively, represent the numbers of competitive and monopolistic firms in period t. Thus, we may define a market structure in period t as the vector of those numbers, (N_{t-1}, Z_{t-1}).

As (43) shows, the demand for new technologies in period t, Z_t, depends on the market structure labor demand for the market structure, (N_{t-1}, Z_{t-1}). The market structure affects through term $\mu_t = \frac{\xi Z_{t-1}+N_{t-1}}{\zeta Z_{t-1}+N_{t-1}}$. Since ξ and ζ depends on ϕ, the effect of a market structure depends on the level of intellectual property protection; if they are not protected (i.e., if $\phi = 0$), $\xi = \zeta$, which implies $\mu_t = 1$.

3.3 Time in Innovation and Market Structure

As is discussed in the Introduction, the development of two-dimensional constrained chaos in our study is motivated by the fact that time in innovation brings the second state variable Z_t in addition to N_t. In order to explain the role of time in innovation, it is useful to think of a model in which innovation, production, and consumption are carried out in one single time frame.

In order to build the atemporal-innovation version of our model, we assume that innovation takes place in the same period as production and consumption. In that case, (35) becomes

$$\phi P_{tj} \leq w_t \kappa.$$

Thus, by (34), an innovation firm's willingness to pay for a new product, (38), becomes, by (40) and (41),

$$\omega_t = \phi\theta \frac{\left(\frac{\lambda w_t}{1-\theta}\right)^{1-1/\theta} X_t}{w_t}.$$

Moreover, since the labor market clearing condition becomes

$$X_t = \frac{\bar{E}\alpha^t - \kappa Z_t}{\lambda^{1-1/\theta}\left(N_t + \zeta Z_t\right) w_t^{-1/\theta}},$$

the marginal willingness to pay for a new technology becomes

$$\omega_t = \phi\theta\eta \frac{\bar{E} - \kappa Z_t/\alpha^t}{N_t/\alpha^t + \zeta Z_t/\alpha^t}, \tag{50}$$

which may be called an atemporal demand function for new technologies.

As is shown below, this demand function, (50), gives rise to the same dynamical system as that of Judd (1985). As Deneckere and Judd (1992) explain, the model of Judd (1985), as well as Deneckere-Judd's model, abstracts from saving/investment decisions. The above analysis formally demonstrates that Judd's model can be interpreted in the atemporal version of the present model.

The comparison between the atemporal-innovation demand, (50), and the time-in-innovation demand, (45), shows that the introduction of time in innovation introduces two additional terms, β and $\mu_t = \frac{\xi Z_{t-1}+N_{t-1}}{\zeta Z_{t-1}+N_{t-1}}$. Term β reflects the fact that the monopolistic sector has to pay interest in investing in new technologies, the output of which will be born one period later. Demand function (45) shows that time in innovation brings about of the effect of market structure on the demand for new technologies, ω_t, through term μ_t.

Without time in innovation, the demand for new technologies, Z_t, does not depend on the market structure, μ_t, but on the number of the existing products, $N_t = N_{t-1} + Z_{t-1}$. If time in innovation is incorporated, as (45) shows, the demand for new technologies, Z_t, depends on market structure (N_{t-1}, Z_{t-1}) in period $t-1$, in which input decisions are made for introducing new products in period t. An important question that we deal with in the latter half of this study is whether this market structure term, $\mu_t = \frac{\xi Z_{t-1}+N_{t-1}}{\zeta Z_{t-1}+N_{t-1}}$, dampens or stimulates chaotic dynamics observed by Judd (1985) and Deneckere and Judd (1992).

4 Chaotic Industrial Takeoffs

In this section, we demonstrate that the equilibrium model in the previous section can be transformed into a two-dimensional system in the form of (2). By using Theorem 2, we provide a complete characterization for ergodic chaos for the case in which the elasticity of substitution, $1/\theta$, is sufficiently large. The general case of θ is discussed separately in Section 6.

4.1 Equilibrium Dynamical System

In order to relate our equilibrium model to system (2), we introduce two new state variables

$$x_t = \frac{N_t}{\phi\theta\alpha^t} \tag{51}$$

and

$$y_t = \frac{1}{\alpha}\left(\frac{N_t}{\phi\theta\alpha^t} + \frac{Z_t}{\phi\theta\alpha^t}\right) \tag{52}$$

Variable x_t may be thought of as capturing the size of existing products that are invented in the past. In contrast, $y_t - x_t$ may be thought of as that of new products that are invented at the beginning of a period. Because we assume that new products in this period are invented by using research activities in the previous period, x_t and $y_t - x_t$ represent different types of stock variables (state variables).

With these state variables , (24), (25), (39), and (45) can be transformed into an autonomous system as follows:

$$\begin{cases} x_{t+1} &= y_t \\ y_{t+1} &= \max\left\{ \frac{1}{\alpha} y_t, \ \frac{1}{\alpha} y_t + \frac{1}{\alpha} \frac{\frac{\bar{E}\beta\eta}{\kappa} \frac{\xi(\alpha y_t - x_t) + x_t}{\zeta(\alpha y_t - x_t) + x_t} - y_t}{\phi\beta\theta\eta \frac{\xi(\alpha y_t - x_t) + x_t}{\zeta(\alpha y_t - x_t) + x_t} + \xi} \right\}. \end{cases} \tag{53}$$

The initial condition is $\bar{x} = \bar{N}/\phi\theta$ and $\bar{y} = \frac{1}{\alpha}(\bar{N}/\phi\theta + \bar{Z}/\phi\theta)$. A quick comparison between (2) and (53) shows that the parameters of (2) can be written as $\psi = \frac{\bar{E}\beta\eta}{\kappa}$, $\delta = \xi$, $\gamma = \zeta$, and $\rho = \phi\beta\theta\eta$. The plots in *Figure* 1, therefore, suggest that innovation dynamics captured in our model is in fact chaotic.

The equilibrium dynamical system, (53), has a unique fixed point, which is

$$y_S^\theta = \frac{\frac{\bar{E}\beta\eta}{\kappa}}{(\alpha - 1)\phi\beta\theta\eta + \zeta(\alpha - 1) + 1}. \tag{54}$$

As the next lemma shows, this steady state can be a saddle point (a proof requires a tedious sequence of calculations, which is omitted here).

Theorem 3 *System* (53) *has a unique steady state that is generically unstable and around which any equilibrium path fluctuates if and only if*

$$\beta < \frac{\alpha - \phi(\alpha - 1)\zeta}{\alpha - \phi(\alpha - 1)\xi} \frac{\xi - \frac{\alpha}{\alpha+1}\frac{1}{\phi}}{\xi\theta} - \frac{2\alpha}{(\alpha + 1)(\alpha - \phi(\alpha - 1)\xi)}. \tag{55}$$

It can easily be checked that this condition is satisfied if the elasticity of substitution, $1/\theta$, is sufficiently large, which implies that a high substitutability between differentiated products can be a cause of Kondratieff-like long waves.

Proposition 1 *If condition* (55) *is satisfied, innovation activities on every equilibrium path but the saddle path exhibit slow cycles like those of Kondratieff's long waves.*

Proposition 1 shows a result similar to that of Matsuyama (2001), who demonstrates the existence of a locally unstable steady state in a model in which endogenous innovation is incorporated into the standard optimal growth model.

4.2 Basic Assumptions for Theorem 2

In order to characterize chaotic dynamics in our equilibrium model, (53), we will make it sure that the basic assumptions introduced in Section 2 are satisfied. In the equilibrium model above, functions A and B in Section 2 have the following forms.

$$A(x_t) = \frac{1}{\alpha}x_t, \tag{56}$$

and

$$B_\theta(x_t, y_t) = \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi(\alpha y_t - x_t)+x_t}{\zeta(\alpha y_t - x_t)+x_t} - y_t}{\phi\beta\theta\eta\frac{\xi(\alpha y_t - x_t)+x_t}{\zeta(\alpha y_t - x_t)+x_t} + \xi}. \tag{57}$$

See *Figure* 6. Equation (56) implies that Assumption 1 is satisfied in the equilibrium model. The upper boundary of the core of feasible activities, C, is the ray from the origin with the slope equal to $1/\alpha$, which is depicted by line A. If and only if $B_\theta(x_t, y_t) > 0$, innovation takes place. The region above the no-innovation line A illustrates the innovation phase. Line $y = A_\theta(x)$ describes the no-innovation phase. The region above line A may be called the innovation phase.

In order to check that Assumption 2 is also satisfied, note $B_\theta(x_t, y_t) > 0$ in the innovation phase. By using $\xi > \zeta$, we may prove that $B_\theta(x_t, y_t) \geq 0$ if and only if $y_t \leq \bar{B}_\theta(x_t)$ where

$$\begin{aligned}\bar{B}_\theta(x) &= \frac{1}{2\alpha\zeta}\Bigg\{-((1-\zeta)x - \frac{\bar{E}\beta\eta}{\kappa}\xi\alpha) \\ &\quad + \sqrt{\left((1-\zeta)x - \frac{\bar{E}\beta\eta}{\kappa}\xi\alpha\right)^2 + 4\frac{\bar{E}\beta\eta}{\kappa}\zeta\alpha(1-\xi)x}\Bigg\}\end{aligned} \tag{58}$$

In the following analysis, it is important to know the precise structure of this function, $y = \bar{B}_\theta(x)$. It is monotone decreasing, and its inverse can be written as

$$\bar{B}_\theta^{-1}(y) = x_1(y) + x_2(y) \tag{59}$$

where

$$x_1(y) = \frac{\alpha\left(\frac{\bar{E}\beta\eta}{\kappa}\right)^2\frac{\xi-\zeta}{1-\zeta}\frac{1-\xi}{1-\zeta}}{(1-\zeta)y - \frac{\bar{E}\beta\eta}{\kappa}(1-\xi)} \tag{60}$$

and

$$x_2(y) = -\frac{\alpha\zeta}{1-\zeta}y + \alpha\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi-\zeta}{(1-\zeta)^2}. \tag{61}$$

This implies that curve $y = \bar{B}_\theta(x)$ is the horizontal sum of a downward sloping rectangular hyperbola, $x = x_1(y)$ illustrated by curves G and G' and a downward sloping line, $x = x_2(y)$, illustrated by line H as is shown in *Figure* 6. Thus, curve $y = \bar{B}_\theta(x)$ is asymptotic to the horizontal line $y = \frac{\bar{E}\beta\eta}{\kappa}\frac{1-\xi}{1-\zeta}$ and the downward sloping line $x = x_2(y)$. This implies that Assumption 2 in Section 2 is also satisfied.

Since line $y_t = A(x_t)$ is positively sloped, their intersection between $y = \bar{B}_\theta(x)$ and $y = A(x)$ is uniquely determined at

$$y_H^\theta = \frac{\bar{E}\alpha\beta\eta}{\kappa}. \tag{62}$$

It is easy to check that the steady state, y_S^θ, obtained in (54), lies in the interior of C; that is,

$$A(y_S^\theta) < y_S^\theta < \bar{B}_\theta(y_S^\theta), \tag{63}$$

which implies

$$y_S^\theta < \bar{B}_\theta(y_H^\theta) = A(y_H^\theta) < y_H^\theta. \tag{64}$$

Define

$$C_\theta = \{(x_t, y_t) > 0 : A(x_t) \leq y_t \leq \bar{B}_\theta(x_t)\}. \tag{65}$$

As in the general model, (5), chaotic dynamics in our equilibrium model, (53), can be captured in a double-period dynamical system with functions R, G, L and T in (10), (11), (83) and (16). That is, the specific form for function R is given by

$$R_\theta(y) = -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta+\xi} - 1\right)y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta+\xi}. \tag{66}$$

Given that R_θ^{-1} exists, the specific forms of G and L can be expressed as follows:

$$G_\theta(y) = \frac{1}{\alpha}y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)) + \frac{1}{\alpha}R_\theta^{-1}(y)}{\zeta\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)} - y}{\phi\beta\theta\eta\frac{\xi\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)}{\zeta\left(\alpha y - \frac{1}{\alpha}R_\theta^{-1}(y)\right) + \frac{1}{\alpha}R_\theta^{-1}(y)} + \xi}. \tag{67}$$

and

$$L_\theta(y) = \frac{1}{\alpha}G_\theta(y) + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi(\alpha G_\theta(y) - y) + y}{\zeta(\alpha G_\theta(y) - y) + y} - G_\theta(y)}{\phi\beta\theta\eta\frac{\xi(\alpha G_\theta(y) - y) + y}{\zeta(\alpha G_\theta(y) - y) + y} + \xi} \tag{68}$$

Finally, T is given by

$$T_\theta(x) = \min_x \{L_\theta(x), R_\theta(y)\}. \tag{69}$$

With T_θ, we may define $y_C^\theta = \arg\max_x T_\theta(x)$, $y_{\max}^\theta = \max_x T_\theta(x)$, and $y_L = R^{-1}(y_{\max})$ by following Theorem 2.

Next, we will obtain conditions under which Assumption 3 (i.e., $L'_\theta > 1$ and $R'_\theta < -1$) is guaranteed. As (66), (67), and (68) show, R'_θ can be obtained explicitly whereas L'_θ is intractable. However, L'_θ can be characterized for the case in which θ is sufficiently small. For this purpose, note that, as $\theta \to 0$,

$$\eta \to \frac{1}{e},\ \xi \to 1 - \phi(1 - \frac{1}{e}),\ \text{and } \zeta \to 1 - \phi(1 - \frac{1}{e}), \tag{70}$$

where e is the base for natural logarithm.

In order to prove $L'_\theta > 1$ for the case of $\theta \to 1$, we bound the domain of our double-period system. Towards this end, focus on the case of $0 < \theta \leq 1/2$. By (46), (47), and (48), then, it holds that $1/e < \eta \leq 1/2$, $1 - \phi/2 \leq \xi < 1 - \phi(1 - e)$ and $1 - \frac{3}{4}\phi < \zeta \leq 1 - \phi(1 - e)$. Thus, by (55),

$$y_S^\theta > \frac{1}{e((1 - \phi(1 - \frac{1}{e}))(\alpha - 1) + 1)} \frac{\bar{E}\beta}{\kappa} \equiv \bar{y}_S \tag{71}$$

and

$$y_H^\theta \leq \frac{\alpha}{2} \frac{\bar{E}\beta}{\kappa} \equiv \bar{y}_H. \tag{72}$$

We will focus on interval $[\bar{y}_S, \bar{y}_H]$.

What is important to know in characterizing L'_θ and G'_θ is that the steady states of L'_θ and G'_θ are not only mutually different from each other but also different from the steady state of the original system, (53), i.e., y_S^θ. In proving the next lemma, this fact has to be taken care of.

Lemma 2 *There is $\theta' > 0$ such that $0 < \theta \leq \theta'$ implies $G'_\theta < -1$ and $L'_\theta > 1$ on $[\bar{y}_S, \bar{y}_H]$ if and only if*

$$\frac{1}{1 - \phi(1 - 1/e)} - 1 > \alpha. \tag{73}$$

Proof. Let $y_{t+1} = \frac{1}{\alpha} R_\theta^{-1}(y_{t+2})$. Then, (66) implies

$$y_{t+1} = -\frac{\alpha(\phi\beta\theta\eta + \xi)}{1 - (\phi\beta\theta\eta + \xi)} y_{t+2} + \frac{\frac{\bar{E}\beta\eta}{\kappa}}{1 - (\phi\beta\theta\eta + \xi)}. \tag{74}$$

By definition, $y_{t+3} = G_\theta(y_{t+2})$ and $y_{t+4} = L_\theta(y_{t+2})$ are given by the the system of equation (74) and the following equations.

$$y_{t+3} = \frac{1}{\alpha}\frac{\frac{\psi\beta\eta}{\kappa}z_{t+1} - y_{t+2}}{\phi\beta\theta\eta z_{t+1} + \xi} + \frac{1}{\alpha}y_{t+2}; \tag{75}$$

$$y_{t+4} = \frac{1}{\alpha}\frac{\frac{\psi\beta\eta}{\kappa}z_{t+2} - y_{t+3}}{\phi\beta\theta\eta z_{t+2} + \xi} + \frac{1}{\alpha}y_{t+3}; \tag{76}$$

$$z_{t+1} = \frac{\xi\left(\alpha y_{t+2} - y_{t+1}\right) + y_{t+1}}{\zeta\left(\alpha y_{t+2} - y_{t+1}\right) + y_{t+1}}. \tag{77}$$

$$z_{t+2} = \frac{\xi\left(\alpha y_{t+3} - y_{t+2}\right) + y_{t+2}}{\zeta\left(\alpha y_{t+3} - y_{t+2}\right) + y_{t+2}}; \tag{78}$$

This implies y_{t+1}, y_{t+3} y_{t+4}, z_{t+2}, and z_{t+4} satisfy (74), (75), (76), (77), and (78) if and only if $y_{t+2} = R_\theta(y_t)$ and $y_{t+4} = L_\theta(y_{t+2})$.

Then, z_{t+1}, z_{t+2}, y_{t+3}, y_{t+4} and y_{t+1} may be thought of as functions of y_{t+2}. By differentiating (74) through (78) with respect to y_{t+2}, we obtain the following:

$$y'_{t+1} = -\frac{\alpha^2(\phi\beta\theta\eta + \xi)}{1 - (\phi\beta\theta\eta + \xi)};$$

$$y'_{t+3} = \frac{1}{\alpha}\frac{(\xi\frac{E\beta\eta}{\kappa} + \phi\beta\theta\eta y_{t+2})z'_{t+1}}{(\phi\beta\theta\eta z_{t+1} + \xi)^2} - \frac{1}{\alpha}\left(\frac{1}{\phi\beta\theta\eta z_{t+1} + \xi} - 1\right);$$

$$\begin{aligned} y'_{t+4} \quad = \quad & \frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta z_{t+2} + \xi} - 1\right)^2 \\ & -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta z_{t+1} + \xi} - 1\right)\frac{(\xi\frac{\psi\beta\eta}{\kappa} + \phi\beta\theta\eta y_{t+2})}{(\phi\beta\theta\eta z_{t+1} + \xi)^2}z'_{t+1} \\ & +\frac{1}{\alpha}\frac{(y_{t+3}\phi\beta\theta\eta + \xi\frac{\psi\beta\eta}{\kappa})}{(\phi\beta\theta\eta z_{t+2} + \xi)^2}z'_{t+2}; \end{aligned}$$

$$z'_{t+1} = \alpha(\xi - \zeta)\frac{y_{t+1} - y_{t+2}y'_{t+1}}{(\zeta\left(\alpha y_{t+2} - y_{t+1}\right) + y_{t+1})^2};$$

$$z'_{t+2} = \alpha(\xi - \zeta)\frac{y_{t+2}y'_{t+3} - y_{t+3}}{(\zeta\left(\alpha y_{t+3} - y_{t+2}\right) + y_{t+2})^2}.$$

By $y_{t+2} \in [\bar{y}_S, \bar{y}_H]$, the lemma follows from (70). ■

By (66) together with (46) and (47), we have the following characterization for R'_θ.

Lemma 3 *It holds that* $R'_\theta < -1$ *if and only if*

$$\frac{1}{\phi\beta\theta(1-\theta)^{1/\theta-1} + 1 - \phi(1-(1-\theta)^{1/\theta-1})} > \alpha^2. \tag{79}$$

Moreover, there is $\theta' > 0$ *such that* $0 < \theta \le \theta'$ *implies* $R'_\theta < -1$ *on* $[\bar{y}_S, \bar{y}_H]$ *if and only if*

$$\frac{1}{1-\phi(1-1/e)} - 1 > \alpha^2. \tag{80}$$

4.3 Two-dimensional Constrained Chaos

In this subsection, we will provide a complete characterization for system (53) to be ergodically chaotic. For this purpose, by Theorem 2, it suffices to prove $y_L^\theta < L_\theta(y_L^\theta)$ and that $R_\theta(y) < \bar{B}_\theta(A(y))$ implies (18) and (19).

For this purpose, first, note the following:

Lemma 4 *There is* $\theta' > 0$ *such that if* $0 < \theta < \theta'$, $y_C^\theta \in [\bar{y}_S, \bar{y}_H]$ *and* $y_{\max}^\theta \in [\bar{y}_S, \bar{y}_H]$.

Proof. By Lemmas 2 and 3, y_C^θ is determined by $R_\theta(y_C^\theta) = L_\theta(y_C^\theta) = y_{\max}^\theta$. Then, by (11), (67), (68), and (70), $y_C^\theta \to \frac{E\beta}{e\kappa}$ and $y_{\max}^\theta \to \frac{E\beta}{e\kappa}(\alpha e - (e-1))$ as $\theta \to 0$. Thus, the lemma holds for y_C^θ and $y_{\max}^\theta$. ■

The next lemma is concerned with a condition under which $y_L^\theta < L_\theta(y_L^\theta)$.

Lemma 5 *Suppose condition* (80) *is satisfied. Then, there is* $\theta' > 0$ *such that* $0 < \theta \le \theta'$ *implies* $L_\theta(y_L^\theta) > \bar{B}_\theta(G_\theta(y_L^\theta)) > y_L^\theta$ *if and only if*

$$\alpha > \sqrt{\frac{-\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^2 + \sqrt{\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^4 + 4\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^3}}{2}} \tag{81}$$

Proof. By Lemmas 2 and 3, we may choose θ' in such a way that $0 < \theta \le \theta'$ implies $L'_\theta > 1$, $G'_\theta < -1$ and $R'_\theta < -1$ on $[\bar{y}_S, \bar{y}_H]$. Thus, $y_C^\theta = y_{t+2}$, $R(y_2) = L_\theta(y_2) = y_{t+4}$, and $y_L^\theta = R_\theta(y_4) = y_{t+6}$ satisfy the following:

$$y_{t+4} = -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta + \xi} - 1\right) y_{t+2} + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta + \xi}; \tag{82}$$

$$y_{t+6} = -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta+\xi}-1\right)y_{t+4} + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta+\xi}. \tag{83}$$

Thus, y_{t+1}, y_{t+2}, y_{t+3}, y_{t+4}, y_{t+6}, z_{t+1} and z_{t+2} are determined by the system of equations (74), (75), (76), (77), (78), (82), and (83). Moreover, $L_\theta(y_L^\theta)$ is determined by

$$y_{t+8} = -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta+\xi}-1\right)y_{t+6} + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta+\xi}. \tag{84}$$

Boundary equation (59) with (60) and (61) shows that, as $\theta \to 0$, $\bar{B}_\theta(x_t) \to \bar{B} = \frac{\bar{E}\beta}{e\kappa}$ uniformly in $x_t \in [\bar{y}_S, \bar{y}_H]$. Thus, if and only if there is $\varepsilon > 0$ such that $y_{t+8} = L_\theta(y_{t+6}) > \frac{\bar{E}\beta}{e\kappa} + \varepsilon$ and $\frac{\bar{E}\beta}{e\kappa} - \varepsilon > y_{t+6}$ for any θ, $0 < \theta \leq \theta'$, it holds that $L_\theta(y_{t+6}) > \bar{B}(G_\theta(y_{t+6})) > y_{t+6}$.

In order to prove this, take the limit case of $\theta = 0$. Since $\xi = \zeta$ in the limit case, by (77) and (78), $z_{t+1} = z_{t+2} = 1$. By using this fact, we may solve the system of (76) and (82) to obtain $y_C = y_{t+2} = \frac{\bar{E}\beta}{e\kappa}$. By this together with (83) and (84), in $\theta = 0$, we have

$$y_{t+6} = \frac{\bar{E}\beta\eta}{\kappa}\left(\frac{1}{\alpha^4}\left(\frac{1}{\xi}-1\right)^2 - \frac{1}{\alpha^3}\left(\frac{1}{\xi}-1\right)\frac{1}{\xi} + \frac{1}{\alpha}\frac{1}{\xi}\right)$$

This implies $y_{t+6} - \frac{\bar{E}\beta\eta}{\kappa} < 0$, given (80) and (81). Moreover,

$$\begin{aligned} y_{t+8} &= \frac{1}{\alpha^2}\frac{\bar{E}\beta}{\kappa e}((e\alpha - (e-1)e) \\ &\quad +(e-1)^2\left(\frac{1}{\alpha^4}\left(e\alpha^3 - (e-1)\,e\alpha + (e-1)^2\right)\right). \end{aligned}$$

Thus, $y_{t+8} > \frac{\bar{E}\beta}{e\kappa}$ if and only if

$$\begin{aligned} &(\alpha - (\frac{1}{1-\phi(1-\frac{1}{e})} - 1)) \\ &\left(\alpha^4 + \left(\frac{1}{1-\phi(1-\frac{1}{e})} - 1\right)^2\alpha^2 - \left(\frac{1}{1-\phi(1-\frac{1}{e})} - 1\right)^3\right) < 0. \end{aligned}$$

Given (73), this implies that there are $\theta' > 0$ and $\varepsilon > 0$ such that $y_{t+8} = L_\theta(y_{t+6}) > \frac{\bar{E}\beta}{e\kappa} + \varepsilon$ for any θ, $0 < \theta \leq \theta'$. This establishes the lemma. ■

$$\frac{\phi(1-\frac{1}{e})}{1-\phi(1-\frac{1}{e})} > \alpha$$

$$\frac{1}{1-\phi(1-\frac{1}{e})}-1>\alpha$$

The main theorem of this study, which is presented below, demonstrates that long innovation waves in our equilibrium model may be generated by an ergodic chaos.

Theorem 4 *Let* $(\bar{x},\bar{y})$ *satisfy* $y_C^\theta<\bar{y}<y_{\max}^\theta$ *and* $\bar{y}>\bar{B}_\theta(\bar{x})$. *There is* $\theta'>0$ *such that* $0<\theta<\theta'$ *implies that the two-state variable equilibrium system,* (53), *from* $(\bar{x},\bar{y})$ *follows the double period system* $y_{t+2}=T_\theta(y_t)$ *that is ergodically chaotic if and only if*

$$\sqrt{\frac{-\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^2+\sqrt{\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^4+4\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^3}}{2}}$$
$$<\alpha<\sqrt{\frac{1}{1-\phi(1-1/e)}-1} \tag{85}$$

Proof. The theorem can be proved by establishing (i) that T_θ is well defined as a dynamical system and (ii) that the equilibrium path from $(\bar{x},\bar{y})$ satisfies conditions (18) and (19) of Theorem 2. Towards this end, fix a θ satisfying (85).Then, by Lemmas 2 and 3, $L'_\theta(y)>1$ and $R'_\theta(y)<-1$ on $[\bar{y}_S,\bar{y}_H]$.

By Lemma 5, $L_\theta(y_L^\theta)>\bar{B}_\theta(y_L^\theta)>y_L^\theta$. Let $x_S^\theta=L_\theta(x_S^\theta)$. Then, by (11), (67), (68), and (70), $\left|y_S^\theta-x_S^\theta\right|\to 0$. Since $\bar{y}_S<\lim_{\theta\to 0}y_S^\theta=\frac{E\beta}{\kappa}\frac{1}{(\alpha-1)+e}<\bar{y}_H$, $\left|y_S^\theta-x_S^\theta\right|\to 0$ implies $\bar{y}_S<x_S^\theta<\bar{y}_H$. Since, by Lemma 5, $L_\theta(y_L^\theta)>\bar{B}_\theta(y_L^\theta)>y_L^\theta$, $\left|y_S^\theta-x_S^\theta\right|\to 0$ implies $y_L^\theta>\bar{y}_S$. Thus, $\bar{y}_S<y_L^\theta<L_\theta(y_L^\theta)<y_{\max}^\eta<\bar{y}_H$ by Lemma 4. This implies that T_θ is a dynamical system on $[y_L^\theta,y_{\max}^\theta]$ onto itself with $L'_\theta>1$ and $R'_\theta<-1$.

In order to prove (ii), take an equilibrium path, (x_t,y_t), solving (53) from $(\bar{x},\bar{y})$. Suppose $y_t\le y_{\max}^\theta$ and $y_t>\bar{B}(y_{t-1})$. This implies $y_{t+1}=\frac{1}{\alpha}y_t<y_{\max}^\theta$. Moreover, since $y_{t+1}=\frac{1}{\alpha}y_t$, $y_t<y_{\max}^\theta<y_H^\theta$ implies implies $y_{t+1}<\bar{B}_\theta(y_t)$, which implies $y_{t+2}=r_\theta(y_{t+1})$ by Lemma 6 (1). We will prove that if $y_{t+2}<\bar{B}_{t+1}(y_{t+1})$, (18) and (19) are satisfied.

Let $y_{t+2}<\bar{B}_\theta(y_{t+1})$. Then, $y_{t+3}=G_\theta(y_{t+2})$, as is shown in the proof of Theorem 2. Moreover, $y_{t+2}<\bar{B}_\theta(y_{t+1})$ implies $y_L^\theta\le y_{t+2}<R_\theta(y_D^\theta)$, where y_D^θ is given by $R_\theta(y_D^\theta)=\bar{B}_\theta(\frac{1}{\alpha}y_D^\theta)$. Let $z_S^\theta=G_\theta(z_S^\theta)$. Then, by (11), (67) and (70), $\left|y_S^\theta-x_S^\theta\right|\to 0$, as $\theta\to 0$. Thus, $y_L^\theta\le y_{t+2}$ implies $z_S^\theta<y_{t+2}$. Since $G'_\theta<-1$, this implies $G_\theta(y_{t+2})<z_S^\theta$. Since $G'_\theta<-1$, the graphs of $y_{t+3} = \bar{B}_\theta(y_{t+2})$ and $y_{t+3} = G_\theta(y_{t+2})$ intersects only once. These facts together with $y_L^\theta \leq y_{t+2} < R_\theta(y_D^\theta)$ implies that $G_\theta(y_{t+2}) = y_{t+3} < \bar{B}_\theta(y_{t+2})$. Thus, condition (18) is satisfied.

Moreover, $y_{t+4} = l_\theta(y_{t+3}) = l_\theta(G_\theta(y_{t+2})) = L_\theta(y_{t+2})$. Since, by Lemma 5, $L_\theta(y_L^\theta) > \bar{B}(G_\theta(y_L^\theta))$, and since $y_{t+4} \geq y_L^\theta$, $L_\theta(y) > \bar{B}(G_\theta(y))$. Thus, condition (19) is satisfied. ■

Condition (85) characterizes the roles of exogenous growth and intellectual property protection in chaotic industrial takeoffs for the case of sufficiently large elasticity of substitution. *Figure* 7 illustrates the region in which this condition is satisfied. That is, curve Φ_0, Φ_1, and Φ_2 illustrate the following:

$$\text{Curve } \Phi_0: \quad \alpha = \frac{1}{1-\phi(1-1/e)} - 1;$$

$$\text{Curve } \Phi_1: \quad \alpha = \sqrt{\frac{1}{1-\phi(1-1/e)} - 1};$$

Curve Φ_2:

$$\alpha = \sqrt{\frac{-\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^2 + \sqrt{\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^4 + 4\left(\frac{1}{1-\phi(1-\frac{1}{e})}-1\right)^3}}{2}}.$$

In the region between Φ_2 and Φ_1, by Lemmas 2 and 3, the slopes of R_θ and L_θ are both larger than 1 in absolute value. In this case, equilibrium system (53) is chaotic and involves explosive takeoffs.

In the region between Φ_1 and Φ_0, by Lemma 3, the slopes of L_θ is greater than 1. However, that of R_θ is smaller than 1 in absolute value. In this case, the fixed point of R_θ is a stable cyclical point of period 2.

Below Φ_0, the slopes of both L_θ and R_θ are smaller than 1 in absolute value. In this region, the first inequality of (93) does not hold, or, in other words, $L_\theta(y_L^\theta) < y_L^\theta$. In this case, the number of differentiated products will shrink to the fixed point of L_θ, if exists. There is a case in which L_θ does not have a fixed point. In that case, the number of products will shrink to zero.

As this shows, the stronger intellectual protection (ϕ) and the weaker exogenous growth factor (α), the more likely the equilibrium system exhibits sharp non-linearity. If intellectual properties are fully protected ($\phi = 1$), chaotic industrial takeoffs emerge if

$$\sqrt{\frac{-(e-1)^2+\sqrt{(e-1)^4+4(e-1)^3}}{2}} = 1.103 \; < \alpha < \; \sqrt{e-1} = 1.310 \;,$$

If the length of a single period is assumed to be 10 years, chaotic industrial takeoffs emerge if the annual trend growth rate is between 1 percent ($1.01^{10} \approx$

1.105) and 2.8 percent ($1.028^{10} \approx 1.318$). In that case, a sufficient protection of intellectual property is necessary for the emergence of massive industrial takeoffs in the process of long innovation waves.

5 Industrial Takeoffs

It may be said that the economy is in the no-innovation phase if $y_{t+1} = \frac{1}{\alpha} y_t$ whereas it is in the innovation phase if $y_{t+1} > \frac{1}{a} y_t$. Thus, if $y_{t+1} = \frac{1}{\alpha} y_t$ and $y_{t+2t} > \frac{1}{\alpha} y_{t+1}$, we may say that an industrial take-off occurs in the period between time $t+1$ and $t+2$.

In this section, we demonstrate that two distinct types of takeoffs emerge in the double-period chaotic dynamics. One type involves an explosive take-off followed by another take-off into the phase in which innovation activities fluctuate over several periods. The other type involves small scale takeoffs alternating the no-innovation phase and the innovation phase every other period.

In the case in which the equilibrium system, (53), is an ergodic chaos, by von-Neumann-Birkov's theorem, it is possible to determine that the probability with which each type of take-off may occur along our equilibrium path. As is noted above, the length of a single period in our model is assumed to be long enough for the monopolistic control over a newly invented product to last. If that length is about 10 years, our result shows that, on average, an explosive industrial take-off may occur once in about 130 years; a general take-off, in contrast, occurs very 50 to 60 years.

5.1 "Chaotic Drift"

In order to explain these types of takeoffs, it is desirable to describe what single period dynamics underlies the double period chaos. In this section, we explain that even if double period dynamics can be described by an autonomous dynamical system, single period dynamics obeys a "non-autonomous system of period 4." This period-4 system generates "chaotic drifts" of a state variable off the double-period dynamics.

In order to explain this, it is useful to investigate single-period dynamics underneath of the double-period system, $y_{2(\tau+1)} = T_\theta(y_{2\tau})$. For this purpose, define the following:

$$r_\theta(y) = -\frac{1}{\alpha}\left(\frac{1}{\phi\beta\theta\eta + \xi} - 1\right) y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta + \xi}; \tag{86}$$

$$l_\theta(y) = \frac{1}{\alpha}y + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi\left(\alpha y - G_\theta^{-1}(y)\right)+G_\theta^{-1}(y)}{\zeta\left(\alpha y - G_\theta^{-1}(y)\right)+G_\theta^{-1}(y)} - y}{\phi\beta\theta\eta\frac{\xi\left(\alpha y - G_\theta^{-1}(y)\right)+G_\theta^{-1}(y)}{\zeta\left(\alpha y - G_\theta^{-1}(y)\right)+G_\theta^{-1}(y)} + \xi}. \tag{87}$$

The next lemma holds:

Lemma 6 *Let path* (x_t, y_t) *solve* (53)*. If* $y_{t+1} = A(y_t)$*, the following holds:*

$$(1) \quad \textit{If } y_{t+1} \leq \bar{B}_\theta(y_t),\ y_{t+2} = r_\theta(y_{t+1}). \tag{88}$$

$$(2) \quad \textit{If } y_{t+2} \leq \bar{B}_\theta(y_{t+1}),\ y_{t+3} = G_\theta(y_{t+2}). \tag{89}$$

$$(3) \quad \textit{If } y_{t+3} \leq \bar{B}_\theta(y_{t+2}),\ y_{t+4} = l_\theta(y_{t+3}). \tag{90}$$

Proof. Statement (89) follows from Theorem 2. If $y_{t+1} \leq \bar{B}_\theta(y_t)$, by Theorem 2, $y_{t+2} = R_\theta(y_t) = R_\theta(\alpha y_{t+1}) = r_\theta(y_{t+1})$, which follows from (66) and (86). This shows (88); (90) may be proved in a similar manner. ■

This lemma implies that, given $y_{t+1} = A(y_t)$ along an equilibrium path $y_{t+2} = R_\theta(y_t)$ and $y_{t+4} = L_\theta(y_t)$ if and only if $y_{t+2} = r_\theta(y_t)$, $y_{t+3} = G_\theta(y_{t+2})$, and $y_{t+4} = l_\theta(y_{t+3})$. An important fact is that, as in L_θ and G_θ, the steady states of r_θ and l_θ are different not only from each other but also from the steady state, y_S^θ. Their distance from the steady state, y_S^θ, converges to zero as $\theta \to 0$; moreover, the graphs of r_θ, l_θ and G_θ become identical in the limit.

The two patterns of takeoffs emerge along a single equilibrium path. Which pattern is realized depends on whereabouts of the state variable, y_t. See *Figure* 8. Note that y_C^θ is at the intersection between curves L and R, that $y_{\max}^\theta$ is the level of the peak, i.e., $y_{\max}^\theta = R_\theta(y_C^\theta)$, that y_L^θ is the minimum value for the state variable, $y_{2\tau}$; i.e., $y_L^\theta = R_\theta(y_{\max}^\theta)$. Let $y_D^\theta = R_\theta^{-1}(R_\theta^{-1}(y_C^\theta))$. The first pattern, involving minor takeoffs that occur every other period, is observed when $y_D^\theta \leq y_t \leq y_{\max}^\theta$, given $y_{t+1} = \frac{1}{\alpha}y_t$. In this case, (y_t, y_{t+1}) is in the no-innovation phase (i.e., segment A), which will move to (y_{t+1}, y_{t+2}) in the innovation phase on line r_θ; i.e., $y_{t+2} = r_\theta(y_{t+1}) > \frac{1}{\alpha}y_{t+1}$. Since (y_{t+1}, y_{t+2}) lies above curve $\bar{B}$, by Lemma 6, (y_{t+2}, y_{t+3}) will move back to the no-innovation phase on segment A; i.e., $y_{t+3} = \frac{1}{\alpha}y_{t+2}$. This implies that in double-period dynamics, (y_t, y_{t+2}) lies on segment R.

The two-period process of $y_{t+1} = \frac{1}{\alpha}y_t$ and $y_{t+2} = r_\theta(y_{t+1})$, followed by $y_{t+3} = \frac{1}{\alpha}y_{t+2}$ and $y_{t+4} = r_\theta(y_{t+3}), \ldots$, involves short innovation phases, in which innovation activities last only one period. The closer y_t lies to the fixed point of segment R, the longer this pattern of dynamics lasts. Within a finite number of periods, however, dynamics falls into the second pattern.

The second pattern, involving an explosive take-off, occurs when $y_C^\theta \leq y_t \leq y_E^\theta$, given $y_{t+1} = \frac{1}{\alpha} y_t$. In this case, (y_t, y_{t+1}) in the no-innovation phase will move into the innovation phase at (y_{t+1}, y_{t+2}) on line r_θ; i.e., $y_{t+2} = r_\theta(y_{t+1})$. Since (y_{t+1}, y_{t+2}) lies below curve $\bar{B}$, $(y_{t+2,} y_{t+3})$ will move to a point on curve G in the innovation phase; i.e., $y_{t+3} = G_\theta(y_{t+2}) > \frac{1}{\alpha} y_{t+2}$. Since (y_{t+2}, y_{t+3}) lies below curve $\bar{B}$, again, $(y_{t+3,} y_{t+4})$ will move to a point on curve l in the innovation phase; i.e., $y_{t+4} = l_\theta(y_{t+3}) > \frac{1}{\alpha} y_{t+3}$. Since (y_{t+3}, y_{t+4}) lies above curve $\bar{B}$, (y_{t+4}, y_{t+5}) will move back to the no-innovation phase on segment A; i.e., $y_{t+5} = \frac{1}{\alpha} y_{t+4}$. The four period process of $y_{t+1} = \frac{1}{\alpha} y_t$, $y_{t+2} = r_\theta(y_{t+1})$, $y_{t+3} = G_\theta(y_{t+2})$, and $y_{t+4} = l_\theta(y_{t+3})$ constitutes a long innovation phase in our sense, in which innovation occurs in the three periods, between time 1 and 2, time 2 and 3 and time 3 and 4.

The next theorem characterizes the phase of explosive takeoffs.

Theorem 5 *Suppose that $(\bar{x}, \bar{y})$ satisfy $y_C^\theta < \bar{y} < y_{\max}^\theta$ and $\bar{y} > \bar{B}(\bar{x})$ and that condition (85) is satisfied. Then,*

$$y_C^\theta = \frac{\bar{E}\beta\eta}{\kappa} \frac{1 - \sqrt{1 - 4\frac{(1-\xi)\alpha(\phi\beta\theta\eta+\xi-\zeta)}{(\alpha\phi\beta\theta\eta+1-\zeta)^2}}}{2\frac{\alpha(\phi\beta\theta\eta+\xi-\zeta)}{\alpha\phi\beta\theta\eta+1-\zeta}} \tag{91}$$

An explosive take-off occurs in the period between time $t+1$ and $t+2$ if in the prior period, the state variable satisfies $y_{t+1} = \frac{1}{\alpha} y_t$ and

$$y_C^\theta \leq y_t \leq y_D^\theta \tag{92}$$

where $y_D^\theta = R_\theta^{-1}(R_\theta^{-1}(y_C^\theta))$. A normal take-off, alternating between the no-innovation phase and the innovation phase every other period more than once occurs if the state variable satisfies $y_{t+1} = \frac{1}{\alpha} y_t$ and

$$y_D^\theta \leq y_t \leq y_E^\theta$$

where $y_D^\theta = R_\theta^{-1}(y_C^\theta)$.

Proof. Since $y = \bar{B}_\theta(\frac{1}{\alpha} x)$ is asymptotic to the horizontal line as $x \to \infty$, $R_\theta(x) = \bar{B}_\theta(x)$ has two real solutions, one of which is negative and the other positive; let y^θ be the positive solution. Since $y = \bar{B}_\theta(\frac{1}{\alpha} x)$ is convex towards below, for any $x < y^\theta$, $R_\theta(y) > \bar{B}_\theta(y)$. This implies that if $y_t < y^\theta$ and $y_{t+1} = \frac{1}{\alpha} y_t$, $y_{t+2} = R_\theta(y_t)$ and $y_{t+4} = R_\theta(y_{t+2})$. If $y_t > y^\theta$, $y_{t+2} = R_\theta(y_t)$ and $y_{t+4} \neq R_\theta(y_{t+2})$, in which case, by Theorem 2, $y_{t+4} = L_\theta(y_{t+2})$. Thus, if $L_\theta(y_{t+2}) = R_\theta(y_{t+2})$, $y_t = y^\theta$. Since $R_\theta(y_t) = y_{t+2}$, this implies $y_{t+2} = y_C^\theta$. By construction, if $y_C^\theta \le y_t \le y_E^\theta$, $y_{\max}^\theta \ge y_{t+1} \ge y^\theta$. This implies $y_{t+2} < \bar{B}_\theta(y_{t+1})$. Thus, by Lemma 6, $y_{t+2} = r_\theta(y_{t+1})$, $y_{t+3} = G_\theta(y_{t+2})$, and $y_{t+4} = l_\theta(y_{t+3})$.

As this shows, $y_{t+1} = y_C^\theta$ is the smaller positive solution to the following equations.

$$y_{t+2} = -\frac{1}{\alpha^2}\left(\frac{1}{\phi\beta\theta\eta+\xi} - 1\right) y_t + \frac{1}{\alpha}\frac{\frac{\bar{E}\beta\eta}{\kappa}}{\phi\beta\theta\eta+\xi};$$

$$\frac{\bar{E}\beta\eta}{\kappa}\frac{\xi\left(\alpha y_{t+2} - \frac{1}{\alpha}y_t\right) + \frac{1}{\alpha}y_t}{\zeta\left(\alpha y_{t+2} - \frac{1}{\alpha}y_t\right) + \frac{1}{\alpha}y_t} - y_{t+2} = 0.$$

The solution is given by (91). ■

5.2 Likelihood of Explosive Industrial Takeoffs

The discount factor, β, and the trend growth rate, α, reflect the length of a single period in the model. For example, suppose that the annual discount factor of future utilities and the annual trend growth rate are $\beta = 0.95$ and 1.6 percent, respectively. (The 1.6 percent rate is chosen because, according to Maddison (2010), the total Western European per capita GDP grew at about 1.6 percent from 1820 through 2008.[6]) If, at the same time, the length of a single period is 10 years, we have $\alpha = 1.016^{10} \approx 1.17$ and $\beta = 0.95^{10} \approx 0.6$.

The parameter values (3) for system (2) are chosen with these considerations. That is, those values can be translated into

$$(\phi, \alpha, \beta, \theta, \bar{E}/\kappa) \approx (1, 1.17, 0.6, 0.05, 10). \tag{93}$$

The values for L', R', y_L, y_C, $y_{\max}$, y_H in (21) and (22) are calculated with these parameter values.

As is noted at the outset, an ergodic chaos can be associated with a probability distribution that is independent of initial conditions and describes the relative frequency with which the state variable solving the system, y_t, $t = 0, 1, 2, ...$, falls in each interval (or a Borel set). While this probability distribution is generally intractable analytically, it may be characterized by plotting a solution to the system.

Figure 9 shows the histogram for a solution to system (2) with parameter values (3) with 10000 iterations. As it shows, a solution never reaches the area around the fixed point of R_θ; this is because, as *Figure* 8 shows, curve L at y_L starts above the fixed point on line R_θ.

[6] According to various empirical studies, the average long-run growth rate of per capta GDP is about 2 percent over the last one hundred years (Kaldor (1961), Jones and Romer (2010), and Maddison (2010)).

The relative frequency with which a solution falls in the interval of explosive industrial takeoffs (i.e., $y_t \in [y_C^\theta, y_D^\theta] = [2.276, 2.286]$) is about 16 percent or, more precisely, 1592 times out of 10000 iterations. In contrast, the relative frequency with which a solution falls in the interval of normal industrial takeoffs, alternating the no-innovation phase and innovation phase every other period more than once (i.e., $y_t \in [y_D^\theta, y_{\max}^\theta] = [2.286, 2.364]$) is about 58 percent or, more precisely, 6787 times out of 10000 iterations. In the other period, the economy is in the middle of mild fluctuations with positive inventions.

Recall that we are looking at values of state variables over two periods, $y_{2\tau}$, $\tau = 0, 1, 2, ...$ Thus, if the length of a single period in the model is assumed to be 10 years, the probability with which an explosive take-off occurs in the latter half of every twenty year period is 16 percent. In other words, an explosive industrial take-off occurs once about 130 years. In contrast, industrial takeoffs occur one in approximately 30 years on average.

Proposition 2 *Explosive industrial takeoffs take places in an equilibrium process in which the innovation phase and the no-innovation phase alternate slowly but chaotically over time. This process of chaotic long waves does not occur either if intellectual properties are not protected sufficiently (represented by ϕ) or if the speed of accumulation of worker skills through leaning-by-doing (captured by α) is too fast.*

6 Time in Innovation and Intellectual Properties

This study incorporates time in innovation and intellectual property protection into a standard model of innovation cycles. As is discussed in Section 3.4, there are two channels through which time in innovation affects dynamics. The first is the determination of an interest rate, governed by parameter β. The second is the endogenous determination market structure, which emerges through term $\mu_t = \frac{\xi Z_t + N_t}{\zeta Z_t + N_t}$ in (45). Since $\xi = 1 - \phi(1 - (1 - \theta)^{1/\theta - 1})$ and $\zeta = 1 - \phi(1 - (1 - \theta)^{1/\theta})$ by (47) and (48), market structure μ_t is affected by the elasticity of substitution, $1/\theta$, and the rate of intellectual property protection, ϕ.

In what follows, we will examine how these parameters, ϕ, β, and θ, interact with one another to bring about chaotic innovation dynamics. For this purpose, we need to extend Theorem 4 for the general case of θ; Theorem 4 is concerned with the case in which θ is sufficiently small. Although, as

Lemma 2 shows, the derivative of L_θ is analytically intractable, we may derive a necessary condition for chaotic dynamics, which is that the peak of tent map $T_\theta = \min\{L_\theta, R_\theta\}$ lies above the 45 degree line.

In order to capture the role of market structure, it is necessary to derive an analytical condition for chaotic dynamics in our model. Because of the complexity of our model, a complete characterization is intractable. For this reason, we derive a necessary condition for chaotic dynamics. As is shown below, this suffices to capture the role of time in innovation. For this purpose, defined

$$\Phi(\alpha,\beta,\theta) = \frac{\frac{\alpha^2}{\alpha^2-1}\left(1-\frac{2\frac{\beta\theta(1-\theta))^{\frac{1}{\theta}-1}+(1-\theta)^{1/\theta-1}-\left((1-\theta)^{1/\theta}\right)}{\alpha\beta\theta(1-\theta))^{\frac{1}{\theta}-1}+1-(1-\theta)^{1/\theta}}}{1-\sqrt{1-4\frac{\left(1-(1-\theta)^{\frac{1}{\theta}-1}\right)\left(\beta\theta(1-\theta))^{\frac{1}{\theta}-1}+(1-\theta)^{1/\theta-1}-(1-\theta)^{1/\theta}\right)}{\left(\alpha\beta\theta(1-\theta))^{\frac{1}{\theta}-1}+1-(1-\theta)^{1/\theta}\right)^2}\alpha}}\right)}{1-(1+\beta\theta)(1-\theta)^{\frac{1}{\theta}-1}} \tag{94}$$

The next theorem characterizes this condition.

Theorem 6 *The peak of T_θ lies above the 45 degree line, i.e., $L_C^\theta(y_C^\theta) > y_C^\theta$, if and only if*

$$\phi > \Phi(\alpha,\beta,\theta). \tag{95}$$

Proof. Since y_C^θ is given by (91), and since $y_{\max}^\theta = R_\theta(y_C^\theta)$, by (68), $L_\theta(y_C^\theta) > y_C^\theta$ holds if and only if

$$\frac{1}{\phi\beta\theta\eta+\xi} > \frac{1}{\alpha}\left(\frac{1}{\phi\beta\theta\eta+\xi}+\alpha^2-1\right)\frac{1-\sqrt{1-4\frac{(1-\xi)\alpha(\phi\beta\theta\eta+\xi-\zeta)}{(\alpha\phi\beta\theta\eta+1-\zeta)^2}}}{2\frac{\alpha(\phi\beta\theta\eta+\xi-\zeta)}{\alpha\phi\beta\theta\eta+1-\zeta}}.$$

Although the last fraction appears to depend on ϕ, it is not. By using this fact, this inequality can be transformed into the first part of max in (95). The second part is equivalent to $R'_\theta < -1$. ■

Another important condition for chaotic T_θ is that the slope of R_θ is steeper than 1. By Lemma 2, this condition is given by

$$\phi > \frac{1-\frac{1}{\alpha^2+1}}{1-(\beta\theta+1)(1-\theta)^{\frac{1}{\theta}-1}}. \tag{96}$$

Figure 10 shows, for the case of $\alpha = 1.17$ and $\beta = 0.6$, the region of (θ, ϕ) in which conditions (95) and (96) are satisfied. See the shaded region. Condition (95) holds above curve Φ whereas (96) holds above curve Φ_R.

Our calculations show that at most points in the shadowed region, the equilibrium system, T_θ, is ergodic chaos. That is, on curve Φ_R to the left of curve Φ, the slope of the decreasing side of tent map T_θ is -1 whereas that of the increasing side is greater than 1. It is easy to check that conditions (18) and (19) are satisfied and that $L'_\theta > 1$. This implies that, by continuity, at a point slightly above curve Φ_R, T_θ is chaotic. It is difficult to determine the slope of L_θ (the increasing side of T_θ) at a point on curve Φ to the left of curve Φ_R, since L_θ is non linear. Although our calculation shows that L'_θ is almost equal to 1, it does not guarantee $L'_\theta > 1$. However, at points sufficiently above curve Φ, it may be checked that T_θ is ergodic chaos.

For example, if $\theta = 0.112027$, $\Phi = 1$ approximately, given $\alpha = 1.17$ and $\beta = 0.6$. In that case, $R'_\theta < -1$ and $y^\theta_C = L_\theta(y^\theta_C)$. However, L'_θ could be larger or smaller than 1. If however, ϕ is given a slightly smaller value, $\theta = 0.11$, then, $L'_\theta > 1$. In that case, given $\bar{E}/\kappa = 10$,

$$R'_\theta \approx -1.02884, \; 1.063450 < L'_\theta < 1.063452. \tag{97}$$

In the case of these parameter values, we have

$$(y^\theta_L, y^\theta_C, y^\theta_D, y^\theta_E, y^\theta_{\max}) \approx (2.36736, 2.36758, 2.36778, 2.37465, 237485. \tag{98}$$

In this case,

$$y^\theta_L \approx 2.36737 < \bar{B}_\theta(y^\theta_L) \approx 2.36866 < L_\theta(y^\theta_L) \approx 2.37463. \tag{99}$$

Similarly, it may be checked that T_θ is chaotic if $(\theta, \phi) = (0.10, 0.99), (0.09, 0.98), (0.85, 0.97)$, which are just off curve Φ. It is chaotic also if $(\theta, \phi) = (0.07, 0.96), (0.057, 0, 95)$, $(0.04, 0.94)$, $(0.026, 0.93)$, $(0.009, 0.02)$, which line up along curve Φ_R. Theorem 4 implies that if $\theta > 0$ is sufficiently close to zero, T_θ is chaotic if ϕ is between 0.915 and 1. These results suggest that it may be safe to conclude that at almost every point in the shaded region in Figure 10, T_θ is chaotic.

Figure 11 shows, for the case of $\alpha = 1.17$ and $\theta = 0.05$, the region of (β, ϕ) in which conditions (95) and (96) are satisfied. See the shaded region. Condition (95) holds above curve Φ whereas (96) holds above curve Φ_R. As this shows, in this case, (96) is always satisfied so long as (95) is satisfied.

In order to see the effect of time in innovation, we compare the above results with those in the case of the atemporal demand function, (50). If no time is involved in innovation, by $N_{t+1} - N_t = Z_t$, $\omega_t = \kappa$ for the interior solution, and (50), our model, (53), boils down to

$$n_{t+1} = \max\left\{\frac{1}{\alpha}\frac{\phi\theta\eta\frac{\bar{E}}{\kappa} - \phi\left(1-(1-\theta)^{\frac{1}{\theta}-1}\right)n_t}{1-\phi(1-(1-\theta)^{\frac{1}{\theta}-1})}, \frac{1}{\alpha}n_t\right\}, \tag{100}$$

where $n_t = N_t/\alpha^t$. The first term on the right-hand side of this system is much the same as equation (31) of Judd (1985).[7]

The necessary and sufficient condition for this system (100) to be chaotic is

$$\phi > \frac{1-\frac{1}{\alpha^2+1}}{1-(1-\theta)^{\frac{1}{\theta}-1}}. \tag{101}$$

In Figure 10, the region in which this condition is satisfied is indicated by the region above the dotted curve, J. This demonstrates that time in innovation has a stabilizing effect on innovation dynamics by narrowing down the region in which chaotic dynamic emerges. A similar fact is captured in Figure 11; that is, the discounting of future utilities does not affect condition (101).

6.1 Summary

In addition to introduce a new characterization for two-dimensional chaos, this study obtains several interesting findings on innovation dynamics.

1. Explosive takeoffs from the no-innovation phase and milder long innovation wages may occur along a chaotic equilibrium path. The frequency with which explosive takeoffs occur may be once in more than one hundred years; if takeoffs of smaller scales are included, they occur once in fifty years or so.

2. Intellectual property protection is a necessary condition for explosive takeoffs from the no-innovation phase and mider long innovation waves.

3. Time in innovation tends to stabilize this dynamics, although the possibility of chaotic takeoffs from cycles persists.

4. This stabilizing effect is attributable to an endogenous change in market structure the effect of which time in innovation brings about.

7 Concluding Remarks

In this study, we have incorporated time needed for R&D investments to materialize into a standard model of atemporal innovation. In that model,

[7]Deneckere and Judd (1992) deal with a similar model, in which the exogenous growth factor is replaced by the depreciation of innovation.

market structure and innovation interact with each other, thereby resulting in a chaotic process of recurrent industrial takeoffs from the no-innovation phase. In this process, what we call explosive and normal takeoffs alternate. We demonstrate that those chaotic industrial takeoffs may be explained by three factors: intellectual property protection, the monopolistic profit accruing to R&D, and an exogenous productivity increase through learning by doing. If no innovation takes place for a long span of time, the exogenous growth factor raises the productivity of workers, which makes R&D more attractive for the innovation industry. If the market become crowded by too many new products, the monopolistic profits reduces, which stops R&D activities. This process of intermittent takeoffs is intensified by intellectual property protection. Unless intellectual properties are protected sufficiently, chaotic industrial takeoffs do not recur.

This provides a theoretical underpinning for Yano's hypothesis that the recurrence of industrial revolutions since the 18th century may be a result of the endogenous interaction between innovation and market quality (see Yano, 2009). Because our model is highly stream-lined, it does not fully capture the rich aspect of an industrial revolution and the broad feature of market quality, which Yano (2009) explains.[8] In our stream-line model, however, the relative size of the competitive sector to the monopoly sector may be thought of as market quality. The higher market quality, the larger the monopolistic profit out of an R&D activity, which implies the larger incentive for R&D investments. In contrast, the lower market quality, the smaller incentive for innovation. This will in turn shrink the monopolistic sector, thereby raising market quality, as the labor productivity rises exogenously.

Yano (2009) argues that, in the real world, a fall in market quality may be associated with various undesirable activities such as misappropriation of market power and abuse of informational advantage, which were observed during the period leading to the Great Depression. If that is the case, a deep decline of market quality would be undesirable. Our result, in contrast, shows that the stronger intellectual property protection, the larger these waves of innovation and market quality. This could be another reason to support careful application of intellectual property protection as argued by Boldrin and Levine (2008).

This study shows the use of our two-dimensional chaotic dynamics in a system with a constrained domain. As is shown above, this method is

[8] Yano (2009) shows that market quality is supported by what he calls market infrastructure or the entire network of social arrangements in which a market functions (Furukawa and Yano 2014). The literature has examined the role of market infrastructure from various aspects; for recent studies, see Akiyama, Furukawa, and Yano (2011), Dastidar (2017), Dastidar and Yano (2017), and Furukawa, Lai, and Sato (2018).

useful not just for showing the possibility of chaotic dynamics but also for explaining the structure behind a chaotic phenomenon. The method may be extended for the case of multi-dimensional state variables with a constrained domain, in which rich applications of our method may be obtained.

References

[1] Acemoglu, D., S. Johnson, and J. Robinson, 2005. "Institutions as a Fundamental Cause of Economic Growth," In *Handbook of Economic Growth*, ed. Philippe Aghion and Steven Durlauf, 385–465. Amsterdam: Elsevier.

[2] Aghion, P., and P. Howitt, 1992. "A Model of Growth through Creative Destruction," *Econometrica* 60, 323–351.

[3] Akiyama, T., Y. Furukawa, and M. Yano, 2011. "Private Defense of Intellectual Properties and Economic Growth," *International Journal of Development and Conflict* 1, 355–364.

[4] Baierl, G., K. Nishimura, and M. Yano, 1998. "The Role of Capital Depreciation in Multi-Sector Models," *Journal of Economic Behavior and Organization* 33, 467–479.

[5] Benhabib, J., and R. Day, 1980. "Erratic Accumulation," *Economic Letters* 6, 113–117.

[6] Benhabib, J., and K. Nishimura, 1979. "The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimum Economic Growth," *Journal of Economic Theory* 21, 421–444.

[7] Benhabib, J., and K. Nishimura, 1985. "Competitive Equilibrium Cycles," *Journal of Economic Theory* 35, 284–306.

[8] Bhattacharya, R., and M. Majumdar, 2007. *Random Dynamical Systems: Theory and Applications*, Cambridge University Press, Cambridge.

[9] Birkhoff, George David, 1931. "Proof of the Ergodic Theorem," *Proceedings of the National Academy of Sciences of the United States of America* 17, 656–660.

[10] Boldrin, M., and L. Montrucchio, 1986. "On the Indeterminacy of Capital Accumulation Paths," *Journal of Economic Theory* 40, 26–39.

[11] Boldrin, M., and D. K. Levine, 2008. *Against Intellectual Monopoly,* Cambridge: Cambridge University Press.

[12] Dastidar, K., 2017, *Oligopoly, Auctions and Market Quality*, Springer.

[13] Dastidar, K., and M. Yano, 2017. "Corruption, Market Quality and Entry Deterrence in Emerging Economies," *RIETI Discussion Paper Series* 17-E-010.

[14] Deneckere, R., and K. Judd, 1992. "Cyclical and Chaotic Behavior in a Dynamic Equilibrium Model," in *Cycles and Chaos in Economic Equilibrium*, ed. by J. Benhabib. Princeton: Princeton University Press.

[15] Dixit, A., and J. Stiglitz, 1977. "Monopolistic Competition and Optimum Product Diversity," *American Economic Review* 67, 297–308.

[16] Ethier, W., 1982. "National and International Returns to Scale in the Modern Theory of International Trade," *American Economic Review* 72, 389–405.

[17] Furukawa, Y., and M. Yano, 2014. "Market Quality and Market Infrastructure in the South and Technology Diffusion," *International Journal of Economic Theory* 10, 139–146.

[18] Furukawa, Y., T.-K., Lai, and K. Sato, 2018. "Novelty-Seeking Traits and Innovation," *RIETI Discussion Paper Series* 18-E-073.

[19] Gale, D., 1996. "Delay and Cycles," *Review of Economic Studies* 63, 169-198.

[20] Grandmont, J.-M., 1985. "On Endogenous Competitive Business Cycles," *Econometrica* 53, 1985, 995–1045.

[21] Grandmont, J.-M., 1986. "Periodic and Aperiodic Behavior in Discrete One-Dimensional Dynamical Systems," in *Contributions to Mathematical Economics: In Honor of Gerard Debreu*, ed. by W. Hildenbrand and A. Mas-Colell. Amsterdam: North-Holland, 44–63.

[22] Grandmont, J.-M., 2008. "Nonlinear Difference Equations, Bifurcations and Chaos: An Introduction," *Research in Economics* 62, 122–177.

[23] Grossman, G., and E. Helpman, 1991. "Quality Ladders in the Theory of Growth, *Review of Economic Studies* 58, 43–61.

[24] Helpman, E., 1993. "Innovation, Imitation, and Intellectual Property Rights," *Econometrica* 61, 1247–1280.

[25] Helpman, E. 2008. *Institutions and Economic Performance*, Cambridge Mass. Harvard University Press.

[26] Jones, C. I., and P. M. Romer, "The New Kaldor Facts: Ideas, Institutions, Population, and Human Capital," *American Economic Journal*: Macroeconomics, January 2010, 2 (1), 224–245.

[27] Kaldor, Nicholas. 1961. "Capital Accumulation and Economic Growth." In The Theory of Capital, ed. F. A. Lutz and D. C. Hague, 177–222. New York: St. Martins Press.

[28] Judd, K., 1985. "On the Performance of Patents," *Econometrica* 53, 567–586.

[29] Khan, A., and T. Mitra, 2005. "On Choice of Technique in the Robinson-Solow-Srinivasan Model," *International Journal of Economic Theory* 1, 83–110.

[30] Khan, A., and T. Mitra, 2012. "Long-run Optimal Behavior in a Two-sector Robinson-Solow-Srivanisan Model," *Macroeconomic Dynamics* 16, 70–85.

[31] Kondratieff, N., 1926. "Die langen Wellen der Konjunktur," *Archiv fur Sozialwissenschajt und Sozialpolitik* 56, 573–609. English translation by W. Stolper, 1935. "The Long Waves in Economic Life," *Review of Economics and Statistics* 17, 105–115.

[32] Lasota, A., and A. Yorke, 1973. "On the Existence of Invariant Measures for Piecewise Monotonic Transformations," *Transactions of the American Mathematical Society* 186, 481–488.

[33] Maddison, A., 2010, "Statistics on World Population, GDP and Per Capita GDP, 1-2008 AD," last version, downloaded on January 19, 2019, at http://www.ggdc.net/MADDISON/oriindex.htm, University of Groningen.

[34] Matsuyama, K., 1999. "Growing through Cycles," *Econometrica* 67, 335–347.

[35] Matsuyama, K., 2001. "Growing through Cycles in an Infinitely Lived Agent Economy," *Journal of Economic Theory* 100, 220–234.

[36] Mitra, T., 1996. "An Exact Discount Factor Restriction for Period-three cycles in dynamic optimization models," *Journal of Economic Theory* 69, 281–305.

[37] Mitra, T., and G. Sorger, 1999. "Rationalizing Policy Functions by Dynamic Optimization," *Econometrica* 67, 375–392.

[38] Mokyr, J., 2009. "Intellectual Property Rights, the Industrial Revolution, and the Beginnings of Modern Economic Growth," *American Economic Review: Papers & Proceedings* 99, 349–355.

[39] Nishimura, K., G. Sorger, and M. Yano, 1994. "Ergodic Chaos in Optimal Growth Models with Low Discount Rates," *Economic Theory* 4, 705–717.

[40] Nishimura, K., and M. Yano, 1994. "Optimal Chaos, Nonlinearity and Feasibility Conditions," *Economic Theory* 4, 689–704.

[41] Nishimura, K., and M. Yano, 1995a. "Nonlinear Dynamics and Chaos in Optimal Growth: An Example," *Econometrica* 63, 981–1001.

[42] Nishimura, K., and M. Yano, 1995b. "Durable Capital and Chaos in Competitive Business Cycles," *Journal of Economic Behavior and Organization* 27, 165–181.

[43] Nishimura, K., and M. Yano, 1996. "On the Least Upper Bound of Discount Factors that are Compatible with Optimal Period-three Cycles," *Journal of Economic Theory* 69, 306–333.

[44] North, D., 1981. *Structure and Change in Economic History*, New York: Norton.

[45] North, D., 1990. *Institutions, Institutional Change and Economic Performance*, Cambridge: Cambridge University Press.

[46] Romer, P., 1990. "Endogenous Technological Progress," *Journal of Political Economy* 98, s71–s102.

[47] von Neumann, J., 1932. "Physical Applications of the Ergodic Hypothesis," *Proceedings of the National Academy of Sciences of the United States of America* 18, 263–266.

[48] Shleifer, A., 1986. "Implementation Cycles," *Journal of Political Economy* 94, 1163-1190.

[49] Yano, M., 1998. "On the Dual Stability of a Von Neumann Facet and the Inefficacy of Temporary Fiscal Policy," *Econometrica*, 427–451.

[50] Yano, M., 2009. "The Foundation of Market Quality Economics," *Japanese Economic Review* 60, 1–32.

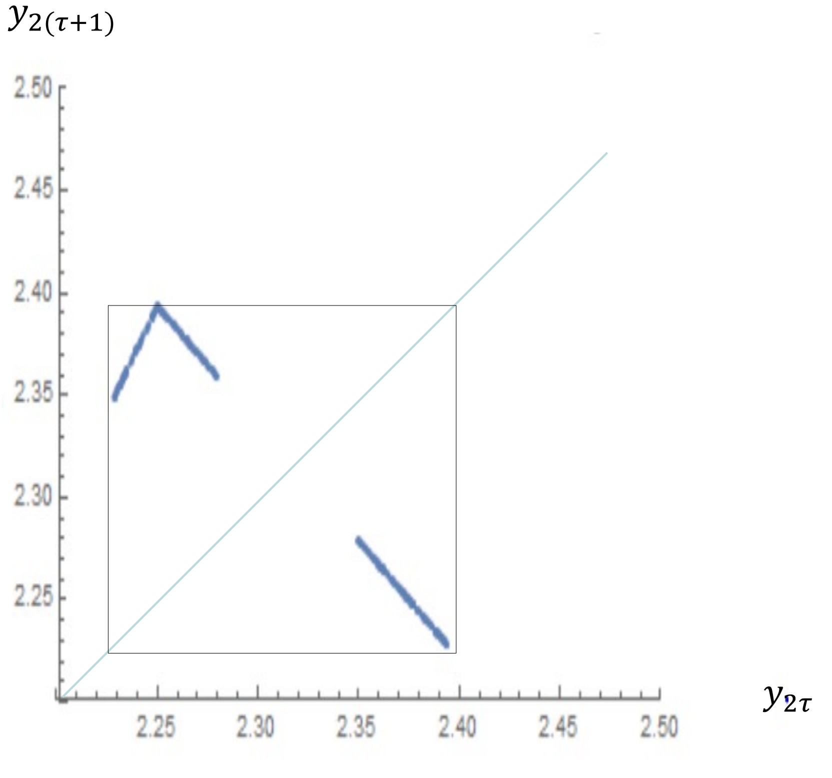

Figure 1: Chaos in Innovation Dynamics

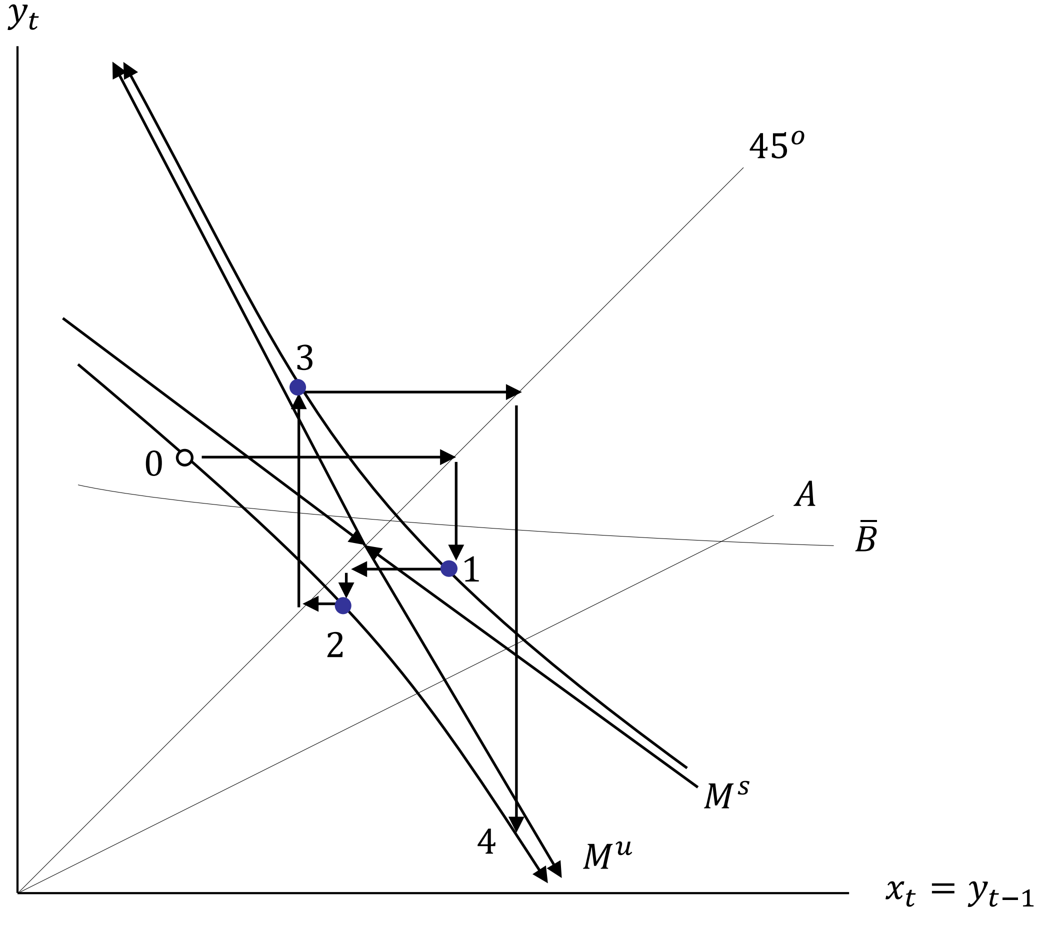

Figure 2: Unconstrained Two-Dimensional Dynamics: Example

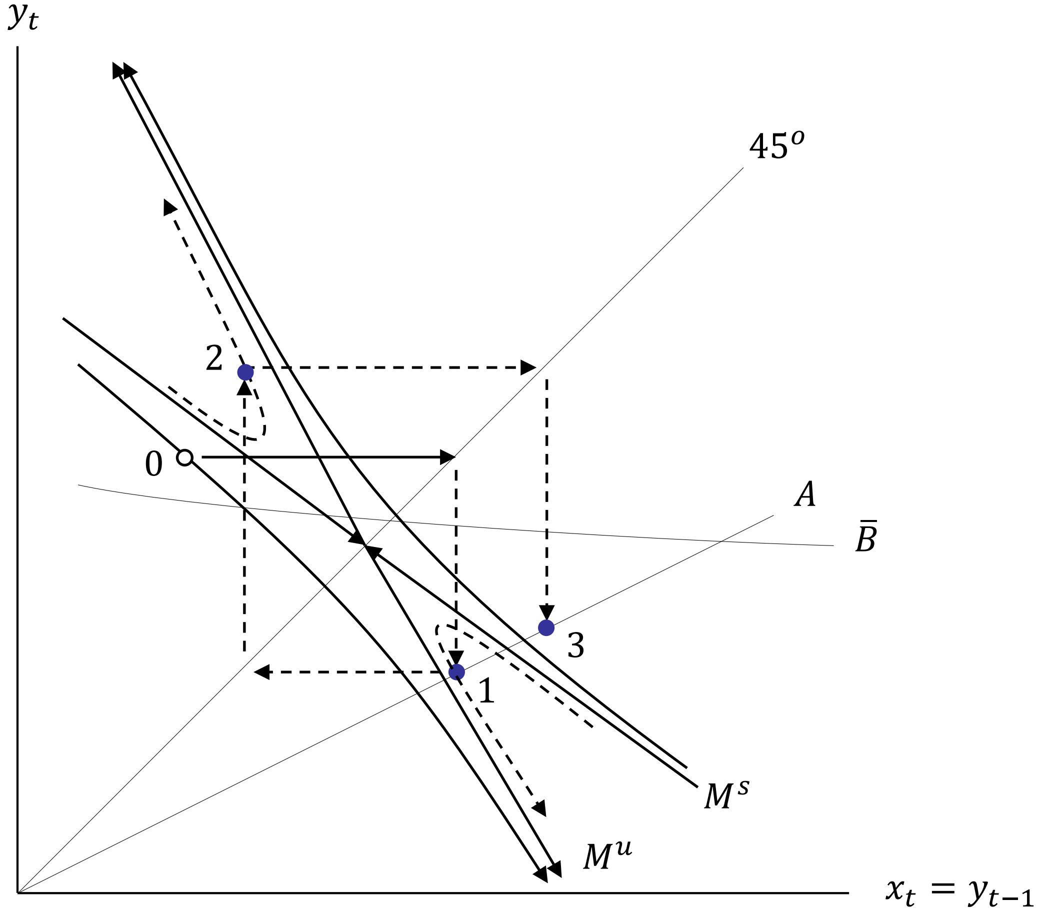

Figure 3: Constrained Two-Dimensional Dynamics: Example

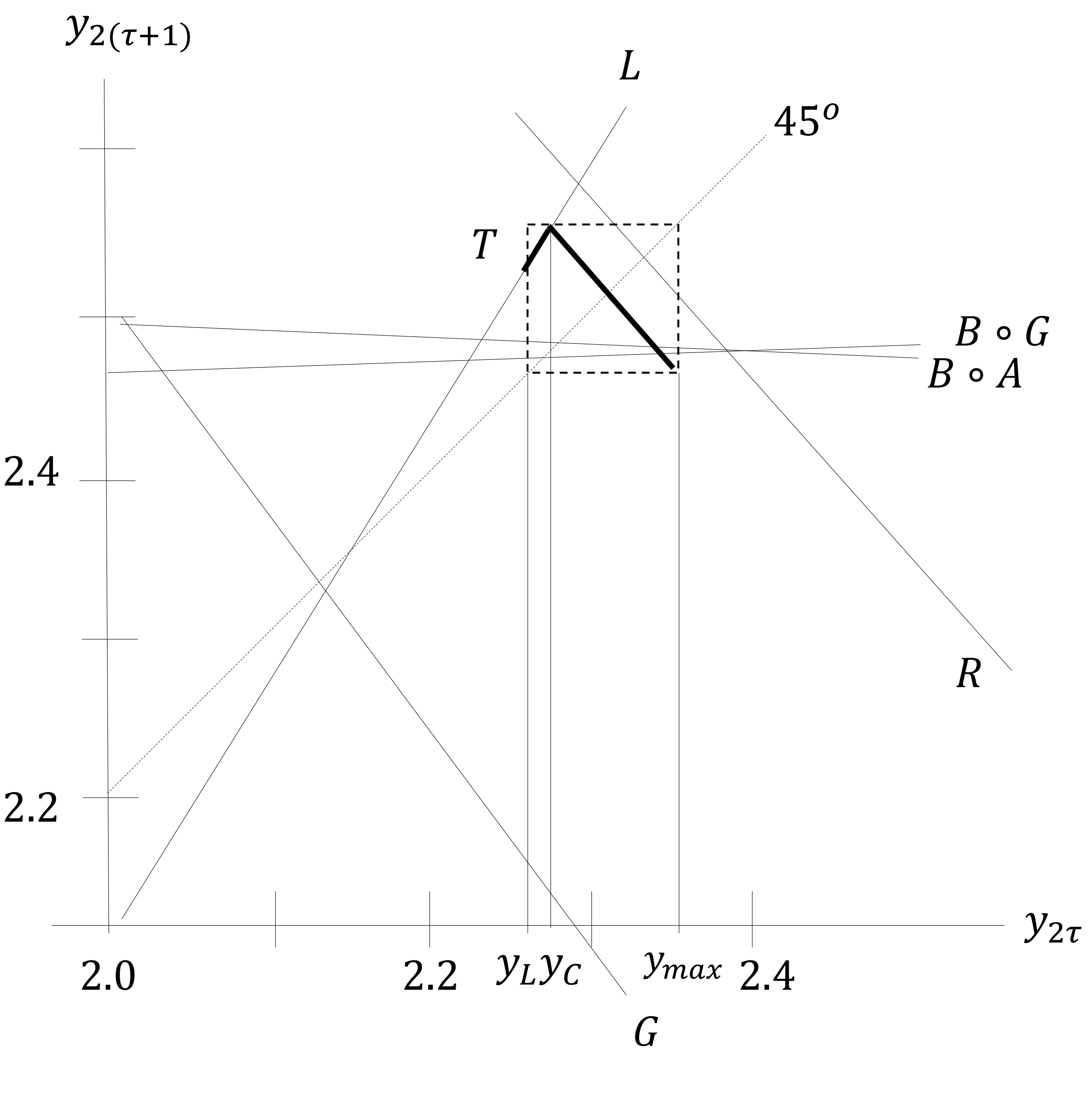

Figure 4: Two-Dimensional Constrained Chaos

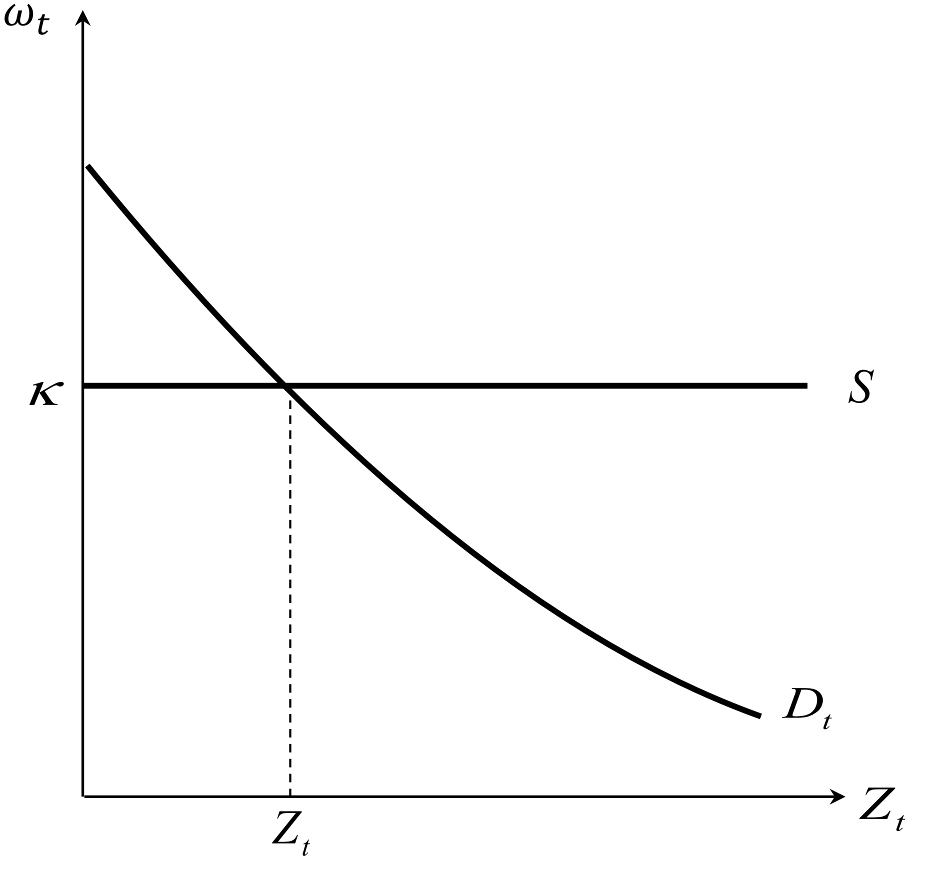

Figure 5: Demand for Inventions

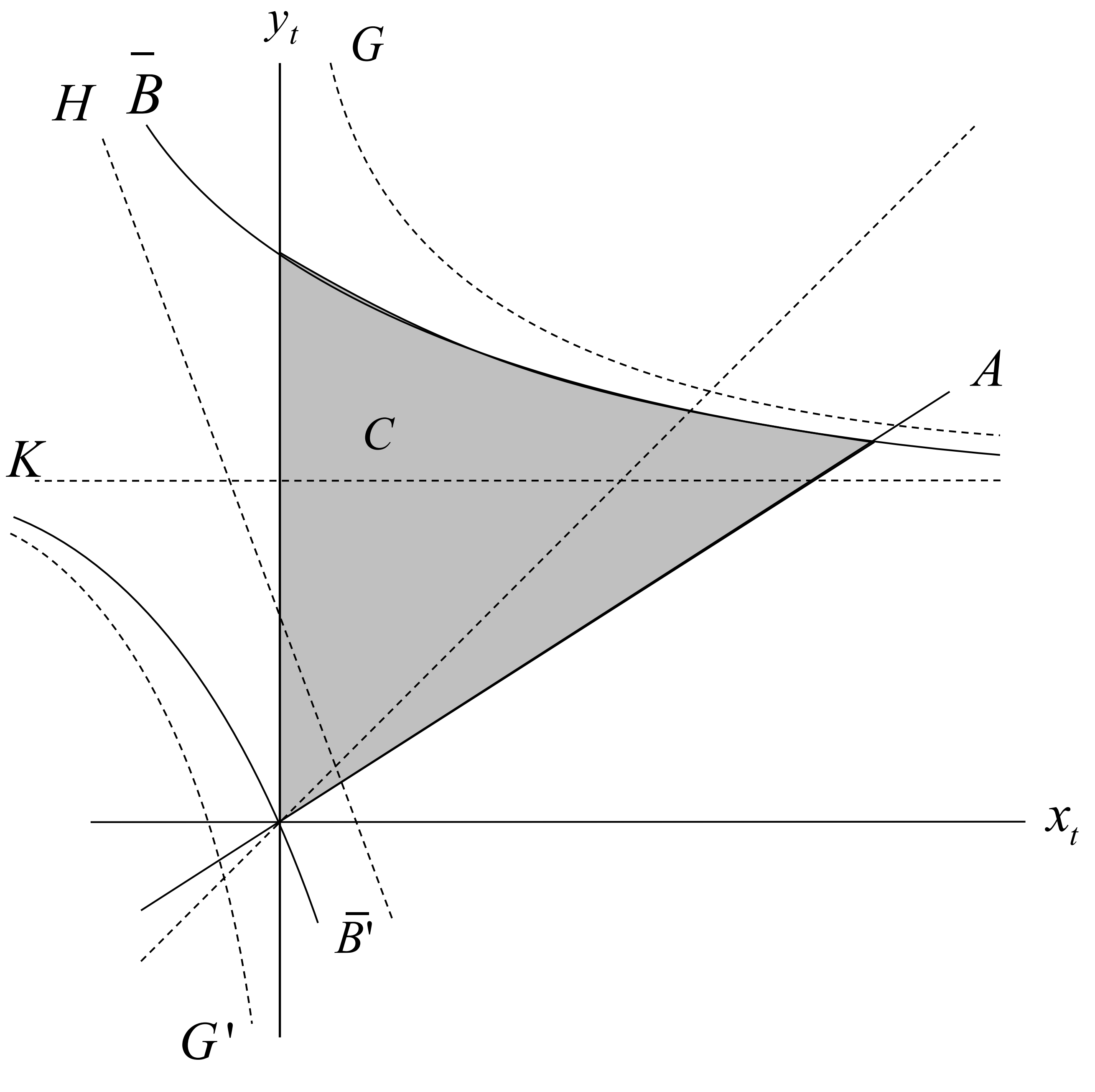

Figure 6: Constrained Domain and Its Core

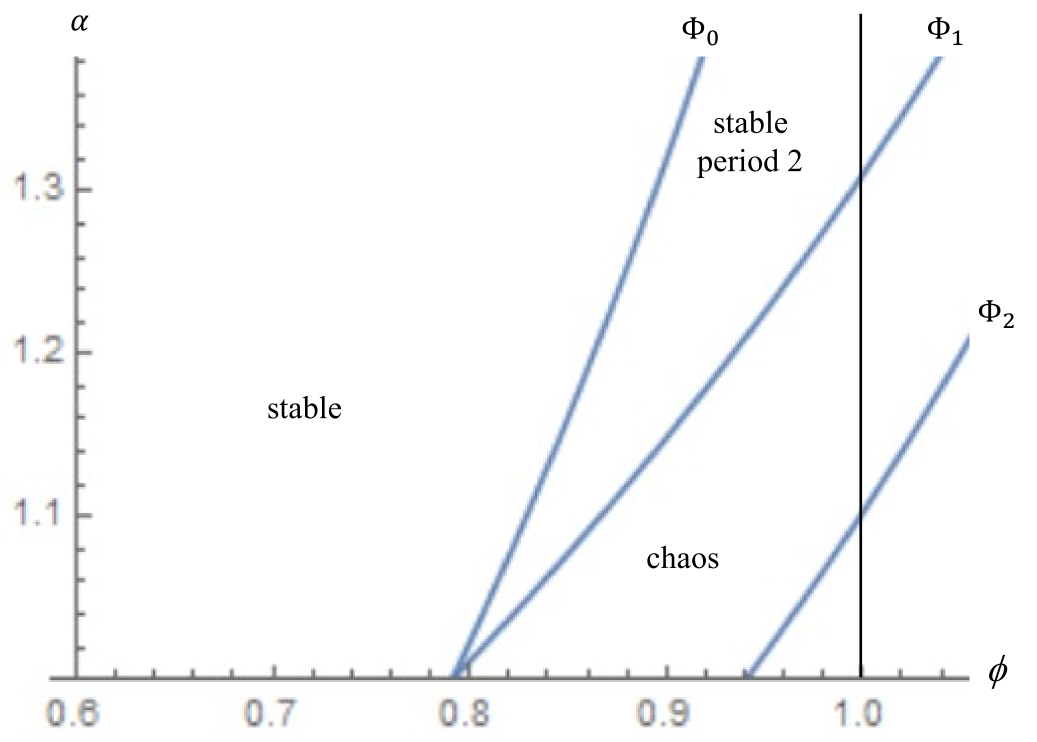

Figure 7: Complete Characterization for Chaos
in the Case of Small q

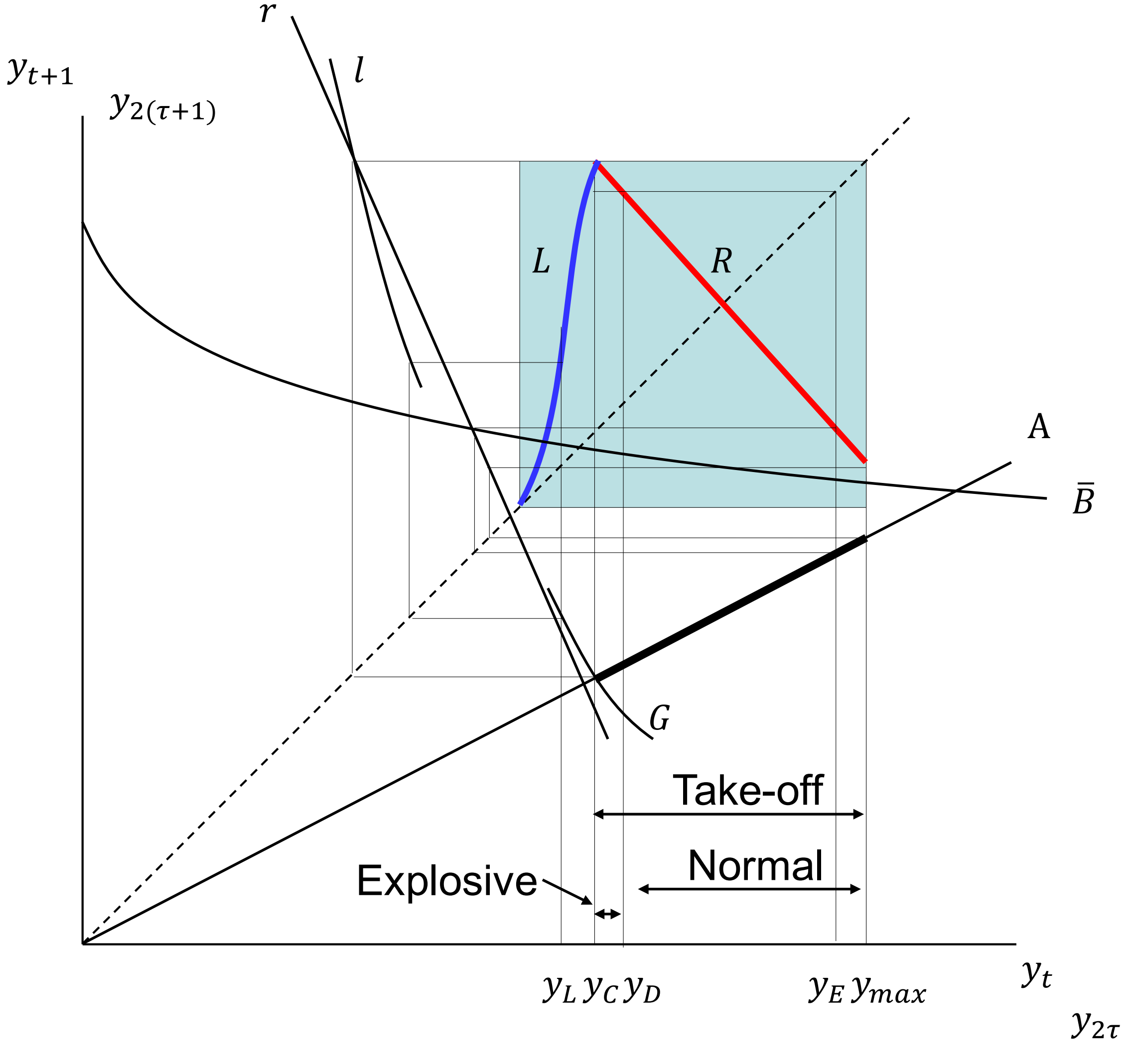

Figure 8: Single-period Dynamics and Explosive Take-offs

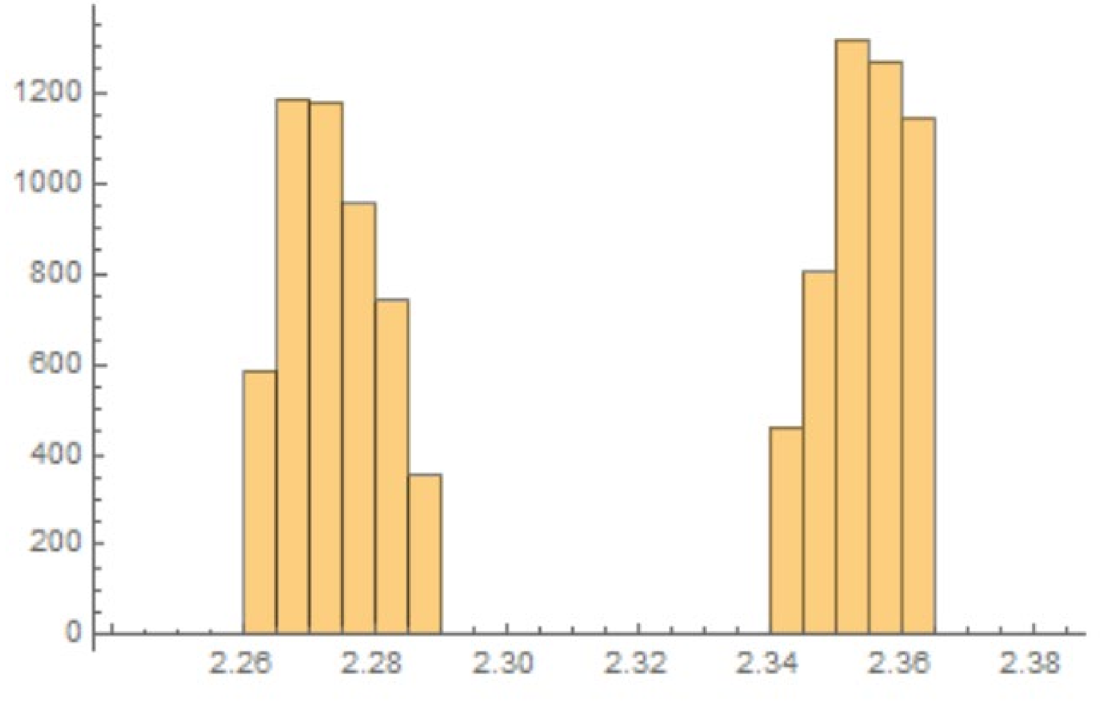

Figure 9: Ergodic Distribution of a Chaotic Paths

$\alpha = 1.17,\ \beta = 0.6,\ \theta = 0.05,\ \frac{E}{\kappa} = 10$

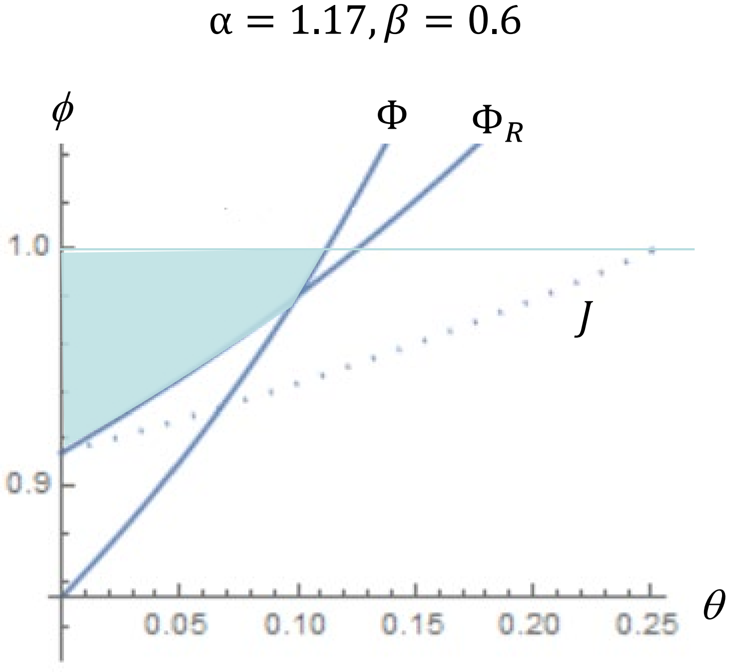

Figure 10: Almost Complete Characterization for Chaos
with Respect to Substitutability (θ) and Intellectual Property Protection (ϕ)

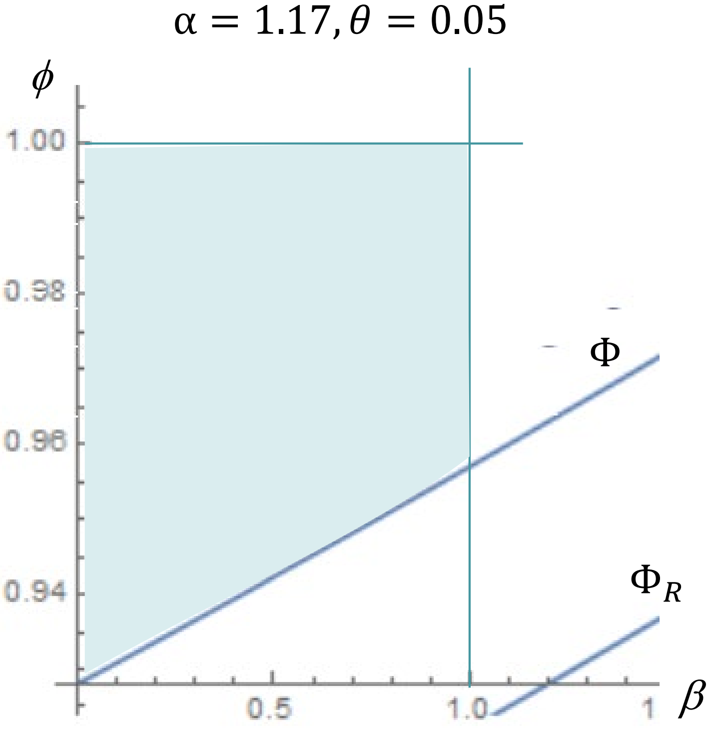

Figure 11: Almost Complete Characterization for Chaos
with Respect to Discount Factor (β) and Intellectual Property Protection (ϕ)