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Voluntary Provision of Public Goods and Cryptocurrency*

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Abstract

The purpose of this paper is to show how the mechanism of the reward structure for cryptocurrency mining (known as "Proof of Work") is applicable to alleviation of the free rider problem for voluntary public goods provision. This paper presents the following results. First, if each individual reports preferences honestly, then the Samuelson condition can hold. It is possible to set an appropriate level of mining. Second, if the scheme (mechanism) offered by our manuscript is introduced, public goods can theoretically be provided at a Pareto optimal level under certain conditions because each rational individual reports true preferences to the government.

Keywords: Blockchain, Cryptocurrency, Mechanism, Public goods, Voluntary provision

JEL Classification: E50, H40, H42, H44, L50

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1. Introduction

This paper presents the argument that it is possible to apply the mechanism of the cryptocurrency reward structure for miners, known as "Proof-of-Work," to alleviate the long-discussed issue of the free rider problem related to voluntary public good provision.

If each supplies public goods voluntarily, the supply of public goods is less than the Pareto optimal level because of the "Free rider problem" that occurs by a Nash equilibrium. This problem is examined in many studies reported in the literature. Yamashige (2013) and Slavov (2014) explain that voluntary provision of public goods is generally inadequate [1]. Villanacci and Zenginobuz (2006) also examine how the government should intervene in the voluntary provision of public goods.

One means of solving this problem is to levy a lump-sum tax on the individuals and to provide some public goods. However, this engenders a "crowding out effect" by which the public goods provided by individuals are crowded out and the level of public goods does not change completely. This result is related to "equivalence of provision of public goods" as demonstrated by Warr (1982, 1983)[2].

Lindahl (1919) derives the Lindahl mechanism, which is not a lump-sum taxation for the provision of public goods but which affects public goods prices to solve the difficulty described above. In this mechanism, if each reports the preference for the public goods for the government honestly, then the public goods can be supplied at a Pareto optimal level because payment for the cost of a unit of public goods determined by the reported preference has the effect of a constant rate tax.

For instance, Boadway, Pestieau and Wildasin (1989) report that a Lindahl equilibrium can be achieved by a subsidy for the provision of public goods in the model of voluntary provision of public goods even if the public goods are supplied uncooperatively. Moreover, Kesternich, Lange and Sturm (2014) examine how the payment provided by individuals is determined in the provision of public goods with heterogeneous agents. However, if each does not report the preference for public goods honestly, then the "free rider problem" occurs and the Lindahl mechanism cannot obtain a Pareto optimal level of public goods.

Many indications exist to resolve this difficulty. For instance, Clarke (1971), Groves

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[1] In the dynamics model, Slavov (2014) shows that the voluntary provision of public goods achieves an efficient level.

[2] Bergstrom, Blume and Varian (1986) examine the case in which there exists a non-negative constraint, a corner solution, and others for the burden for provision of public goods as the case that equivalence of provision of public goods is not held. For instance, for the voluntary provision of public goods, we can consider the case in which individuals fully pay for the private goods and do not pay for the public goods at all if the individual income is low or if the total amount of public goods provided by other individuals except for oneself is more than their optimal demand level of public goods. In this case, a lump-sum tax or the income transfer brought about by the government has a marked effect on public goods provision.

and Ledyard (1977) and others present the "Clarke--Groves Mechanism"[3]; Varian (1994) presents the "Varian Mechanism."

However, the former (CG Mechanism) cannot finance the necessary cost to provide public goods solely by the burden for each. A problem occurs by which the fiscal deficit exists in part of government. In the latter (V Mechanism), there is no fiscal deficit for the government. However, this mechanism depends on the assumption that each can ascertain the information of others perfectly. Today, no realistic means exists to resolve this difficulty.

In this situation, for instance, Morgan (2000) presents a novel mechanism that incorporates a lottery for the voluntary provision of public goods as a solution that differs from the mechanism presented to date. Concretely, part of the revenue of the lottery is paid as the prize; the reminder of the revenue is paid to supply public goods. However, the mechanism presented by Morgan (2000) has the important shortcoming that unless the lottery prize is huge, the supply of public goods is not equal to a Pareto optimal level because the lottery cost is financed only by the persons who purchase lottery tickets.

The lottery prize is the reward that people can obtain at a certain probability in providing public goods voluntarily. It is an interesting idea that the prize has the mechanism that raises the incentive to provide public goods voluntarily. Does the way to set mechanism something to solve the problem of Morgan (2000) entail the same incentive existing with the lottery?[4]

Then, we would like to present the mechanism of voluntary provision of public goods related to "Cryptocurrency" that has become popular rapidly in recent years. Now, the market of cryptocurrency is increasing rapidly worldwide. As shown by CoinMarketCap.com, the market capitalization reached about 800 billion US dollars at the beginning of 2018 and is still about 200 billion US dollars now (24 August 2018).

Approximately ten years have passed since Bitcoin, the first cryptocurrency, was invented. There already exist thousands of types of cryptocurrencies (about two thousand types) worldwide. Among many cryptocurrencies, Bitcoin, the most well-known and predominant cryptocurrency, relies on a "distributed ledger technology," designated as Blockchain, to record transaction data and thereby avoid double payment attempts. When people wanting to join the Bitcoin system provide the calculation capability of a computer, the Bitcoin system probabilistically rewards the people by providing a certain

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[3] See Groves (1970, 1973) for an initial model of Groves.

[4] In the model of Morgan (2000), there is a critical problem that individuals who never buy lottery tickets can be tempted to free ride on the public goods voluntarily provided by others due to the mechanism that only individuals who buy lottery tickets cover the cost of public goods provision. On the other hand, in our model, the cost of public goods provision is completely covered by inflation tax, and all consumers are compelled to cover the cost. Therefore, we show that it is difficult for individuals to have incentive to free-ride in our model.

level of cryptocurrency based on a certain rule. “Mining” is defined as behavior by which the persons record the valid transaction data for blockchain. This system is generally explained as “proof of work (POW).”[5]

Although the POW of Bitcoin is to confirm whether a transaction is valid or not, there exists a different system of cryptocurrency. For instance, the cryptocurrency designated as Ripple participates in the team of “World Community Grid” and provides a mechanism by which the person can obtain a reward for contribution to cancer research, investigations of new diseases, and other endeavors. Now, cryptocurrencies of some kinds are used for general transactions. However, by virtue of blockchain technology, a system arises by which people can obtain cryptocurrency issued by the platform as a reward for contributions to movies, music, and other cultural endeavors based on a certain rule.

Consequently, although the technology related to cryptocurrency has the possibility of being useful as a reward for the voluntary provision of public goods, there is no related literature describing this analysis. We consider that deepening the analysis of this mechanism is important. Therefore, our paper presents an examination of how the mechanism of voluntary provision of public goods is related to rewards for the mining of cryptocurrency (Proof of work) affects the supply of public goods.

The remainder of the paper comprises the following. First, section 2 sets the basic model (the case in which that the government supplies the public goods) for fundamental analysis. Furthermore, we examine the case in which the cryptocurrency is used for the voluntary provision of public goods and derive the related propositions. Section 4 presents conclusions and avenues for future studies.

2. Basic Model – A Case in which the government provides public goods –

We set the simple model to examine the case of voluntarily provided public goods with cryptocurrency in section 3. There exist number N individuals. The utility function of the individual of j th (j=1, 2, 3,…,N) U_j is assumed as

$$U_j = \log(c_j) + \alpha_j \log(z) \ , \tag{1}$$

where c_j and z respectively denote the consumption of the j th individual and the supply of public goods. Preferences for public goods are heterogeneous among individuals. The preference of j th individual for public goods is given as α_j.

Before analyzing the case of the supply of public goods with cryptocurrency, we consider the case in which the government levies a proportional income tax rate τ on a j th individual to provide the public goods as the standard model. Defining w_j as the income of the j th individual, the budget constraint of the j th individual is shown as

[5] We can present Chiu and Koeppl (2017) as the related literature that sets this system.

$$(1-\tau)w_j = c_j \ . \tag{2}$$

Then, the government budget constraint is shown as follows.

$$\sum_j \tau w_j = \mathrm{z} \tag{3}$$

Considering (2) and (3), the social welfare function $W = \sum_{j=1}^{N} U_j$ is

$$W = \sum_{j=1}^{N} \left\{ \log\left[\left(1 - \frac{z}{\sum_j w_j}\right) w_j\right] + \alpha_j \log(z) \right\} .$$

Also, $\bar{\mathrm{z}}$ is given as the following equation to maximize the social welfare function W.

$$\sum_j \left[-\frac{\frac{w_j}{\sum_j w_j}}{\left(1-\frac{z}{\sum_j w_j}\right) w_j} + \alpha_j \frac{1}{\bar{\mathrm{z}}} \right] = 0 \quad \Leftrightarrow \quad \bar{\mathrm{z}} = \frac{\bar{\alpha}}{1+\bar{\alpha}} \sum_j w_j \tag{4}$$

Therein, $\bar{\alpha} = \sum_{j=1}^{N} \alpha_j / N$ shows the average preference for public goods; (4) shows the Samuelson condition, which is the condition to provide public goods at a Pareto optimal level. If the public goods are provided by government such that (4) holds, social welfare W is maximized. It is most efficient. However, it is too difficult for a government to provide the public goods at the optimal level.

Even if the government can obtain information of income w_j of the j th individual with the report of tax system, it is not easy for the preference for public goods α_j to hold because of asymmetric information between individuals and the government. However, the individuals do not always reveal their own preferences for public goods honestly.

This problem is "the problem of revealed preference." For instance, if the j th individual that has preference α_j maximizes utility U_j, then the following equation is expected to hold for the optimal public goods level z^j for the j th individual.

$$\frac{\partial U_j}{\partial z^j} = \frac{\partial}{\partial z^j} \left\{ \log\left[\left(1 - \frac{z^j}{\sum_j w_j}\right) w_j\right] + \alpha_j \log\left(z^j\right) \right\} = 0 \quad \Leftrightarrow \quad z^j = \frac{\alpha_j}{1+\alpha_j} \sum_j w_j,$$

With $\alpha_j > \bar{\alpha}$, we obtain $z^j > \bar{\mathrm{z}}$. If the government holds information about the preference of each with the report from individuals, then it is less than the public goods level of z^j to maximize utility U_j in honest report of preference α_j. Therefore, the j th individual reports a preference that is greater than true preference α_j dishonestly to raise $\bar{\alpha} = \sum_{j=1}^{N} \alpha_j / N$ because the individuals have an incentive to be close to the true preference for the public goods of the j th individual.

Similarly, the j th individual who has preference $\alpha_j < \bar{\alpha}$ reports a preference for the public goods that is less than the true preference level α_j to reduce $\bar{\alpha}$ and to be close to the true preference α_j. Generally, as demonstrated by Lindahl (1919) and others, if the individuals intend to obtain benefit with a dishonest report, then the free-ride problem occurs. The supply of Pareto optimal level of public goods fails.

Moreover, if the supply of public goods is decided by the majority voting, then considering the median voter theorem, the supply of public goods z^m is determined politically by preference α_j of the median voter.

$$z^m = \frac{\alpha_m}{1+\alpha_m} \sum_j w_j, \tag{5}$$

In that equation, α_m given in (5) is not always equal to $\bar{\alpha}$ given in (4). Therefore, the level of public goods determined by majority voting does not always fulfill the Samuelson condition. It is clear that the public goods can not be provided at a Pareto optimal level[6].

3. Voluntary Provision of Public Goods with Cryptocurrency

In this section, we examine the case of voluntary provision with a cryptocurrency. As described above, Bitcoin and many other cryptocurrencies have a system by which individuals can obtain a reward for recording transaction data, based on the rule, to enforce the security that prevents double payment, for instance. The correct recording of the transaction for blockchain to obtain the reward is defined as mining. This system is called Proof of Work.

By virtue of the mining technology, we can set a rule by which the individuals can obtain a certain level of cryptocurrency as a reward for individual voluntary provision of the public goods. Then, the model in this section assumes that if the j th individual provide a unit of public goods voluntarily, then the individual can obtain β units of cryptocurrency as the reward ($0 < \beta < 1$ is assumed): a j th individual who provides q_j units of public goods voluntarily can obtain βq_j units of the cryptocurrency.

Then, the budget constraint of the j th individual is shown as

$$w_j + \beta q_j = q_j + p c_j, \tag{6}$$

where p and Ω respectively denote the price index level and the initial endowment of cryptocurrency. The amount of cryptocurrency obtained after voluntarily providing q_j units of public goods by individuals is given as $\Omega + \beta \sum_j q_j$. This model includes the assumption that the normal monetary stock is given by M. We can change a unit of cryptocurrency and a unit of monetary stock. Moreover, there exists only the cryptocurrency and normal monetary stock as currency used as a transaction tool in this model. With the Quantity Theory of Money, price index level p can be represented as

$$p = \frac{M+\Omega+\beta \sum_j q_j}{M+\Omega}. \tag{7}$$

[6] Itaya and Schweinberger (2006) consider political equilibrium in a model in which the public goods are financed by the income tax. They derive Pareto improvement as brought about by a decrease in the income tax rate.

In addition, the aggregate supply of the public goods z is given as the following.

$$\sum_j q_j = \mathrm{z}. \tag{8}$$

Under the setting described above, we consider the maximization condition of utility (1). Substituting (6)–(8) into (1), we can obtain the following function:

$$U_j = \log\left\{\frac{M+\Omega}{M+\Omega+\beta\sum_j q_j}\left[w_j+(\beta-1)q_j\right]\right\} + \alpha_j\, log\left(\sum_j q_j\right).$$

Therefore, the optimization condition for an individual voluntarily providing the public goods is given as shown below. The following proposition can be established:

$$\frac{\partial U_j}{\partial q_j} = \frac{\beta-1}{w_j+(\beta-1)q_j} - \frac{\beta}{M+\Omega+\beta\sum_j q_j} + \alpha_j\frac{1}{\sum_j q_j} = 0. \tag{9}$$

Proposition 1

If the initial endowment of normal monetary stock and cryptocurrency $(\Omega + M)$ are sufficiently large and $\beta\sum_j q_j \ll \Omega + M$ holds, then the aggregate supply of public goods z and the public goods that individuals provide voluntarily q_j can be shown as

$$z = \frac{1}{1-\beta}\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}, \tag{10}$$

$$q_j = \frac{1}{1-\beta}\left(w_j - \frac{1}{\alpha_j}\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}\right). \tag{11}$$

Proof

With $\Omega + \beta\sum_j q_j \ll \Omega + M$, (9) changes to $\frac{1-\beta}{\alpha_j}z = w_j - (1-\beta)q_j$. For the summation about j=1, 2, 3…N, we obtain $\sum_j\frac{1-\beta}{\alpha_j}z = \sum_j w_j - (1-\beta)z$; also, (10) is obtainable by the reduced form about z. Substituting (10) into (9), we obtain (11). (Q.E.D.)

Equation (10) shows that the voluntary provision of public goods is a function of the reward of mining β. With $\beta = 0$, we obtain $z = \frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}$. Comparing this equation with (4), one can infer that the public goods shown by (10) with $\beta = 0$ are less than the public goods shown by (4). This result demonstrates that if each provides the public goods voluntarily, then free riding occurs because of the Nash equilibrium. Then the supply of public goods is less than the Pareto optimal level. However, (10) shows that an increase in the reward for mining raises the level of voluntary provision of public goods. Conversely, a decrease in the reward for mining reduces the voluntary provision of public goods. Therefore, if setting the reward for mining β at the appropriate level, then equation (10) shows that the voluntary provision of public goods can be equal to the Samuelson condition given by (4) and can achieve maximization of social welfare W. Then, the following proposition can be established.

Proposition 2

If the reward for mining is set as follows, then (10) holds for the Samuelson condition.

$$\beta = 1 - \frac{1+\frac{1}{\bar{\bar{\alpha}}}}{1+\sum_j \frac{1}{\alpha_j}}. \tag{12}$$

Proof

Substituting (4) into z given by (10), we obtain $(1-\beta)\frac{\bar{\alpha}}{1+\bar{\alpha}}\sum_j w_j\left(1+\sum_j \frac{1}{\alpha_j}\right) = \sum_j w_j$. Then, we obtain (12) by solving for β. (Q.E.D.)

Actually, (12) shows that we can obtain the Samuelson condition and Pareto optimal supply of public goods not only by the voluntary supply by the setting of appropriate level of the reward for mining if each j th individual (j=1, 2, 3…N) reports preference α_j honestly. However, "asymmetric information" exists between the government and the individuals. For that reason, knowing preference α_j of the j th individual is difficult. It must depend on a report from each for the government to ascertain the preference α_j of each j th individual. If each j th individual (j=1, 2, 3…N) reports strategy related to preference α_j, then we can examine how an individual should report preference α_j.

If α_j is the true preference of the j th individual, then α_j' denotes the preference that reported by the j th individual. The reward for mining is given as $\beta' = 1-(1+1/\bar{\alpha}')/(1+\sum 1/\alpha_j')$ from (12). If the endowment of normal monetary stock and cryptocurrency is sufficiently large, then we consider maximization of the following utility of j th individual (j=1, 2, 3…N) for a given reward for mining β'.

$$U_j = \log\left[w_j-(1-\beta')q_j\right]+\alpha_j\log\left(\sum_j q_j\right). \tag{13}$$

The condition to maximize the utility is

$$\frac{1-\beta'}{w_j-(1-\beta')q_j}=\alpha_j\frac{1}{\sum_j q_j}. \tag{14}$$

We can obtain $z=\frac{1}{1-\beta'}\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}$ from (14). The following equation can be derived by substitution into (14):

$$\frac{1-\beta'}{\alpha_j}\frac{1}{1-\beta'}\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}=w_j-(1-\beta')q_j.$$

Substituting this equation into (13), the utility of the *j* th individual can be derived as shown below.

$$\begin{aligned} U_j &= \log\left(w_j-(1-\beta')q_j\right)-\alpha_j\log(1-\beta')+\alpha_j\log\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}} \\ &= -\alpha_j\log(1-\beta')+\log\frac{1}{\alpha_j}\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}+\alpha_j\log\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}} \\ &= -\alpha_j\log(1-\beta')+\left(1+\alpha_j\right)\log\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha_j}}-\log(\alpha_j) \end{aligned} \tag{15}$$

Equation (15) is a function of the reward for mining β'. If *N* is sufficiently large, then $\partial\beta'/\partial\alpha_j{}'$ is shown by the approximation form as shown below.[7]

$$\frac{\partial\beta'}{\partial\alpha_j{}'}=\frac{\partial}{\partial\alpha_j{}'}\left(1-\frac{1+\frac{1}{\alpha'}}{1+\sum_j\frac{1}{\alpha_j{}'}}\right)\approx\frac{N}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\sum_{k\neq j}\alpha_k'}\right)\frac{1}{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left[\left(\frac{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}{\sum_{k\neq j}\alpha_k'}\right)-\frac{1}{\left(\alpha_j{}'+\frac{1}{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\right)\alpha_j{}'}\right]$$

Because the part of $\mathrm{g}(\alpha_j')\equiv\left(\alpha_j{}'+\frac{1}{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\right)\alpha_j{}'$ is a quadratic function and a monotonic increasing function, the sign of $\frac{\partial\beta'}{\partial\alpha_j{}'}$ is negative in $\alpha_j'<\gamma$ and is positive in $\alpha_j'>\gamma$, Also, β' is a downward convex function if we define γ as α_j' such that $\left(\frac{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}{\sum_{k\neq j}\alpha_k'}\right)=1/\mathrm{g}(\alpha_j')$ holds.

Therefore, if the lower limit of the preference α_j of the *j* th individual is $\underline{\theta}$ and the upper limit is $\overline{\theta}$, then the reward for mining β' is maximized at the lower limit $\underline{\theta}$ or the upper limit $\overline{\theta}$. We assume that the share of σ of (*N*—1) individuals except for the *j* th individual select the lower limit $\underline{\theta}$ and that the share of $1-\sigma$ of (*N*—1) individuals selects the upper limit $\overline{\theta}$. If *N* is sufficiently large, then the reward for mining β' is

[7] See Appendix for a detailed proof.

approximated as[8]

$$\beta' \approx 1-\frac{1+\frac{1}{\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left[1-\frac{1}{N}f(\alpha_j')\right]+ \text{(Const.)}\ ,$$

where $f(\alpha_j') \equiv \frac{\alpha_j'}{\left[1+\frac{1}{\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}\right]\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{\frac{1}{\alpha_j{}'}}{\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}$.

The reward for mining β' is maximized where α_j is the lower limit $\underline{\theta}$ or the upper limit $\overline{\theta}$. The sign of $f(\underline{\theta})-f(\overline{\theta})$ must be checked to ascertain at which β' is maximized: $\underline{\theta}$ or $\overline{\theta}$. For instance, if the sign of $f(\underline{\theta})-f(\overline{\theta})$ is always positive in spite of σ $(0 \le \sigma \le 1)$, then the reward for mining β' is maximized by $\alpha_j=\underline{\theta}$. Then, (15) shows that the utility of the j th individual can be maximized if the individuals report the preference as $\alpha_j'=\underline{\theta}$.

Which is the real sign, positive or negative? Generally, the sign of $f(\underline{\theta})-f(\overline{\theta})$ depends on σ. It is not easy to check the sign. However, if the upper limit $\overline{\theta}$ is sufficiently large, then we can show that the sign of $f(\underline{\theta})-f(\overline{\theta})$ is positive at $0 \le \sigma < 1$. This result demonstrates that most individuals select the lower limit $\underline{\theta}$ if there exist a few individuals that report the upper limit $\overline{\theta}$ and if $\overline{\theta}$ is sufficiently large. Then, the following proposition can be established.

Proposition 3

We define the lower limit and the upper limit of preference α_j of the j th individual as $\underline{\theta}$ and $\overline{\theta}$, respectively. If N and $\overline{\theta}$ are sufficiently large, then each reports the preference $\alpha_j'=\underline{\theta}$. In addition, the reward for mining is shown as presented below:

$$\beta' = 1-\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{N}{\underline{\theta}}}=\frac{N-1}{\underline{\theta}+N}. \tag{16}$$

If the government-set reward for mining is (16), then preference $\alpha_j{}'=\underline{\theta}$ deviates from true preference α_j . Moreover, there exists the problem that the public goods can not be provided at a Pareto optimal level. We consider the following scheme to solve the problem under the situation that the government can ascertain the wage w_j of j th individual (j=1, 2, 3…N) by virtue of a taxpayer identification number system.

(Step 1) First, the government makes each j th individual (j=1, 2, 3…N) report the preference. Then, we consider that the reported preference by the j th individual is α_j' (j=1, 2, 3…N).

[8] See Appendix for a detailed proof.

(Step 2) Based on preference α_j' explained above and (12), the government sets the reward for mining β' and collects information related to voluntary supply q_j (j=1, 2, 3…N) by the j th individual.

(Step 3) Substituting q_j and β' into (14), we count backward to the preference of each α_j in (14) by solving the N size of simultaneous equations. We define the preference derived by the counting backward as α_j''.

(Step 4) If the preference reported by the j th individual α_j' differs from the preference derived by the counting backward α_j'', the government gives a certain level or an infinite level of penalty.[9]

Proposition 4 By virtue of the scheme explained above, each j th individual reports the true preference α_j for the government.

Proof If the j th individual reports false preference $\alpha_j'(\neq \alpha_j)$ at Step 1, then the individual selects the only strategy to avoid the penalty at Step 4. That is the strategy by which the individual behaves as though true preference α_j is equal to the false reported preference α_j' to maximize utility. Then, (14) is given as shown below.

$$\frac{1-\beta'}{w_j-(1-\beta')q_j}=\alpha'_j\frac{1}{\sum_j q_j}$$

Substituting this equation and the reward for mining $\beta'=1-(1+1/\bar{\alpha}')/(1+\sum 1/\alpha_j')$ into (13), the utility given by (13) changes to the expressions presented below.

$$\begin{aligned}U'_j&=-\alpha_j\log(1-\beta')+(1+\alpha_j)\log\frac{\sum_j w_j}{1+\sum_j\frac{1}{\alpha'_j}}-\log(\alpha'_j)\\&=-\alpha_j\log\frac{1+\frac{1}{\bar{\alpha}'}}{1+\sum_j\frac{1}{\alpha'_j}}+(1+\alpha_j)\log\frac{1}{1+\sum_j\frac{1}{\alpha'_j}}-\log(\alpha'_j)+(1+\alpha_j)\log\sum_j w_j \qquad (17)\\&=-\alpha_j\log(1+\frac{1}{\bar{\alpha}'})-\log(1+\sum_j\frac{1}{\alpha'_j})-\log(\alpha'_j)+(1+\alpha_j)\log\sum_j w_j \quad .\end{aligned}$$

The j th individual can ascertain how individual k ($\neq j$ except for the j th individual selects the strategy or behaves, each $\alpha'_k(k\neq j)$ shown at (17) is expected by the j th individual. If N is sufficiently large, then the partial derivative of (17) is

$$\frac{\partial U'_j}{\partial\alpha'_j}=\alpha_j\frac{\frac{1}{\bar{\alpha}'^2}}{1+\frac{1}{\bar{\alpha}'}}\frac{1}{N}+\frac{\frac{1}{\alpha'^2_j}}{1+\sum_j\frac{1}{\alpha'_j}}-\frac{1}{\alpha'_j}\approx\frac{1}{\alpha'_j}\left(\frac{\frac{1}{\alpha'_j}}{1+\sum_j\frac{1}{\alpha'_j}}-1\right)<0.$$

This equation shows that the j th individual has an incentive to report preference $\alpha_j'=\underline{\theta}$ in spite of the strategy selected by the individual k except for j if the j th

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[9] For instance, the government sets $\beta'=0$ ex post.

individual selects a strategy to report a false preference at Step 1. If all individuals report false preference $\alpha_j' = \underline{\theta}$, then (17) is

$$U'_j = -\alpha_j \log\left(1 + \frac{1}{\underline{\theta}}\right) - \log\left(1 + \frac{N}{\underline{\theta}}\right) - \log(\underline{\theta}) + (1 + \alpha_j)\log\textstyle\sum_j w_j. \tag{18}$$

However, if each j (j=1, 2, 3…N) reports true preference α_j, then (15) is

$$U_j = -\alpha_j \log\left(1 + \frac{1}{\bar{\alpha}}\right) - \log\left(1 + \textstyle\sum_j \frac{1}{\alpha_j}\right) - \log(\alpha_j) + (1 + \alpha_j)\log\textstyle\sum_j w_j.$$

The difference between U_j given by (15) and U'_j given by (18) is

$$\varphi(\alpha_j) \equiv U_j - U'_j = \alpha_j \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \log\frac{1+\frac{N}{\underline{\theta}}}{1+\sum_j \frac{1}{\alpha_j}} + \log\frac{\underline{\theta}}{\alpha_j}. \tag{19}$$

If N is sufficiently large, then the partial difference of (19) with respect to α_j is

$$\varphi'(\alpha_j) \equiv \frac{\partial(U_j - U'_j)}{\partial \alpha_j} = \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \alpha_j \frac{\frac{1}{\bar{\alpha}^2}}{1+\frac{1}{\bar{\alpha}}}\frac{1}{N} + \left(\frac{\frac{1}{\alpha_j}}{1+\sum_j \frac{1}{\alpha_j}} - 1\right)\frac{1}{\alpha_j}$$
$$\approx \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \left(\frac{\frac{1}{\alpha_j}}{1+\sum_j \frac{1}{\alpha_j}} - 1\right)\frac{1}{\alpha_j}\ . \tag{20}$$

$$\varphi''(\alpha_j) \equiv \frac{\partial^2(U_j - U'_j)}{\partial \alpha_j \partial \alpha_j} \approx -\frac{\frac{2}{\alpha_j^3}}{1+\sum_j \frac{1}{\alpha_j}} + \frac{\frac{1}{\alpha_j^4}}{\left(1+\sum_j \frac{1}{\alpha_j}\right)^2} + \frac{1}{\alpha_j^2} = \frac{1}{\alpha_j^2}\left(1 - \frac{\frac{1}{\alpha_j}}{1+\sum_j \frac{1}{\alpha_j}}\right)^2 > 0. \tag{21}$$

As shown by (21), $\varphi'(\alpha_j)$ of (20) is a monotonic increasing function of α_j. Given the lower limit and upper limit of α_j as $\underline{\theta}$ and $\bar{\theta}$, respectively, the following equation is obtainable in (20).

$$\varphi'(\alpha_j) \approx \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \left(\frac{\frac{1}{\alpha_j}}{1+\sum_j \frac{1}{\alpha_j}} - 1\right)\frac{1}{\alpha_j} < \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \left(\frac{\frac{1}{\underline{\theta}}}{1+\frac{N}{\underline{\theta}}} - 1\right)\frac{1}{\underline{\theta}}.$$

Therefore, if N is sufficiently large, then the necessary and sufficient condition to have $\varphi'(\alpha_j) < 0$ is $\log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} - \frac{1}{\underline{\theta}} < 0$. It is equal to the following inequality:

$$\frac{1}{\bar{\alpha}} > \frac{1+\frac{1}{\underline{\theta}}}{\exp\left(\frac{1}{\underline{\theta}}\right)} - 1 \quad \Leftrightarrow \quad \frac{1}{\bar{\alpha}} > f(t) \equiv \frac{1+\log t}{t} - 1 \quad (t > 1).$$

Brief calculations can be used to derive $f(t) < 0$ and $\varphi'(\alpha_j) < 0$. Therefore, we obtain $\varphi(\alpha_j) > 0$ because $\varphi(\alpha_j)$ of (19) is a monotonic decreasing function of α_j and $\left(1 + \frac{1}{\underline{\theta}}\right) / \left(1 + \frac{1}{\bar{\alpha}}\right) > 1$ holds.

$$\max_{\sigma\to\infty} \varphi(\sigma) \approx \max_{\sigma\to\infty}\left(\sigma \log\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}} + \log\frac{\underline{\theta}}{\sigma}\right) = \log\left(\max_{\sigma\to\infty}\left(\frac{1+\frac{1}{\underline{\theta}}}{1+\frac{1}{\bar{\alpha}}}\right)^{\sigma}\frac{\underline{\theta}}{\sigma}\right) > 0.$$

That is, we obtain $U_j > U'_j$. This shows that the individual has no incentive to report a false preference $\alpha_j{}' = \underline{\theta}$ and reports true preference α_j for the government. (Q.E.D.)

4. Conclusions and Future Research

The aims of this paper are to present an examination of whether the mechanism for the voluntary provision of public goods related to the reward for mining of cryptocurrency ("Proof of Work") can solve "the free ride problem." The results obtained using this analysis are presented as the following three points.

First, if each person reports their own preferences honestly, then the Samuelson rule holds in the case of voluntary provision of public goods by setting the reward for mining as the appropriate level. The theoretical possibility exists that the public goods can be provided at a Pareto optimal level under a Nash equilibrium (Proposition 2).

In our model, each has an incentive to report false preferences for public goods. Therefore, the reward for mining deviates from the optimal level described above. Moreover, there exists the theoretical possibility that the public goods cannot be provided at a Pareto optimal level (Proposition 3).

However, if the scheme or mechanism shown by our paper under the case in which the government can ascertain the wage of each with a taxpayer identification number system, then each rational individual has an incentive to report the true preference for public goods under certain conditions. Therefore, it increases the theoretical probability that the public goods can be provided at a Pareto optimal level (Proposition 4).

Future studies that remain, surmised from results presented herein, must address the following three points.

The first problem is the relation between the endowment of a normal monetary stock and cryptocurrency and the cryptocurrency issued as a reward for mining. In this model, we assume that $\Omega + M$ is sufficiently large and that $\beta \sum_j q_j \ll \Omega + M$ holds. However, if the Monetary Quantity Theorem holds, then this assumption does not hold because the price index rises considerably if the cryptocurrency issued by the voluntary provision of public goods increases rapidly. If this assumption changes, then the manner in which Propositions 1--4 should be revised must be examined. We consider that the contribution of this analysis is large.

A second problem is the lower limit and the upper limit of the preference. We consider that the assumption that there exists a lower limit and the upper limit of preference for public goods does not lose generality of analysis in this model because this assumption is realistic. However, our paper derives the propositions in the model that everyone has the same lower limit and upper limit of preference for the public goods. It is important to consider how Propositions 3 and 4 should be revised if this assumption were to change.

A third problem is the relation between the theoretical mechanism and the empirical experiment. For instance, Chen and Plott (1996) and Chen and Tang (1998) examine the empirical experiment of CG mechanism and indicate the possibility that a Nash equilibrium can not be achieved as long as the desirable nature (Super Modularity) holds. Moreover, the mechanism presented in our manuscript offers a penalty for the false reporting of the respective preferences. However, the empirical experiment of voluntary provision game of public goods explained by Walker and Halloran (2004) and Sefton, Shupp and Walker (2002) points to the possibility that the reward is more effective than the penalty for an increase in the contributed amount of public goods.

The empirical experiment explained by Fehr and Gachter (2000) points to the possibility that the penalty is more effective for the amount of contribution of public goods. Therefore, we need a cautious decision for the consistency and interpretation for the theory and empirical experiment including Nash equilibrium and the effect of a penalty. Moreover, we must assess the possibility of application for realistic policy by repeating detailed empirical experiments.

References

- Bergstrom, T., Blume, L. and Varian, H. (1986) "On the private provision of a public goods," *Journal of Public Economics* Vol.29, pp.25-49
- Boadway, R., Pestieau P. and Wildasin, D.E. (1989) "Non-cooperative Behavior and Efficient Provision of Public Goods," *Public Finance* Vol.44(1), pp.1-7.
- Chen, Y. and Plott, C. R. (1996) "The Groves-Ledyard Mechanism: An Experimental Study of Institutional Design," *Journal of Public Economics* Vol.59, pp.335-64
- Chen, Y. and Tang, F. (1998) "Learning and Incentive Compatible Mechanisms for Public Goods Provision: An Experimental Study," *Journal of Political Economy* Vol.106, pp.633-662
- Chiu, J., and Koeppl, T. (2017) "The Economics of Cryptocurrencies — Bitcoin and Beyond," *Working Papers* 1389, Queen's University, Department of Economics.
- Clarke, E. H. (1971) "Multipart Pricing of Public Goods," *Public Choice* Vol.11, pp 17-33
- Fehr, E. and Gachter, S. (2000) "Cooperation and Punishment in Public Goods Experiments," *American Economic Review* Vol.90, pp.980-994
- Groves, T. (1970) *The allocation of resources under uncertainty*. Doctoral dissertation, University of California, Berkeley.
- Groves, T. (1973) "Incentives in Teams." *Econometrica* Vol.41(4), pp.617-631
- Groves, T. and J. Ledyard (1977) "Optimal Allocation of Public Goods: A Solution to the 'Free Rider' Problem," *Econometrica* Vol.45, pp.783-811.
- Itaya, J. and Schweinberger A.G. (2006) "The Public and Private Provision of Pure Public Goods and The Distortionary Effects of Income Taxation: A Political Economy Approach," *Canadian Journal of Economics* Vol. 39(3), pp.1023-1040.
- Kesternich, M., Lange, A., and Sturm B. (2014) "The Impact of Burden Sharing Rules on The Voluntary Provision of Public Goods," *Journal of Economic Behavior & Organization* Vol. 105(C), pp.107-123.
- Lindahl, E. (1919) "Just Taxation: a positive solution," Reprinted in part in Classics in Theory of Public Finance, R. Musgrave and A. Peacock, Eds., New York, McMillan 1958, pp.168–176.
- Morgan, J. (2000) "Financing Public Goods by Means of Lotteries," *The Review of Economic Studies* Vol.67(4), pp.761–784
- Sefton, M., Shupp, R. and Walker, J. (2007) "The effect of Rewards and Sanctions in Provision of Public Goods," *Economic Inquiry* Vol.45(4), pp.671-690
- Slavov, S. N. (2014) "Public Versus Private Provision of Public Goods," *Journal of Public Economic Theory* Vol.16(2), pp.222-258.

・Varian, H. (1994) “A Solution to the Problem of Externalities when Agents are Well-Informed,” *The American Economic Review* Vol.84(5), pp. 1278-1293
・Villanacci, A., and Zenginobuz Ü. (2006) "Pareto Improving Interventions in A General Equilibrium Model with Private Provision of Public Goods," *Review of Economic Design* Vol. 10(3), pp.249-271.
・Walker, J. M. and Halloran, M. A. (2004) “Rewards and Sanctions and the Provision of Public Goods in One-Shot Settings,” *Experimental Economics* Vol. 7, pp.235-247
・Warr, P. G. (1982) “Pareto optimal redistribution and private charity,” *Journal of Public Economics* Vol.19, pp.131-138
・Warr, P. G. (1983) “The private provision of a public goods is independent of the distribution of income,” *Economic Letters* Vol.13, pp.207-211
・Yamashige S.(2013) “Economic Analysis of Family and Society Change of Japan Society and Policy Response (Kazoku to Shakai no Keizai Bunseki Nihonteki Shakai no Henyou to Seisakuteki Taiou in Japanese)” University of Tokyo Press

Appendix A

From (12), if N is sufficiently large, then the partial differentiation of the reward for mining β' $(\partial\beta' / \partial\alpha_j{}')$ can be expressed as shown below.

$$\frac{\partial\beta'}{\partial\alpha_j{}'} = \frac{\partial}{\partial\alpha_j{}'}\left(1 - \frac{1+\frac{1}{\bar{\alpha}'}}{1+\sum_j \frac{1}{\alpha_j{}'}}\right) = \frac{1}{1+\sum_j \frac{1}{\alpha_j{}'}}\left(\frac{1}{\bar{\alpha}'}\right)^2 \frac{1}{N} - \left(1+\frac{1}{\bar{\alpha}'}\right)\left(\frac{1}{1+\sum_j \frac{1}{\alpha_j{}'}}\right)^2 \left(\frac{1}{\alpha_j{}'}\right)^2$$

$$= \frac{1}{1+\sum_j \frac{1}{\alpha_j{}'}}\left[\left(\frac{1}{\bar{\alpha}'}\right)^2 \frac{1}{N} - \left(1+\frac{1}{\bar{\alpha}'}\right)\frac{1}{1+\sum_j \frac{1}{\alpha_j{}'}}\left(\frac{1}{\alpha_j{}'}\right)^2\right]$$

$$\approx \frac{N}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left[\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right)^2 - \left(\frac{1}{N}+\frac{1}{\sum_{k\neq j}\alpha'_k}\right)\frac{1}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\alpha_j{}'}\right)^2\right]$$

$$\approx \frac{N}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right)\left[\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right) - \frac{1}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\alpha_j{}'}\right)^2\right]$$

$$\approx \frac{N}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right)\left\{\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right) - \frac{1}{\left[\left(1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}\right)\alpha_j{}'+1\right]\alpha_j{}'}\right\}$$

$$\approx \frac{N}{1+\frac{1}{\alpha_j{}'}+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left(\frac{1}{\sum_{k\neq j}\alpha'_k}\right)\frac{1}{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\left[\left(\frac{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}{\sum_{k\neq j}\alpha'_k}\right) - \frac{1}{\left(\alpha_j{}'+\frac{1}{1+\sum_{k\neq j}\frac{1}{\alpha_k{}'}}\right)\alpha_j{}'}\right]$$

Appendix B

From (12), if N is sufficiently large, then the reward for mining β' can be shown as presented below.

$$\beta' = 1 - \frac{1+\frac{1}{[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]\frac{N-1}{N}+\frac{\alpha'_j}{N}}}{1+\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right](N-1)+\frac{1}{\alpha_j{}'}} = 1 - \frac{1}{N}\frac{1+\frac{1}{[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]+\frac{1}{N}\{\alpha'_j-[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]\}}}{\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]+\frac{1}{N}\left\{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]\right\}}$$

$$= 1 - \frac{1}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\frac{1+\frac{1}{[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]\left\{1+\frac{1}{N}\{\alpha'_j-[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]\}/[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}]\right\}}}{1+\frac{1}{N}\left\{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]\right\}/\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}$$

$$\approx 1-\frac{1}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left\{1+\frac{1-\frac{1}{N}\frac{\alpha_j'-\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}{\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}}{\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}\right\}\left\{1-\frac{1}{N}\frac{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}\right\}$$

$$\approx 1-\frac{1}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left\{\left\{1+\frac{1}{\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}\right\}-\frac{\alpha_j'-\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}\frac{1}{N}\right\}\left\{1-\frac{1}{N}\frac{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}\right\}$$

$$\approx 1-\frac{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left\{1-\frac{\alpha_j'-\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}\frac{1}{N}\right\}\left[1-\frac{1}{N}\frac{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}\right]$$

$$\approx 1-\frac{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left\{1-\frac{1}{N}\left\{\frac{\alpha_j'-\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{1+\frac{1}{\alpha_j{}'}-\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}\right\}\right\}$$

$$\approx 1-\frac{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}}{N\left[\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}\right]}\left\{1-\frac{1}{N}\left\{\frac{\alpha_j'}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{\frac{1}{\alpha_j{}'}}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}\right\}\right\}+(\text{Const.})$$

Appendix C

Setting $f(\alpha_j') \equiv \frac{\alpha_j'}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{\frac{1}{\alpha_j{}'}}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}$, $f(\underline{\theta})-f(\overline{\theta})$ changes to the following.

$$f(\underline{\theta})-f(\overline{\theta})>0$$

$$\Leftrightarrow \quad \frac{\underline{\theta}}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{\frac{1}{\underline{\theta}}}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}>\frac{\overline{\theta}}{\left\{1+\frac{1}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]}\right\}\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2}+\frac{\frac{1}{\overline{\theta}}}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}$$

$$\Leftrightarrow \quad \frac{\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}}{\frac{\sigma}{\underline{\theta}}+\frac{(1-\sigma)}{\overline{\theta}}}>\frac{\overline{\theta}-\underline{\theta}}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2+\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}$$

$$\Leftrightarrow \quad \frac{\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}}{\frac{1}{\overline{\theta}}+\left(\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}\right)\sigma}>\frac{\overline{\theta}-\underline{\theta}}{\left[\sigma\underline{\theta}+(1-\sigma)\overline{\theta}\right]^2+\sigma\underline{\theta}+(1-\sigma)\overline{\theta}}$$

$$\Leftrightarrow \quad \frac{\frac{1}{\overline{\theta}}+\left(\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}\right)\sigma}{\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}}<\frac{\left[\overline{\theta}-(\overline{\theta}-\underline{\theta})\sigma\right]^2+\overline{\theta}-(\overline{\theta}-\underline{\theta})\sigma}{\overline{\theta}-\underline{\theta}}$$

$$\Leftrightarrow \quad \frac{\frac{1}{\overline{\theta}}}{\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}}+\sigma<\frac{\left[\overline{\theta}-(\overline{\theta}-\underline{\theta})\sigma\right]^2}{\overline{\theta}-\underline{\theta}}+\frac{\overline{\theta}}{\overline{\theta}-\underline{\theta}}-\sigma$$

$$\Leftrightarrow \quad \frac{\frac{1}{\overline{\theta}}}{\frac{1}{\underline{\theta}}-\frac{1}{\overline{\theta}}}-\frac{\overline{\theta}}{\overline{\theta}-\underline{\theta}}+2\sigma<\frac{\left[\overline{\theta}-(\overline{\theta}-\underline{\theta})\sigma\right]^2}{\overline{\theta}-\underline{\theta}}$$

$$\Leftrightarrow \quad \frac{1}{\frac{\overline{\theta}}{\underline{\theta}}-1}-\frac{\frac{\overline{\theta}}{\underline{\theta}}}{\frac{\overline{\theta}}{\underline{\theta}}-1}+2\sigma<\frac{\left[\overline{\theta}-(\overline{\theta}-\underline{\theta})\sigma\right]^2}{\overline{\theta}-\underline{\theta}}$$

$$\Leftrightarrow \quad 2\sigma - 1 < \frac{\left[\overline{\theta} - (\overline{\theta} - \underline{\theta})\sigma\right]^2}{\overline{\theta} - \underline{\theta}}$$

$$\Leftrightarrow \quad 2\sigma - 1 < \left(\overline{\theta} - \underline{\theta}\right)\sigma^2 - 2\overline{\theta}\sigma + \frac{\overline{\theta}^2}{\overline{\theta} - \underline{\theta}}$$

$$\Leftrightarrow \quad \left(\overline{\theta} - \underline{\theta}\right)\sigma^2 - 2\left(1 + \overline{\theta}\right)\sigma + 1 + \frac{\overline{\theta}^2}{\overline{\theta} - \underline{\theta}} > 0$$

$$\Leftrightarrow \quad \sigma < \frac{1+\overline{\theta} - \sqrt{\left(1+\overline{\theta}\right)^2 - \left(\overline{\theta}-\underline{\theta}\right)\left(1+\frac{\overline{\theta}^2}{\overline{\theta}-\underline{\theta}}\right)}}{\overline{\theta}-\underline{\theta}} \quad \text{or} \quad \sigma > \frac{1+\overline{\theta} + \sqrt{\left(1+\overline{\theta}\right)^2 - \left(\overline{\theta}-\underline{\theta}\right)\left(1+\frac{\overline{\theta}^2}{\overline{\theta}-\underline{\theta}}\right)}}{\overline{\theta}-\underline{\theta}}$$

$$\Leftrightarrow \quad \sigma < \frac{1+\overline{\theta} - \sqrt{1+\overline{\theta}+\underline{\theta}}}{\overline{\theta}-\underline{\theta}} \quad \text{or} \quad \sigma > \frac{1+\overline{\theta} + \sqrt{1+\overline{\theta}+\underline{\theta}}}{\overline{\theta}-\underline{\theta}}$$

Considering $\sigma \in [0,1]$, the equation above is equal to the following.

$$\Leftrightarrow \quad \sigma < \frac{1+\overline{\theta} - \sqrt{1+\overline{\theta}+\underline{\theta}}}{\overline{\theta}-\underline{\theta}}$$

If the upper limit $\overline{\theta}$ is sufficiently large, then the right-hand-side of this inequality can be expressed as shown below.

$$\lim_{\overline{\theta} \to \infty} \frac{1+\overline{\theta} - \sqrt{1+\overline{\theta}+\underline{\theta}}}{\overline{\theta}-\underline{\theta}} = \lim_{\overline{\theta} \to \infty} \frac{\overline{\theta} - \sqrt{\overline{\theta}}}{\overline{\theta}} = 1$$