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Firm Value and Retained Earnings:
Optimal dividend policy with retained earnings

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Abstract

We propose a model of dynamic investment, financing, and risk management with retained earnings. The key contribution of this chapter is to provide a dynamic model which explicitly includes retained earnings and equity issuance costs as friction. To consider the retained earnings explicitly, we describe the dynamics of both the asset and liability section of the balance sheet, i.e., cash holdings and (physical) property in the asset section, and stock and retained earnings in the equity section. Our key results are: (1) the firm retains its earnings when its productivity is high and cash-capital ratio is low, and (2) the optimal rate of cash holdings increases when the volatility of productivity shock is high and decreases when risk-neutral mean productivity shock is low.

Keywords: Dynamic capital structure, Corporate finance, Optimal cash management.

JEL classification number: G32; G35.

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1 Introduction

Since the Modigliani and Miller (1958), a sizable literature has investigated to understand firms' financing policies. Early standard model of investment under uncertainty assume that capital credit markets are friction-less so that firms are always able to secure funding or borrowing for appropriate projects and cash reserves are assumed to be irrelevant to the firm's value. These models faced questions by a large number of empirical studies and this leads to literature treating uncertainties or frictions. A literature to treat uncertainty reveals that if a firm faces liquidity risk, the firm accumulates large cash balances to avoid bankruptcy. Another, but related literature treating frictions devotes main efforts to understand the effects of frictions, such as corporate taxes, credit (for example, Hugonnier et. al. (2015)) and equity market frictions, etc.

Another related and important literature to our work is the dividend policy, sometimes mentioned as the "dividend puzzle". According to de Matos (2001),

... although there is no consensus in the marketplace on the need and importance of payout policies, most managers and some academics believe the policies affect the value of firms. Based on the empirical studies, the answer to the dividend puzzle — namely, to understand why firms insist on paying dividends if they are supposed to be irrelevant to the value of the firm — seems to be that the payment of dividends has a natural market among the inframarginal investors who can make some tax-based arbitrage profit.

In this paper, we address both the financial structure problem and the dividend puzzle by introducing retained earnings, which relates closely with dividend, explicitly into current dynamic capital structure model.

The major reason to focus the retained earnings is current corporate financing structure in macro level. Despite many related literature, not many researches set its focus on the retained earnings which attract interests in the global, especially in advanced economies. For instance, OECD (2015) pointed out the importance to consider the retained earnings as follows;

In advanced economies between 1995 and 2010, it is estimated that on average 66% of corporate investments were financed by shareholder capital in the form of retained earnings. In emerging market economies on the other hand, only 25% of corporate investments were financed by retained earnings.

Although the retained earnings occupies majority of the actual corporate finance, the current corporate financial theory does not provide any theoretical framework to explain the phenomenon. In this paper we address this issue by modifying the dynamic capital structure model. The retained earnings are usually defined by a cumulative corporate profit that are not paid out to the shareholders as a dividends or share buybacks kept in the company. The retained earnings are usually defined by a cumulative corporate profit that are not paid out to the shareholders as a dividends or share buybacks kept in the company. The inclusion of the retained earnings to the currently established model enables us to describe the dynamic and long term firm's growth policy by introducing dividend rate and other related variables.

Papers most closely related to ours is Bolton et. al. (2011) and Décamps et. al. (2011). Both our paper and Bolton et. al. (2011) employs simple AK model and assume the firm's cumulative productivity evolves with a standard Brownian motion under the risk-neutral measure.

Bolton et. al. (2011) assume financial frictions directly instead of explicitly modeling an agent problem. The frictions assumed in Bolton et. al. (2011) are related to the features of the optimal contract that motivate effort. Specifically, they assume that the firm maintains a cash balance and that it is costly to issue new equity when the firm runs out of cash. In addition, they assume that it is costly to keep cash inside the firm instead of paying it out to shareholders. On the other hand, Décamps et. al. (2011) analyzed model of a firm facing internal agency costs of free cash flow and equity issuance cost such as professional fee, commissions, etc.

The feature of our model is that we consider a firm facing explicit external financing cost and allowed to reserve retained earnings with the dividend policy (or, payout policy) taking into account.

Also, the analysis on the heterogeneity in the drift of productivity shock would be another feature of our analysis. In the numerical calculation, Bolton et. al. (2011) sets the drift of productivity shock as homogeneous across firms. Bolton et. al. (2013) changed this setting by considering it as a state variable, but the drift term has only 2 options to take. As our concern is macroeconomic dynamics under heterogeneous agent, it is natural to eliminate the restriction on the level of the drift of productivity shock. To consider the aggregation, it would be reasonable to consider that there is a distribution on the drift of productivity shock, or growth, and calculate macroeconomic variable by integrating across the distribution. So here it should be again emphasized that we consistently focus the heterogeneity of agent and calculate macro variable by integrating across the heterogeneity.

The reminder of this chapter proceeds as follows. Section 2 sets up our baseline model. Section 3 presents the model solution. Section 4 continues with quantitative analysis and Section 5 concludes.

2 The Model

2.1 Definition of the Variables

This section presents our model of dynamic capital structure choice with uncertain productivity shock. The basic framework closely follows Bolton et. al. (2011). In our model, we add a subsection of the "Firm's Asset and Capital" and consider the capital structure explicitly.

2.1.1 Production and Investment

We consider a financially constrained firm with stochastic productivity evolution, as considered in Bolton et. al. (2011). Firstly we describe the firm's physical production and investment process.

$$dK_t = (I_t - \delta K_t)\, dt \tag{2.1}$$

Secondly we assume the firm's revenue at time t to evolve proportionally to its capital stock, just assumed in the simple AK model. The dynamics of A is governed by two terms; a constant growth term described in μ and a stochastic term.

$$dA_t = \mu dt + \sigma dZ_t \tag{2.2}$$

where Z is a standard Brownian motion.

2.1.2 Firm's Asset and Capital

Consider a firm whose asset is composed of cash inventory (W_t) and property (K_t), e.g., productive facilities, and its financial resource is composed of stock, or namely shareholder's equity (S_t) and retained earnings (E_t). This definition of variables leads an identical equation for the asset section and shareholder's equity section of the balance sheet as;

$$W_t + K_t = S_t + E_t \;\; (dW_t + dK_t = dS_t + dE_t)\,. \tag{2.3}$$

The schematic image of the asset and equity section is in the Figure 2.1. The firm uses its property (K_t) for the production and reserves the cash inventory (W_t) for the risk management. The financial resource of these asset is composed of the shareholder's equity (S_t) and retained earnings (E_t).

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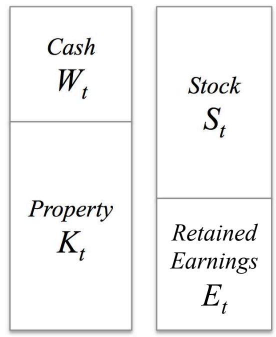

Figure 2.1: Simplified Balance Sheet without Liability

2.1.3 Profit and Firm Value Maximization

The increase in firm's cash flow (dW_t) and net income ($Y_t dt$) during a time period dt can be described as

$$\begin{aligned} dW_t &= K_t dA_t + rW_t dt - I_t dt - G\left(I_t, K_t\right) dt + \tfrac{1}{p} dS_t - dL_t \\ Y_t dt &= K_t dA_t + rW_t dt - \delta K_t dt - G\left(I_t, K_t\right) dt - \left(1 - \tfrac{1}{p}\right) dS_t \end{aligned} \tag{2.4}$$

where r is an interest rate, δK_t ($\delta \geqq 0$) represents a depreciation of the physical stock K_t, $G\left(I_t, K_t\right)$ is the additional adjustment cost that firm incurs in the investment process, dL_t is the dividend process and p (> 1) is equity issuance cost as a friction for each dollar of new shares issued as is assumed in Décamps et. al. (2011). As supposed in other literature such as Demarzo et. al. (2012), we assume that the adjustment cost satisfies $G\left(0, K_t\right) = 0$, is smooth and convex in investment I_t, and is homogeneous of degree one in I_t and K_t. We assume that there is no tax non-operational revenue nor expenditure for the firm at first. Then the net income is distributed to the shareholder as a dividend (dL_t) and reserved in the firm as a retained earnings (dE_t).

$$Y_t dt = dL_t + dE_t \tag{2.5}$$

where L_t is the cumulative dividend process. Following Décamps et. al. (2011), the value of the firm is the difference between the expected present value of all future dividends and the expected present value of all future gross issuance process, that is,

$$V = max \left[E_0 \int_0^{\tau} e^{-rt} (dL_t - dS_t) \right] + e^{-r\tau} \left(K_\tau + W_\tau\right) \tag{2.6}$$

where τ is the liquidation time and E_0 is the expectation operator induced by the firm's maximization process starting at $t = 0$. If $\tau = \infty$, then the firm never chooses to liquidate. A firm may liquidate when the cost of financing is too high, or it faces the failure of heirs, etc.

2.2 Optimal Choice; Capital Expansion and Payout Policy

The firm can choose the ratio of the shareholder's equity S_t and retained earnings E_t. We assume that the firm chooses the ratio of shareholder's equity (dS_t) and retained earnings (dE_t) during a time period dt so that the summation of each value equals to the summation of dW_t and dK_t. Defining the ratio of dS_t and dE_t at time t as $\alpha_t$to describe the dependence as follows;

$$dE_t = \alpha_t \left(dW_t + dK_t\right), \; dS_t = \left(1 - \alpha_t\right)\left(dW_t + dK_t\right). \tag{2.7}$$

If the ratio of S_t and E_t converges to the certain value, the ratio of dS_t and dE_t also converges to the same value and therefore $E_\infty = \frac{\alpha_\infty}{1-\alpha_\infty} S_\infty$.

Secondly, a firm also reserve a right to choose its payout policy directly by setting the ratio of the dL_t and dE_t, whose summation equals to the firm's net income, according to (2.5). If we define the ratio of the dL_t and dE_t as $\beta_t \left(0 \leq \beta_t \leq 1\right)$ for every time t, the retained earnings and dividend process during a time period dt is described as a function of the firm's net profit and β_t as;

$$dE_t = \beta_t Y_t dt, \; dL_t = \left(1 - \beta_t\right) Y_t dt. \tag{2.8}$$

2.3 Cash Inventory Dynamics and the Firm Value

Hereforward, we first calculate the increase of the cash inventory dW_t. By using (2.7) and (2.8), $dS_t = \left(1 - \alpha_t\right) \frac{\beta_t}{\alpha_t} Y_t dt$. Defining $\frac{\beta_t}{\alpha_t}$ as γ_t, the firm's net profit can be described in;

$$Y_t dt = \frac{1}{1 + \gamma_t \left(1 - \frac{1}{p}\right)\left(1 - \alpha_t\right)} \left\{K_t \left(\mu dt + \sigma dZ_t\right) + rW_t dt - \delta K_t dt - G\left(I_t, K_t\right) dt\right\} \tag{2.9}$$

Also, the firm's cash inventory can be calculated by using (2.4) and (2.8) as;

$$\begin{aligned} dW_t &= Y_t dt - dK_t + dS_t - dL_t \\ &= \gamma_t Y_t dt - \left(I_t - \delta K_t\right) dt \end{aligned} \tag{2.10}$$

Next, we can calculate the difference between the expected present value of all future dividends and the expected present value of all future gross issuance process $dL_t - dS_t \equiv f\left(W_t, K_t\right) dt$ by using (2.4), (2.7) and (2.8) as

$$f\left(W_t, K_t\right) dt = \left(1 - \gamma_t\right) Y_t dt. \tag{2.11}$$

2.4 Simplification and Hamilton-Jacobi-Bellman Equation

Calculations so far revealed that our assumptions leads the firm's maximization problem to be described by 2 stochastic variables (W_t and K_t) and 3 parameters to be defined by maximization condition (α_t, β_t and I_t). However, the firm's maximization problem can be reduced to a 1 stochastic variable problem by exploiting homogeneity, i.e., writing the firm value $V(K,W) = K \cdot v(w)$ where $w = {}^{W}/_{K}$. By defining ${}^{W_t}/_{K_t}$ as w_t and similarly for other variables, the key equations for the time development of the stochastic variable (2.9) can be re-described as:

$$\begin{aligned} dw_t &= d\left(\frac{W_t}{K_t}\right) = \frac{dW_t}{K_t} - w_t\left(i_t - \delta\right)dt = \frac{\gamma_t Y_t dt}{K_t} - \left(i_t - \delta\right)dt - w_t\left(i_t - \delta\right)dt \\ &= \left[\frac{\gamma_t}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\left\{\mu + rw_t - \delta - g\left(i_t\right)\right\} - \left(1+w_t\right)\left(i_t - \delta\right)\right]dt + \frac{\gamma_t\sigma}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}dZ_t \end{aligned} \tag{2.12}$$

here we also assumed that $G_t(I_t, K_t)$ to be ${}^{G_t(I_t,K_t)}/_{K_t} = g\left({}^{I_t}/_{K_t}\right) = g\left(i_t\right) = {}^{\theta i_t^2}/_{2}$. Similarly, the firm's dividend minus issuance process can be reduced into a form by dividing K_t as

$$\begin{aligned} \frac{1}{K_t}f\left(w_t\right)dt &= \frac{1}{K_t}\left(1-\gamma_t\right)Y_t dt \\ &= \frac{1-\gamma_t}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\left\{\mu + rw_t - \delta - g\left(i_t\right)\right\}dt + \frac{\left(1-\gamma_t\right)\sigma}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}dZ_t \end{aligned} \tag{2.13}$$

According to (2.12) and (2.13), the firm value $v(w_t)$ satisfies the following the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} rv\left(w_t\right) = \max_{\alpha_t,\gamma_t,i_t} & \left[\frac{1-\gamma_t}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\left\{\mu + rw_t - \delta - g\left(i_t\right)\right\}\right. \\ & + \left[\frac{\gamma_t}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\left\{\mu + rw_t - \delta - g\left(i_t\right)\right\} - \left(1+w_t\right)\left(i_t-\delta\right)\right]v^{(1)}\left(w_t\right) \\ & \left. + \frac{1}{2}\left\{\frac{\gamma_t\sigma}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\right\}^2 v^{(2)}\left(w_t\right)\right] \end{aligned} \tag{2.14}$$

where $v^{(i)}$ represents $\frac{\partial^i v}{\partial w^i}$. This equation equals to the neoclassical benchmarks when there is no friction, i.e., $\gamma_t = 0$ ($\Longleftrightarrow \beta_t = 0$, i.e., no retained earnings) and $p = 1$.

2.5 Tax Distortion

The effect of tax distortions has been considered since the Modigliani and Miller (1958). The major reason to consider the tax distortion is that if the model include debt for its financial source, the tax benefit of debt appears because the interest is paid before before the taxation. In our model we did not include the debt and therefore no tax distortion is expected. Actually, if we are to consider the taxation in this economy, the equation (2.9) becomes;

$$Y_t dt = \frac{1-\tau}{1+\gamma_t\left(1-\frac{1}{p}\right)\left(1-\alpha_t\right)}\left\{K_t\left(\mu dt + \sigma dZ_t\right) + rW_t dt - \delta K_t dt - G\left(I_t, K_t\right)dt\right\} \tag{2.15}$$

where τ is the corporate tax rate ($0 \leq \tau \leq 1$). The derivation of the Hamilton-Jacobi-Bellman equation is straight forward based on this equation. For the simplicity, we set τ to be zero as the taxation does not provide any distortion in our model.

3 Model Solution

3.1 First Best Benchmark

Before considering the firm's dividend policy with issuance cost, we examine the benchmark case in which such cost is absent. In the case without the financial friction, the equity issuance cost $p = 0$ and there is no noise to hide the agent's action, i.e., $\sigma = 0$. Also, because there are no issuance costs in the benchmark economy, hoarding cash reserves does not bring any benefit to shareholders. It is therefore optimal for the firm not to hold any cash for all the time $w_t = 0 \ for \ ^{\forall}t \geqq 0$. In the absence of other financial frictions, the Modigliani and Miller logic applies. If we go back to the equation (2.1) and assume i to be constant across time, the firm's physical property evolves as $K_t = K_0 e^{(i-\delta)t}$. Next the difference between the expected present value of all future dividends and the expected present value of all future gross issuance process evolves as the following;

$$dL_t - dS_t = \frac{\alpha_t - \beta_t}{\alpha_t} \left(\mu + rw - \delta - g\left(i\right)\right) K_t dt \tag{3.1}$$

where $0 \leq \alpha_t, \beta_t \leq 1$. As $\frac{\alpha_t - \beta_t}{\alpha_t}$ is increasing with respect to α_t and decreasing with respect to β_t, the firm chooses α_t to be 1 and β_t to be 0 to maximize its firm value. This leads to the firm's value to be

$$\begin{aligned} V_{FB} &= \max_{i_t} \int_0^\infty e^{-rt} \left(dL_t - dS_t\right) \\ &= \max_{i_t} \int_0^\infty \left(\mu - \delta - g\left(i\right)\right) K_0 e^{(i-\delta-r)t} dt \\ &= \max_{i_t} \left[\tfrac{K_0}{r+\delta-i} \left(\mu - \delta - g\left(i\right)\right)\right] \end{aligned} \tag{3.2}$$

Here we assumed that $r + \delta - i$ to be positive. In this framework, the Tobin's average q calculated is described as;

$$q_{FB} = \max_{i_t} \left[\frac{\mu - \delta - g\left(i\right)}{r + \delta - i}\right]. \tag{3.3}$$

This expression slightly different from other literature such as Bolton et. al. (2011) and Demarzo et. al. (2012). In the standard previous literature, the numerator of the average Tobin's q is $\mu - i - g\left(i\right)$, despite our model is $\mu - \delta - g\left(i\right)$. The fundamental difference in the expression of numerator is the determining process of the dividend. In the previous literature, the dividend is calculated based on the cash flow of each period. On the other hand, in our model, dividend is calculated based on the profit of the period. This difference arises because our model sets its focus on the retained earnings. In the account process, retained earnings is defined as the profit after dividend payment. To describe this feature, we set firms decision process to determine its dividend not from the current cash flow but from firm's net profit, or profit after taxation.

3.2 Comparative Statistics

The Hamilton-Jacobi-Bellman equation (2.14) describes the optimization control of the firm value under uncertainty and frictions at every time t. In this section we consider the comparative statistics under the financial frictions, i.e., equilibrium analysis of the Hamilton-Jacobi-Bellman equation. As the second order differential equation is highly complicated, it is difficult to provide theoretical solution of the equation. However, it is still possible to provide several parameter restriction with simple calculations.

The first issue to be considered in this section is the range of γ_t. As γ_t is defined as $\frac{\beta_t}{\alpha_t}$ and $0 \leq \alpha_t, \beta_t \leq 1$, it is natural to set the range of γ_t to be $0 \leq \gamma_t < \infty$. However, according to (2.13) the source of the firm value turns to be negative when $\gamma_t > 1$. It is still possible if $dL_t - dS_t$ tentatively turns to be negative due to large productivity shocks or any other accidents, but such action is not sustainable and can not be achieved in equilibrium. Therefore, in the comparative statistics, it is reasonable to set γ_t to be $0 \leq \gamma_t \leq 1$ and therefore $\alpha_t > \beta_t$.

The second issue is the dynamics of the cash holdings, i.e., (2.12). In the comparative statistics, or the equilibrium analysis, it is also reasonable to assume $E_0 \left[dw_t\right]_{t\to\infty}$ to be zero. This restriction actually provide the condition for the investment rate i_t to satisfy;

$$\frac{\gamma_t}{1+\gamma_t\left(1-\frac{1}{p}\right)(1-\alpha_t)}\left\{\mu + rw_t - \delta - g\left(i_t\right)\right\} = \left(1+w_t\right)\left(i_t - \delta\right). \tag{3.4}$$

(3.4) provides the relation between i_t and w_t. The appropriate cash holding rate w_t can be calculated numerically from (2.14) and using (3.4) to provide the appropriate rate of investment in the equilibrium. However, here the problem occurs as there are two independent methodology to determine investment rate, one is from maximization condition of (2.14) and another is the restriction on the dynamics of cash holdings (3.4).

The third issue is the first order condition of (2.14). It is straightforward to calculate the first order condition of (2.14) for 3 variables, α_t, γ_t and i_t. The first order condition for the three variables are as follows;

$$\begin{aligned}
\left[i_t\right]: &\quad i_t = -\frac{1}{\theta}\frac{v'(1+w_t)}{1-\gamma_t+\gamma_t v'}\left\{1+\gamma_t\left(1-\frac{1}{p}\right)(1-\alpha_t)\right\} \\
\left[\alpha_t\right]: &\quad \left(1-\gamma_t+\gamma_t V\right)\left(\mu+rw_t-\delta-g\left(i_t\right)\right) \\
&\quad +\frac{\gamma_t^2\sigma^2}{1+\gamma_t\left(1-\frac{1}{p}\right)(1-\alpha_t)}v'' = 0 \\
\left[\gamma_t\right]: &\quad \left(\mu+rw_t-\delta-g\left(i_t\right)\right)\left\{v'-1-\left(1-\frac{1}{p}\right)(1-\alpha_t)\right\} \\
&\quad +\frac{\gamma_t\sigma^2 v''}{1+\gamma_t\left(1-\frac{1}{p}\right)(1-\alpha_t)} = 0
\end{aligned} \tag{3.5}$$

Again, we have another relation i_t and w_t as (3.5). However, the problem of the equation (3.5) is that the right hand side is negative as long as $v' > \frac{\gamma_t - 1}{\gamma_t}$. In some parameter region the v' not necessarily satisfies the relation and in such case it is optimal to set i_t to be negative. Despite the relation, it is not reasonable to set investment as negative in the comparative statics and in that sense we need further restriction for the parameter γ_t for the firm value to be plausible, which is determined along with the functional type of the firm value $v\left(w\right)$.

4 Quantitative Analysis; Numerical Solution for the HJB Equation

The analytical solution of the HJB equation shall be calculated by solving second order differential equation with maximization condition written in (3.5). However, it is difficult to analyze the second order differential equation (2.14) under (3.5) analytically. Therefore here we conduct numerical analysis to consider the firm value maximization through adjusting α, β and i under the productivity shock delivered with volatility σ.

Here the difference between the analysis of Bolton et. al. (2011) should be emphasized. In standard economic model, a solution after substituting first order condition (3.5) into (2.14) are calculated numerically, just like Bolton et. al. (2011). As a difference from the model of Bolton et. al. (2011), our model includes 3 parameters (i.e. α, β and i) for the firm to adjust and therefore the introduction of the first order condition to the HJB equation becomes much complicated. Taking such situation into consideration, we employed to estimate the appropriate parameters value by scanning through the parameter space. The range of the parameter space could be described as $0 \leq \alpha_t, \gamma_t \leq 1$ and $i_t < r+\delta$ (from equation (3.3)).

Parameters	Symbol	Value
Risk-free rate	r	6%
Rate of depreciation	δ	10%
Risk-neutral mean productivity shock	μ	20%-80%
Volatility of productivity shock	σ	10%-30%
Capital Expansion Policy	α	0%-100%
Payout Policy	β	0%-100%
Investment	i	-100%-16%
Adjustment cost parameter	θ	1.5
Equity issuance cost	p	1.05

Table 1: Summary of Key Parameters

This table summarizes the symbols for the key variables used in the model and the parameter values in the benchmark cases. The risk-free rate and risk-neutral mean productivity shock varies by cases. Regarding the range of the investment, the first best benchmark calculation exerts relation as $i_t < r+\delta$ and according to this relation, $i_t < 16\%$. However, the numerical calculation of the firm value-capital ratio diverges in high investment ratio such as 14%. Therefore we set the upper limit of investment rate as 12%.

Table 1 summarizes the parameter value for the parameter scanning. Most variables are the same as Bolton et. al. (2011) and Décamps et. al. (2011) except for α, β and i . In addition, we allow the one parameter, Risk-neutral mean productivity shock, to change identically within the range written in the Table 1 for the purpose of analyzing the difference under the growth rate.

Also, for the numerical calculation, we have to consider the upper limit of w, or "Payout Region". In Bolton et. al. (2011), the Payout Region is calculated around 0.2, we employ the result and assume $w_t \leq 0.25$.

4.1 Brief Protocol of the Numerical Calculation

The numerical calculation aims to solve the second order differential equation (2.14) under the first order condition (3.5) to determine the level of α, β and i. However, the first order conditions (3.5) are complicate and (2.14) becomes too complicated when we introduce the solutions of (3.5) into (2.14).

To solve this problem, we firstly assume some initial value for α, β and i and solve the differential equation (2.14) to derive the functional type of $v(w)$ by the Runge-Kutta 4th order. Secondly we calculate the first order condition with using the derived $v(w)$ to obtain the solution of the α, β and i. Thereafter we introduce 1) the solution of the first order condition, 2) maximum and minimum value of the α, β and i into (2.14) and derive $v(w)$ again by the Runge-Kutta 4th order. This process yields 3 different value of $v(w)$ and therefore we can identify which α, β and i maximizes the value function. We iterate this process until the change in α, β and i, which maximizes the value function, becomes quite small and below externally given threshold value (in our model, the sum of squares becomes below 10^{-4}). In most cases, 2 to 3 iteration is enough to meet the condition.

In addition to the protocol written above, we set following conditions to proceed the calculation. In detail, we show actual MATLAB code in Appendix.

4.1.1 The Minimum Value of α, γ and i

According to the definition, the domain of α, γ is set as $0 \le \alpha \le 1$ and i is set as $i < 16\%$. In the following numerical calculation, we assume the minimum value of α, γ and i as 10^{-4} for the purpose of avoiding divergence of the firm value. The parameter α, γ and i sometimes becomes the denominator during the numerical calculation (e.g., α becomes denominator to calculate γ, as γ is defined by β/α). Therefore we set the minimum value for α, γ and i as 10^{-4} for the numerical calculations to continue.

4.1.2 Condition for the Coefficient of the $v^{(1)}(w_t)$ in (4.14)

The coefficient of the $v^{(1)}(w_t)$ in (4.14) represents the drift of w_t with respect to the time. In the numerical calculation, we assume the drift of w_t to be non-negative for the following reason. In case of the negative drift, the firm has to dissipate its cash holdings under shoestring operation and will face bankrupt when the firm runs out of its cash. Firms get into such situation when 1) the growth rate of income is low (e.g., low productivity growth or low interest rate), or 2) low γ_t. The first condition corresponds to a condition for the exogenous variable. On the other hand, the second one is an endogenous matter. According to equations (2.7) and (2.8), the relation between the profit, γ_t and the increase in asset becomes $dW_t + dK_t = \gamma_t Y_t dt$. This relation well describes the effect to choose low γ_t. In case of low γ_t, the firm's financial and physical asset does not increase even in a case of profitable company due to high dividend rate. However, it is not reasonable for the firm to pay much dividend above its affordability at the expense of shoestring operation or even bankruptcy.

4.2 Firm Value Capital Ratio

Figure 4.1 plots the calculated Firm Value-Capital Ratio $v(w)$ by solving (2.14) with following previous protocol under $\sigma = 0.10$. As is clearly seen, the firm value increases as w increases and this result corresponds to that of Bolton et. al. (2011). When we compare the firm value growth with respect to w, the firm value clearly decreases as the μ increases.

Figure 4.2 plots the derivative of the firm value with respect to w for different μ, and the dashed line corresponds to $v'(w) = 1$. When $v'(w) < 1$, one unit of the increase in cash holdings results less than one unit of increase in the firm value. In such case, the firm is assumed not to increase the cash holdings but pay the extra cash to the shareholders as a dividend. Therefore the domain of w satisfying $v'(w) < 1$ could be defined as the Payout Region. Although the Payout Region in Bolton et. al. (2011) starts from around 20%, the Payout Region in our model becomes far more away in case of $\mu = 0.20$ and $3\% \sim 9\%$ in other cases. This result indicates that the less the firm's productivity becomes, the more cash holdings the firm requires (high w). This result mainly comes from the level of productivity growth for each cases. In case of high productivity firm, the income rapidly increases without increasing cash holdings, owning to its high productivity growth. On the other hand, in the low productivity growth case, the income growth rate becomes low as long as the firm's cash holdings are not adequate. Therefore the firm increases its cash holdings to increase its income growth by interest.

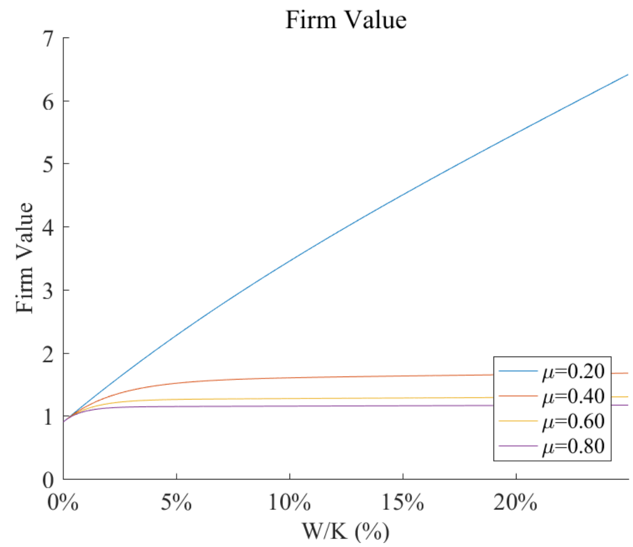

Figure 4.1: Firm Value under $\sigma = 0.1$ and $\mu = 0.2 \sim 0.8$

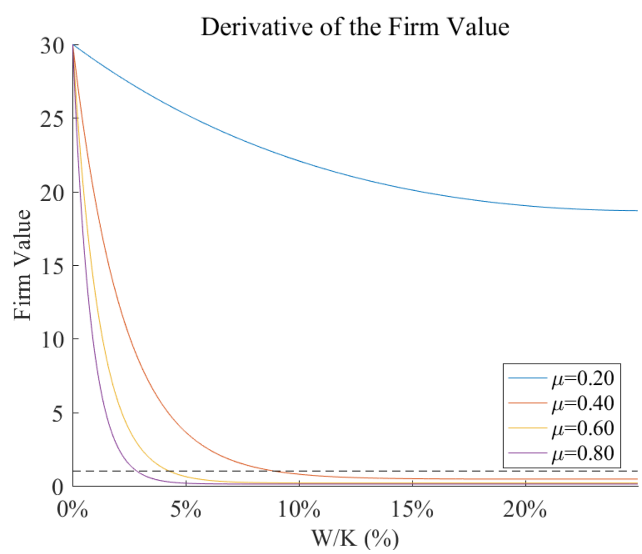

Figure 4.2: Derivative of the Firm Value under $\sigma = 0.1$ and $\mu = 0.2 \sim 0.8$

4.3 The Policy Functions

Next let us discuss the policy functions of α, γ and i. Figure 4.3 to 4.5 represents the policy functions of each parameters with respect to w. It should be noted that the policy function of γ and i becomes constant, and the shape of policy function of α becomes the Heaviside step function. Actually, the value of γ and i is bound to its maximum value under its domain ($0 \leq \gamma_t \leq 1$ and $i_t < r + \delta$). On the other hand, the value of α changes as w increases, except for the case $\mu = 0.20$. In high productivity case ($\mu \geq 0.40$), the firm value maximized when i) α is bounded to its maximum value ($\alpha = 1$) in low w and also ii) bounded to its minimum value ($\alpha \simeq 0$) in high w. This indicates that when the firm's cash inventory decreases by some shocks, the firm i) mainly finances from the retained earnings and

ii) most of profits are not paid as dividend but used as a retained earnings. Also, when the firm's cash inventory increases and exceeds certain limit, the firm i) mainly finances from the market through the capital increase and ii) most profits are paid as dividend.

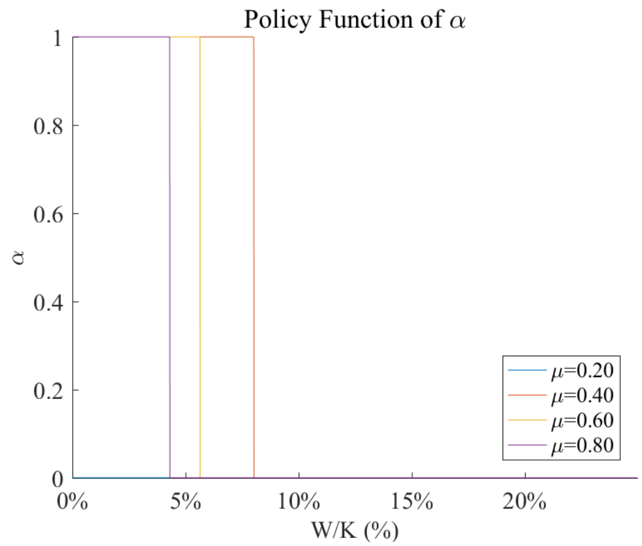

Figure 4.3: Policy function of α for several cases of Productivity ($\mu = 20\% \sim 80\%$)

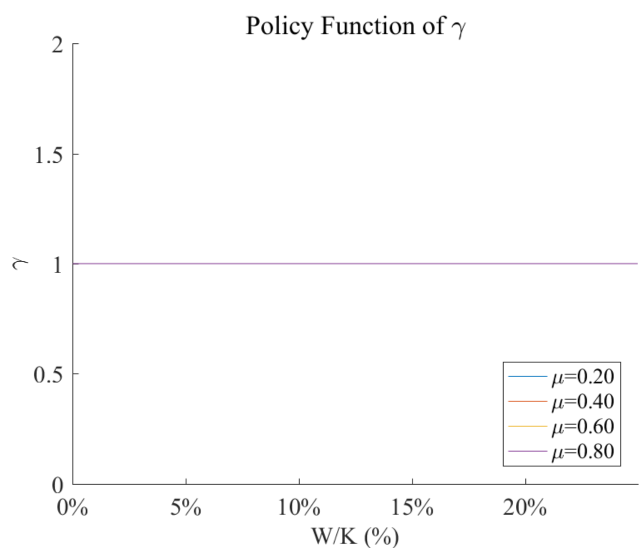

Figure 4.4: Policy function of γ for several cases of Productivity ($\mu = 20\% \sim 80\%$)

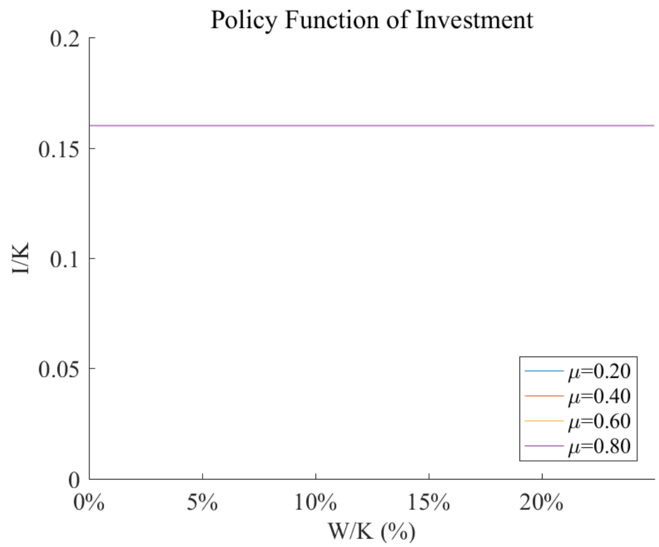

Figure 4.5: Policy function of i for several cases of Productivity ($\mu = 20\% \sim 80\%$)

4.4 The Solutions of the First Order Conditions

As described in the previous section, the policy functions are only composed of the boundary value of each domain, and does not described by the solution of the first order conditions (4.20). However, during the numerical calculation, we firstly calculate the solution of the first order conditions and plug the solution into the differential equation as long as the solution-pair stays within the domain $0 \leq \alpha_t \leq 1$, $0 \leq \gamma_t \leq 1$ and $i_t < r + \delta$. The Figure 4.6 to 4.8 illustrates the solutions of first order conditions in case of $\mu = 20\%$. We set 10 initial value for solving the equations and therefore Figure 4.6 to 4.8 includes 10 lines as solutions. Although the volatility of each solution is high in some range, mainly solutions of α are i) around 100 and ii) around 6, solutions of γ are around 0.2 and around -3, and solutions of i are around 0.4 and -0.4. As is clearly seen, the first order condition for α mostly exceed the defined domain $0 \leq \alpha_t \leq 1$ and this is the reason the policy function does not include the solution of the first order condition.

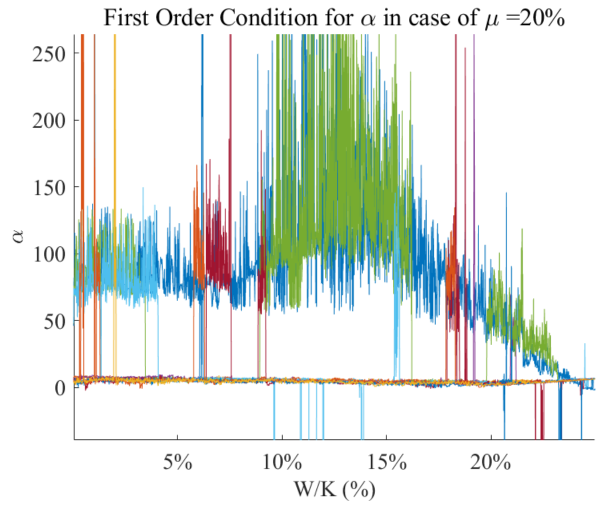

Figure 4.6: Solutions of First Order Conditions on α for several initial value in $\mu = 20\%$

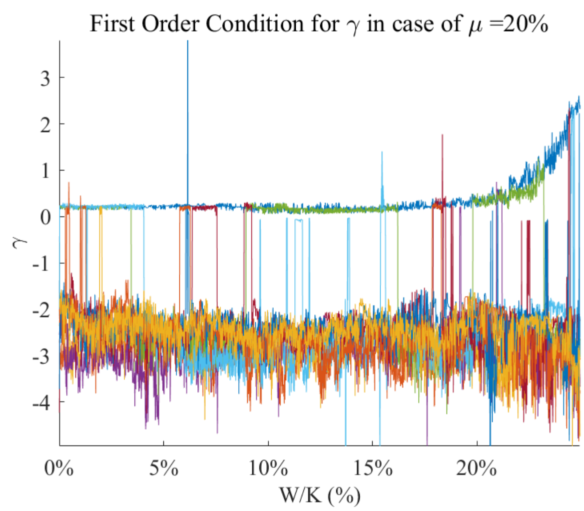

Figure 4.7: Solutions of First Order Conditions on γ for several initial value in $\mu = 20\%$

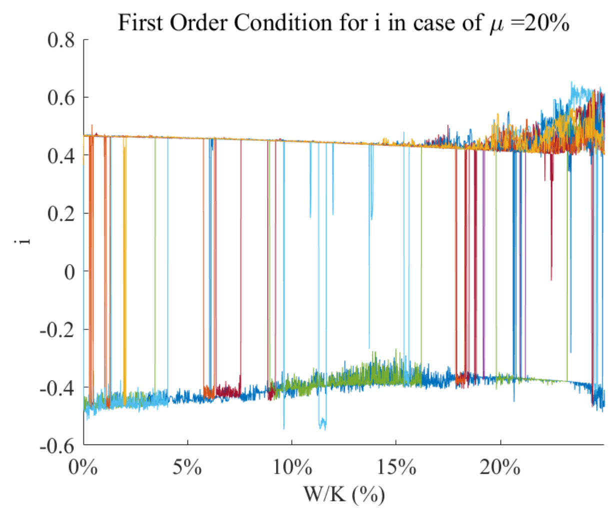

Figure 4.8: Solutions of First Order Conditions on i for several initial value in $\mu = 20\%$

4.5 Firm Value-Capital Ratio for Different σ

It is also meaningful to understand the difference of the dynamics when the volatility of productivity shock changes. The Figure 4.9 and 4.10 illustrates the firm value and its derivative with respect to w under $\sigma = 0.3$. It is interesting the derivative of the firm value sill exceed 1 when $w = 25\%$ for all cases of productivity growth. Comparison of the numerical calculation for different σ reveals that the increase in σ results in the shift of the value function $v(w)$. This result leads when the volatility of the productivity shock increases, the firm requires more cash holdings to maximize its firm value.

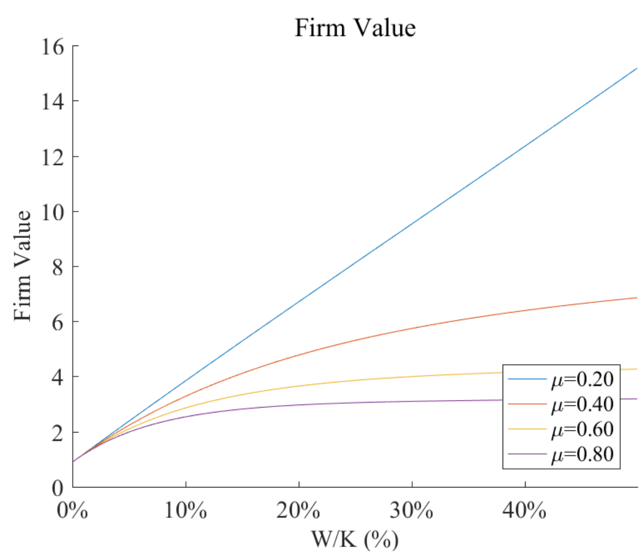

Figure 4.9: Firm Value under $\sigma = 0.1$ and $\mu = 0.2 \sim 0.8$

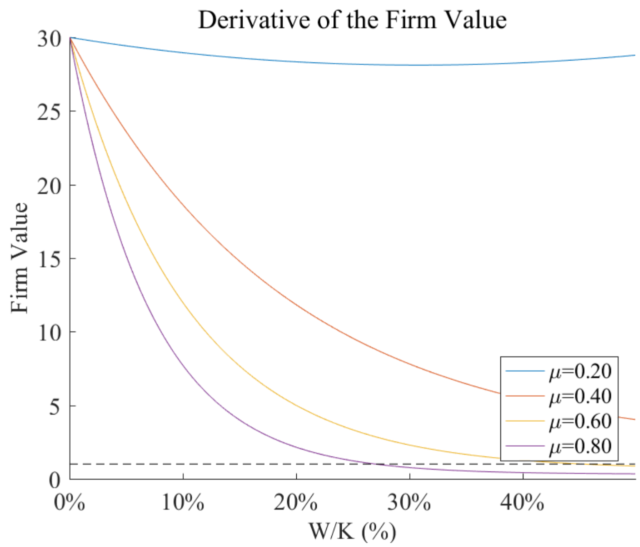

Figure 4.10: Derivative of the Firm Value under $\sigma = 0.1$ and $\mu = 0.2 \sim 0.8$

Next let us discuss the policy functions of α, γ and i in high volatility case. Figure 4.11 to 4.13 represents the policy functions of each parameters in high volatility. The shape of policy functions of γ and i are the same in the case of low volatility case ($\sigma = 0.1$). Similar to the value function, the policy function of α also shifts compared to the low volatility case. However, the basic characteristics preserves. Therefore, even in high volatility case, the firm value maximized when i) $\alpha = 1$ (and $\beta = 1$) in low w and ii) $\alpha \simeq 0$ (and $\beta \simeq 0$) in high w when the mean productivity shock $\mu \geq 0.40$.

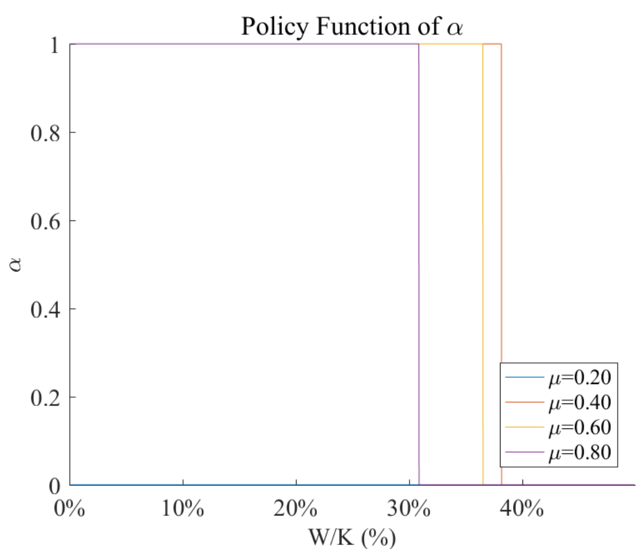

Figure 4.11: Policy function of α for several cases of Productivity ($\mu = 20\% \sim 80\%$)

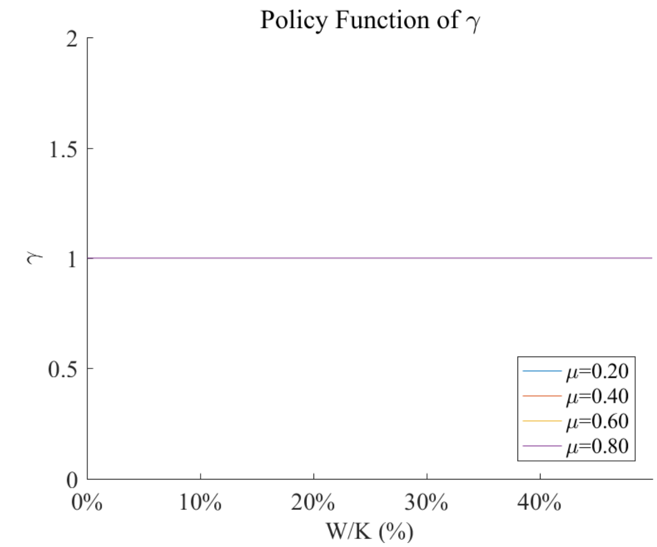

Figure 4.12: Policy function of γ for several cases of Productivity ($\mu = 20\% \sim 80\%$)

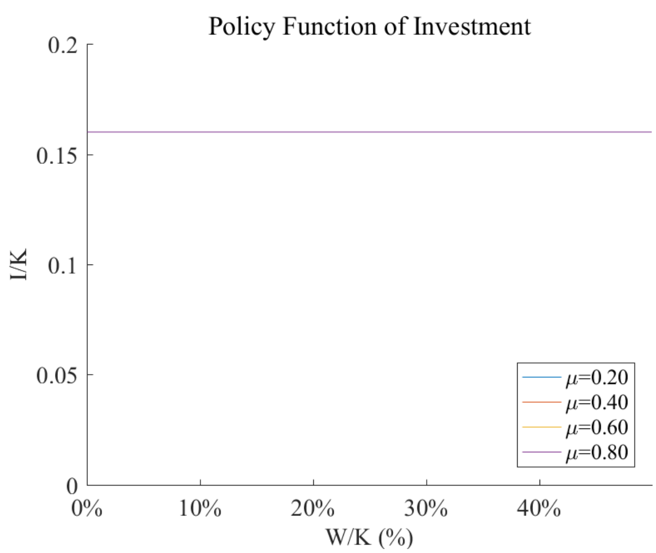

Figure 4.13: Policy function of i for several cases of Productivity ($\mu = 20\% \sim 80\%$)

5 Discussion

In this chapter we proposed a dynamic operational model which includes the retained earnings and cash holdings explicitly by considering the balance sheet of each firm. The major findings of this analysis is that whether the firm accumulates the retained earnings or not is determined by the external parameter, especially μ and σ.

First and foremost, the relation between the Risk-neutral mean productivity shock (μ) and retained earnings should be discussed. As is shown in Figure 4.3 and 4.11, the policy function of α changes as

μ increases. The firm only retains earnings when i) μ is not small (at least larger than 0.20) and ii) low cash-capital ratio (w). This leads that, based on our model, the medium to high growth company with low cash holdings can increase retained earnings without decreasing its firm value. Actually, fast growing companies like Google (Alphabet), Facebook, Alibaba increases the ratio of the retained earnings with respect to the total capital recently.

Secondly the relation between the firm value and volatility of productivity shock (σ) should be discussed. The comparison of the Figure 4.2 and 4.10 reveals the increase in σ results in the shift of the value function $v(w)$. The equation (4.14) shows the volatility of productivity shock appears only in the coefficient of $v^{(2)}(w_t)$, and this is the major reason σ affects by shifting the value function with respect to w. The economic reason for the σ to shift the value function is that when the volatility increases, the firm requires much cash holdings to avoid sudden cash short and bankruptcy. Through the shift of the value function, the change in σ also affects the ideal policy for the retained earnings.

Thirdly, we can also provide a good implication to the cash holdings. If we watch equation (4.14) carefully, in a low cash generating firm (low μ), the firm is required to finance adequate amount of cash to compensate its low turn over by obtaining interest income generated from the cash holdings (rw_t). In the real economy, especially in developed countries, the interest rate is close to zero and therefore the firm reserves its cash holdings in a shape of portfolio investment, or M&A. As long as the expected return from such cash holdings remains high, it might be better for the low growth firms to increase such kind of cash holdings. In the literature of corporate liquidity, the major reason for holding liquid assets or cash are summarized as 1) costly external finance (e.g., Kim et. al. (1998)) and 2) growth opportunity (e.g., Villeneuve et. al. (2014)). In this paper we also considered the relation between the drift of the productivity and this relation could explain current increase in cash holdings especially in developed countries.

In addition to the findings written above, we had better discuss the validity of this numerical model. As is shown in the Figure 4.6 to 4.8, the solutions of the first order conditions are volatile and highly depends on the initial value of the numerical solver (in this paper, we employed MATLAB to conduct the numerical simulation). Moreover, the policy functions sticks on the maximum or minimum value of its domein. Such result may leads to an analysis as what happens when there are no restriction for the domein of parameters α, γ and i. Although the protocol to iterate calculation until the parameter converges to the similar value seems rational in theory and possible to replicate the result of Bolton et. al. (2011), there might be some unexpected numerical problems for searching the solutions. One possible solution for this problem is to modify the protocol. Current protocol is a modification or combination of 1) Newton method and 2) Runge-Kutta method. Another possible way is the combination of 1) steepest descent (or stochastic gradient descent) method and 2) Runge-Kutta method. If we employ this methodology, we do not calculate the first order condition but search most or part of domain through steepest descent or stochastic gradient descent method. Although such methodology does not require the first order condition, the policy functions could be derived by its extreme values.

Lastly, we see a few directions for related future research. One possible direction would be the assumption on the production function which is currently assumed as homogeneous of degree 1 and does not satisfy diminishing returns. If we allow the firm value to be the function of W and K and calculate the 2 variable Hamilton-Jacobi-Bellman equation, we can assume production function with diminishing returns and such expansion should be considered in the next work. Also, to introduce the debt into this model could be another direction for future research.

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