RIETI Discussion Paper Series 18-E-009

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RIETI Discussion Paper Series 18-E-009

Februrary 2018

Wealth Distribution in the Endogenous Growth Model with Idiosyncratic Investment Risk[1]

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Abstract

In this paper, we study the continuous time Uzawa-Lucas growth model with physical and human capital accumulation, and study the relationship between economic growth and wealth inequality. Human capital accumulation is deterministic, but investment in physical capital is subject to the idiosyncratic risk. There exists a unique balanced growth path, and the stationary wealth distribution along the path is double Pareto. We show that the increase in efficiency in human capital accumulation raises economic growth, and further equalizes wealth distribution. We also consider a case with linear tax on risky physical capital, and show that capital tax reduces the degree of wealth inequality.

Keywords: Endogenous growth, Human capital risk, Wealth distribution

JEL classification: E5

[1]This study is conducted as a part of the project "Sustainable Growth and Macroeconomic Policy" undertaken at the Research Institute of Economy, Trade and Industry (RIETI). The author is grateful for helpful comments and suggestions by Hiroshi Yoshikawa (Rissyo Univ.), Takashi Unayama, (Hitotsubashi Univ.) and Discussion Paper seminar participants at RIETI.

1 Introduction

Economic inequality has been one of the important topic for macroeconomists. [1] One of their interests is the relationship between wealth inequality and economic growth. A famous claim by Piketty (2014) is that the gap between the rate of return on capital and the income growth rate plays a very important role in determining the wealth distribution. According to Piketty (2014), if the real interest rate exceeds the economic growth rate, inherited wealth increases faster than labor income and then wealth distribution will become more and more unequal. Piketty and Zucman (2015) validate the claim by setting up the overlapping generations model in which the individuals are subject to the idiosyncratic shocks on their wealth preferences. It is true that some economists criticize Piketty's claims by using neoclassical growth models. For example, Jones (2014, 2015) constructs a continuous-time OLG model with AK type production function. He finds that in the general equilibrium where the interest rate is endogenously determined, the inequality depends *only* on the population growth rate and is independent of the gap between the two returns.[2]

Recently, some papers analyze the wealth distribution in an economy where the individuals are subject to idiosyncratic investment risk. Benhabib, Bisin and Zhu (2016, henceforth BBZ) consider the perpetual youth model in which each individual maximizes their intertemporal utility by investing both safe asset and risky asset. The return on the safe asset is constant, but the one on the risky asset is subject to the idiosyncratic shock, and follows the Brownian motion. The individual gets utility from both consumption and bequest. BBZ then derive the stationary wealth distribution which is double Pareto.

In this paper, we apply the method of BBZ to the two-sector endogenous growth model with physical and human capital accumulation which is developed by Uzawa (1969) and Lucas (1988). Here we assume that physical capital accumulation is deterministic, but investment on human capital is subject to the idiosyncratic risk. We show that a unique

[1] Stiglitz(1966) investigates the income inequality in the Solow growth model.

[2] Krusell and Smith (2015) and Mankiw (2015) find that in the neoclassical growth model, the gap between the interest rate and the economic growth rate equals the discount rate in the long run.

balanced growth path exists, and the stationary wealth distribution along the path is double Pareto just as BBZ. The increase in efficiency of the human capital accumulation raises economic growth, and makes wealth distribution more equal.

The importance of human capital accumulation on economic growth has been pointed out by many authors including Cohen and Soto (2005), and human capital based economic growth model has been studied in many directions. Caballe and Santos (1993) analyze the dynamics of the optimal path. Benhabib and Perli (1994) find that human capital externality generates equilibrium indeterminacy. Jones et al. (1993) studies the optimal tax policy. Josten (2000) studies OLG model of endogenous growth. As far as we know, however, there is no existing literature that deals with the stationary wealth distribution in their framework. Our model is close to Krebs (2003) who investigates two-sector endogenous growth model in which individual is subject to the idiosyncratic risk in human capital accumulation. Although he considers the balanced growth rate and also welfare, he does not investigate the wealth distribution.

This paper is organized as follows. Section 2 describes the basic structure of the model. Section 3 investigates wealth inequality. Section 4 considers several extensions. Section 5 concludes the paper.

2 The Model

In this section, we provide an overview of our model. Time is continuous. In each period, a continuum of individuals is born. The number of newborns at date t is $p > 0$. Death follows the Poisson process with the arrival rate p. The death rate and the birth rate is the same and then the total population is constant. We set the population to one. The population of agents born on date s (henceforth cohort s) is $L_{t,s} = pe^{-p(t-s)}$ in period t.

The treatment of the asset of the individuals who dies in our paper is almost the same as Blanchard (1985) and BBZ. Just like Blanchard (1985), there is a competitive insurance market, and the asset of the dead is re-distributed to the alive. [3]In BBZ, these assets

[3]In Jones (2014), all the assets are redistributed to the newborns by the government as a lump-sum

of the dead is distributed to their children, and then the transfer is not lump-sum. In BBZ, the bequest consists of both accidental bequest and life insurance. Here we assume that there does not exist a asset market for human capital and that human capital of the parents who are dead is redistributed to the child equally.

An individual supplies labor in the labor market, receives wage income in each period, consumes and accumulates both physical and human capital. Just as Lucas (1988), human capital accumulation needs time. Each individual has unit amount of time each period, and if he works l units of time, then the individual accumulate human capital by spending $1-l$ units of time on studying. The increase in human capital is linear function on the time spent on human capital accumulation.

Here we suppose that human capital accumulation is deterministic, but physical capital accumulation is risky. [4] The individual i of cohort s maximizes the following expected intertemporal utility:

$$U^i = \int_s^\infty e^{-(\theta+p)(t-s)} \ln C^i_{t,s} dt, \tag{1}$$

subject to the following budget constraint

$$\begin{aligned} I^i_{t,s} + C^i_{t,s} &= (r_t - \tau + p)K^i_{t,s} + w_t l^i_t H^i_{t,s}, &(2)\\ dK^i_{t,s} &= I^i_{t,s}dt + \sigma K^i_{t,s} dQ^i_{t,s}, &(3)\\ dH^i_{t,s} &= A(1-l^i_t)H^i_{t,s}dt, &(4)\end{aligned}$$

where θ is the discount factor, $C^i_{t,s}$ is the consumption level of individual i in cohort s at time t, $Z^i_{t,s}$ is the bequest holdings of cohort s in period t, r_t is the real interest rate in period t, w_t is the net wage income in period t, $I_{t,s}$ is an input to physical capital accumulation of cohort s in period t, $l^i_{t,s}$ is a time spent on working, $A > 0$ is the efficiency parameter on human capital accumulation, $Q^i_{t,s}$ is a standard Brownian motion, and σ is the standard deviation. Physical capital is subject to a linear tax τ. Tax revenue is redistributed to the newborn individuals by a lump-sum. The shock on physical capital $dQ^i_{t,s}$ is individual specific. If we let $N^i_{t,s} = l^i_t H^i_{t,s}$ denote the effective labor supply of

transfer.

[4] Krebs (2003) assumes that human capital accumulation needs consumption good, not time.

individual i, the human capital is accumulated according to $dH^i_{t,s} = A(H^i_{t,s} - N^i_{t,s})dt$. There are many identical firms. The production function F has constant returns to scale and is given by $F(K_t, N_t)$, where K_t is the aggregate physical capital, and N_t is the aggregate effective labor. The total effective labor supply at time t equals

$$N_t = \int_s L_{t,s} \left(\int_i N^i_{t,s} di \right) ds. \tag{5}$$

The production function per efficiency unit of labor $f(k) = F(k, 1)$ satisfies $f(0) = 0$, $f' > 0$, $f'' < 0$ and $f'(0) = +\infty$.

The representative firm maximizes the profit

$$\pi_t = F(K_t, N_t) - r_t K_t - w_t N_t,$$

where r_t is the gross capital rental rate. We let $k_t = \frac{K_t}{N_t}$ denote the capital per efficiency unit of labor at time t. In the following, we simply call k the capital-labor ratio. Factor markets are perfectly competitive, and the equilibrium wage rate in period t is $w_t = w(k_t) = f(k_t) - k_t f'(k_t)$ and the physical capital rental rate in period t is $r_t = r(k_t) = f'(k_t)$. Physical capital does not depreciate.

We assume that human capital of the parents is transmitted to the newborn individuals equally when they are dead. The total level of the human capital of the dead is equal to pH_t. Since the number of the newborns is equal to the number of the dead, the level of human capital of by the individual i with cohort t who is born at time t is $H^i_{t,t} = H_t$.

The three intertemporal constraints of the individuals are consolidated as follows:

$$dK^i_{t,s} + \frac{w_t}{A} dH^i_{t,s} = (\bar{r}_t K^i_{t,s} + w_t H^i_{t,s} - C^i_{t,s})dt + \sigma K^i_{t,s} dQ^i_{t,s}. \tag{6}$$

where $\bar{r} = r + p - \tau$ is the net rate of return on physical capital.

In the following, we focus on the constant growth path along which the capital rental rate r and the wage rate w is time independent constant. We first characterize the path by assuming that w and r are constant, and later we show that the capital rental rate and the wage rate are in fact constant along the path.

We let $W^i_{t,s} = K^i_{t,s} + \frac{w}{A} H^i_{t,s}$ denote the total wealth, which can be interpreted as a sum of the physical capital and the market value of the human capital. We also follow BBZ and let

$$z_{t,s}^i = \frac{K_{t,s}^i}{W_{t,s}^i} = \frac{K_{t,s}^i}{K_{t,s}^i + \frac{w}{A}H_{t,s}^i} \in (0,1). \tag{7}$$

denote the share of risky physical capital. By definition, $z_{t,s}^i W_{t,s}^i = K_{t,s}^i$ and $(1-z_{t,s}^i)W_{t,s}^i = \frac{w}{A}H_{t,s}^i$. Using z, we can re-express the intertemporal budget constraint as

$$dW_{t,s}^i = \{AW_{t,s}^i + (\bar{r} - A)z_{t,s}^i W_{t,s}^i - C_{t,s}^i\}dt + \sigma z_{t,s}^i W_{t,s}^i dQ_{t,s}^i, \tag{8}$$

where the term $\bar{r} - A$ shows the difference between the rate of returns on risky asset and the one on the safe asset (i.e., human capital accumulation). Hamiltonian-Bellman-Jacobi equation is written as

$$(p+\theta)J(W) = \max_{C,z\in(0,1)}[\ln C + J'(W)\{AW + (\bar{r} - A)zW - C\} + 0.5J''(W)\sigma^2 z^2 W^2], \tag{9}$$

In the following, we assume that the difference in the returns $\bar{r} - A = r - \tau + p - A$ is sufficiently small:

$$r < \sigma^2 + A - p + \tau. \tag{10}$$

This equality ensures that the optimal level of the human capital ratio z is within the interval $(0,1)$. In our model, these returns are endogenously determined and later we check that the equilibrium allocation satisfies Eq. (10).

Proposition 1 *Suppose Eq. (10) holds. Given the constant factor prices r and w, the optimal allocation of the individual i satisfies*

$$C_{t,s}^i = (p+\theta)W_{t,s}^i, \tag{11}$$

$$z_{t,s}^i = \frac{\bar{r} - A}{\sigma^2}, \tag{12}$$

$$K_{t,s}^i = z_{t,s}^i W_{t,s}^i \tag{13}$$

$$H_{t,s}^i = \frac{A}{w}(1 - z_{t,s}^i)W_{t,s}^i \tag{14}$$

The total wealth of the individual evolves according to

$$dW_{t,s}^i = gW_{t,s}^i dt + \sigma z_{t,s}^i W_{t,s}^i dQ_{t,s}^i. \tag{15}$$

where $g = A - p - \theta + \frac{(\bar{r}-A)^2}{\sigma^2}$.

Proof. See the Appendix. ■

As in BBZ, the ratio of risky asset, $z_{t,s}^i$ is constant across time and across individuals. In the following, we denote it simply as z. In the equilibrium path, it depends on the capital labor ratio:

$$z = \frac{r(k) - \tau + p - A}{\sigma^2}. \quad (16)$$

Using z, we can express the growth rate of the individual wealth g as $g = A - p - \theta + \sigma^2 z^2$. It depends on the difference on the return between the human capital and the one on physical capital. In the general equilibrium, these returns are endogenous and depends on z or equivalently the physical capital- human capital ratio.

2.1 Balanced growth path

Here we investigate the aggregate growth path $\{K_t, H_t, C_t, l_t\}$. The average wealth level of cohort s at time t is equal to $E_s[W_{t,s}] = e^{g(t-s)} E_s[W_{s,s}]$. Therefore the aggregate wealth level $W_t = \int_{-\infty}^{t} E_s[W_{t,s}] L_{t,s} ds$ equals

$$W_t = \int_{-\infty}^{t} p E_s[W_{s,s}] e^{(g-p)(t-s)} ds. \quad (17)$$

Since the share of physical capital is constant, $K_t = zW_t$ and $\frac{w}{A} H_t = (1-z)W_t$.

The term $E_s[W_{s,s}]$ is the market value of average wealth of the newborns, and then equals $\frac{w}{A} H_t + \tau K_t = (1 - z + \tau z) W_t$. the total wealth is $W_t = p(1 - z + \tau z) \int_{-\infty}^{t} W_s e^{(g-p)(t-s)} ds$.

The Balanced Growth Path (BGP) is such that the aggregate consumption, physical capital, human capital and the output all grow at the same rate. Along the BGP, the growth rate of the total wealth, say G, satisfies $W_t = p(1 - z + \tau z) W_t \int_0^{\infty} e^{-(G-g+p)s} ds$. This implies that $G - g + p = p(1 - z + \tau z)$, or equivalently

$$G = g - p(1-\tau)z. \quad (18)$$

If the returns r and w are constant, z is constant, and then the growth rate of K, H and C are all equal to G.

We consider the constant returns to scale production function and then if z is constant, then the physical capital to human capital is constant and then the returns r and w are shown to be constant.

Since $K = zW$ and $\frac{w}{A}H = (1-z)W$, $\frac{K}{H} = \frac{z}{1-z}\frac{w(k)}{A}$. Therefore the capital-labor ratio is re-expressed by

$$k = \frac{K}{N} = \frac{K}{H} \cdot \frac{H}{N} = \frac{z}{1-z}\frac{w(k)}{A}\frac{H}{N}.$$

Since human capital does not depreciate by assumption, the aggregate human capital evolves according to $\dot{H}_t = A(H_t - N_t)$. Along the BGP, if it exists, $\hat{H} = \hat{K} = G$. Therefore $\frac{N}{H} = \frac{A-G}{A}$. This implies

$$k = \frac{w(k)}{A}\frac{z}{1-z} \cdot \frac{A}{A-G}. \tag{19}$$

We assume that the production function is Cobb-Douglas type

$$F(K, N) = A_Y K^\alpha N^{1-\alpha},$$

where A_Y denotes the TFP level. In this case, $f(k) = A_Y k^\alpha$. In the Cobb-Douglas case, $w(k) = A_Y(1-\alpha)k^\alpha$ and $r(k) = A_Y \alpha k^{\alpha-1}$. There are two unknowns, k, z, and are determined by Eq. (16) and Eq. (19). Since $k = \frac{\alpha}{1-\alpha}\frac{w}{r}$, $z\sigma^2 z = r - \tau + p - A$, and $G = A - \theta - p(1+z) + \sigma^2 z^2$, these two equations are consolidated as

$$1 = \frac{1-\alpha}{\alpha} \cdot \frac{z}{1-z} \cdot \frac{\sigma^2 z + A - p}{p\{1 + (1-\tau)z\} + \theta - \sigma^2 z^2}. \tag{20}$$

The BGP exists if and only if the solution path exists and satisfies Eq. (10) and $z < 1$.

Proposition 2 *The BGP exists. Along the BGP, physical capital, human capital and output all growth with rate*

$$G = A - \theta - p\{1 + (1-\tau)z\} + \sigma^2 z^2. \tag{21}$$

The equilibrium ratio z is uniquely determined by Eq. (20).

Proof. See the appendix. ■

As is the case with the perfect foresight version of Uzawa-Lucas model, the balanced growth rate is independent of the efficiency parameter of the final goods production and depends only on A. If A increases, it definitely reduces the ratio on physical capital investment z. However, it is not clear whether it will raise G because of the final term $\sigma^2 z^2$.

Proposition 3 *Suppose that the parameters satisfy $p(1-\tau) > 2\sigma^2$. Then the increase in the efficiency parameter of human capital accumulation raises the aggregate economic growth rate G.*

Proof. See the appendix. ■

When we consider a linear capital tax τ, it reduces the incentive of the individual to invest in a risky asset. They instead spend more resources on human capital accumulation. It then raises economic growth rate. Thus we have the following proposition.

Proposition 4 *Suppose that the parameters satisfy $p(1-\tau) > 2\sigma^2$. Then the increase in capital tax τ raises the aggregate economic growth rate G.*

2.2 Case with no investment shock

Here we consider the simplest case with no investment shock. This corresponds to Josten (2000) who investigates a two-sector version of Blanchard type OLG model, although Josten (2000) does not explicitly obtains the balanced growth rate. When $\sigma = 0$, to ensure that the individual accumulates human capital, or equivalently z is positive, Eq. (16) requires the non-arbitrage equation between human capital and physical capital:

$$r(k) = A - p.$$

If there is no uncertainty $\sigma = 0$, Eq. (20) then implies that the value of z is determined by

$$1 = \frac{1-\alpha}{\alpha} \cdot \frac{z}{1-z} \cdot \frac{A-p}{p(1+z)+\theta}. \tag{22}$$

This is a quadratic equation and then the solution is

$$\bar{z} = \frac{-(\alpha^{-1}-1)(A-p) - \theta + \sqrt{\{(\alpha^{-1}-1)(A-p)+\theta\}^2 + 4p(p+\theta)}}{2p}.$$

The balanced growth rate is $G = A - p(1+\bar{z}) - \theta$. If we further set p to zero, the model now becomes the popular Uzawa-Lucas model with infinite horizons, and we have $r = A$ and $G = A - \theta$. It is well-known that in the Uzawa-Lucas model with logarithmic utility, the balanced growth rate equals the gap between human capital efficiency parameter and the discount rate. This formula is found in many textbooks including Barro and Salai-Martin (2000).

3 Wealth distribution

Here we follow BBZ, and derive the wealth distribution. Since the difference between the average individual growth rate and the aggregate wealth grows with rate is $g - G = p(1-\tau)z$, the ratio of the individual to aggregate wealth, $x^i_{t,s} = \frac{W^i_{t,s}}{W_t}$ evolves according to

$$dx^i_{t,s} = p(1-\tau)zx^i_{t,s}dt + \sigma z x^i_{t,s}dQ_{t,s}.$$

For any individual i in cohort s, the initial asset is $W^i_{t,t} = \{1-(1-\tau)z\}W_t$. Thus

$$x^i_{t,t} = 1-(1-\tau)z.$$

Note that in the competitive equilibrium, z is constant and then $x^i_{t,s}$ also equals to the ratio of the individual to aggregate physical capital holdings. In other words, $x^i_{t,s} = \frac{K^i_{t,s}}{K_t}$.

As BBZ show, the stationary distribution $f(x)$, is double Pareto and, satisfies

$$\frac{1}{2}(\sigma z)^2 f''(x) + \{2(\sigma z)^2 - z^2\}xf'(x) + \{(\sigma z)^2 - p(1-\tau)(1+z)\}f(x) = 0. \tag{23}$$

We summarize the derivation in the Appendix. We have the following proposition:

Proposition 5 *The stationary distribution $f(x)$ is determined by the following: 1) If $x \leq 1-z$, then $f(x) = C_1 x^{-a_1}$, and 2) If $x > 1-z$, then $f(x) = C_2 x^{-a_2}$. Here x^*, C_1, and C_2 are positive constant, and a_1 is the smaller root and a_2 is a larger root of the the following quadratic equation*

$$\frac{(\sigma z)^2}{2p(1-\tau)}(a-2)(a-1) = 1 + z - za.$$

The solution always exists, and satisfies $a_1 < 1$ and $a_2 > 2$.

As is clear from the graphic argument, the increase in z always reduces a_2 and the wealth distribution more unequal. Together with the previous argument, we obtain the following proposition.

Proposition 6 *The reduction in human capital accumulation parameter A reduces the economic growth rate and increases the wealth inequality.*

Proposition 7 *Capital tax raises the economic growth rate and makes wealth distribution more equal.*

In our model, human capital accumulation is deterministic. Therefore the reduction in A, the individuals invest more on the risky physical capital and raises the wealth inequality. Linear tax on risky physical capital makes individual spend more time on studying , raises economic growth rate and makes wealth distribution more equal.

4 Robustness

In the previous section, we assumed that human capital needs time to accumulate. In this section, we consider a case where human capital accumulation needs consumption goods just as Krebs (2003) and characterizes the equilibrium path. Here we assume that physical capital accumulation is deterministic, but human capital accumulation is risky. The individual i of cohort s maximizes the expected intertemporal utility U^i subject to the following budget constraint

$$x_{s,t}^{iK} + x_{s,t}^{iH} + C_{s,t}^i = r_t^K K_{s,t}^i + r_t^H H_{s,t}^i, \tag{24}$$

$$dK_{s,t}^i = x_{s,t}^{iK} dt, \tag{25}$$

$$dH_{s,t}^i = x_{s,t}^{iH} dt + \sigma H_{s,t}^i dB_{s,t}^i, \tag{26}$$

where r_t^K is the net real interest rate in period t, r_t^H is the net wage income in period t, $x_{s,t}^K$ is an input to physical capital accumulation, $x_{s,t}^H$ is an input to human capital accumulation, $B_{s,t}^i$ is a standard Brownian motion, and σ is the instantaneous standard deviation.

We let $W_{s,t}^i = K_{s,t}^i + H_{s,t}^i$ denote the total wealth. The budget constraints are consolidated as

$$dW_{s,t}^i = r_t^K W_{s,t}^i + (r_t^H - r_t^K)\omega_{s,t}^i W_{s,t}^i - C_{s,t}^i + \sigma\omega_{s,t}^i W_{s,t}^i dB_{s,t}^i, \tag{27}$$

where $\omega_{s,t}^i = \frac{H_{s,t}^i}{W_{s,t}^i}$. Hamiltonian-Bellman-Jacobi equation is written as

$$(p+\theta)J(W) = \max_{C,\omega}[\ln C + J'\{r^K W + (r^H - r^K)\omega W - C\} + 0.5J''\sigma^2\omega^2 W^2] \tag{28}$$

The optimal allocation satisfies $C_{s,t}^i = (p+\theta)W_{s,t}^i$ and $\omega_{s,t}^i = \omega \equiv \frac{r^H - r^K}{\sigma^2}$. The total wealth of the individual i evolves according to

$$dW_{s,t}^i = gW_{s,t}^i dt + \sigma\omega W_{s,t}^i dB_{s,t}^i. \tag{29}$$

where $g = r^K - p - \tau - \theta + \frac{(r^H - r^K)^2}{\sigma^2}$. The aggregate growth rate is $G = r^K - p - \theta + \frac{(r^H - r^K)^2}{\sigma^2}$.

Along the BGP, if it exists, $k = \frac{K}{H} = \frac{1-\omega}{\omega} = \frac{1}{\omega} - 1$ where $\frac{r^H - r^K}{\sigma^2}$. Thus $\omega = \frac{1}{1+k}$. The capital rental rate $r^K = F_K(K, H)$ and wage rate $r^H = F_H(K, H)$ are equal to

$$r^K = r(k) = f'(k), \tag{30}$$

$$r^H = w(k) = f(k) - kf'(k). \tag{31}$$

Therefore the equilibrium level of ω is determined by

$$\omega = \frac{1}{1+k} = A\frac{(1-\alpha)k^\alpha - \alpha k^{\alpha-1}}{\sigma^2}.$$

Such k exists. Note that once we find $k > 0$, then $\omega = \frac{1}{1+k}$ is always within the interval (0,1). Along the BGP, $G = f'(k) - p - \theta + \frac{(f(k) - kf'(k) - f'(k))^2}{\sigma^2}$. We can easily check that main conclusion above continue to hold in this set-up.

5 Conclusion

In this paper, we study the continuous time overlapping generations model of endogenous growth with physical and human capital accumulation, and studies the relationship between economic growth and wealth inequality. Physical capital accumulation is deterministic, but investment on human capital is subject to the idiosyncratic risk. There exists a unique balanced growth path, and the stationary wealth distribution along the path is Pareto. The increase in human capital efficiency parameter raises economic growth, and makes wealth distribution more equal. We next show that linear tax on wealth makes distribution more equal.

Appendix

The Appendix provides proofs for propositions.

A Proof of Proposition 1

We let $c = \frac{C}{W}$. The problem is re-written as a choice on c and z:

$$(p+\theta)J(W) = \ln W + \max_{c,z}[\ln c + J_W W\{A + (\bar{r} - A)z - c\} + 0.5J_{WW}W^2\sigma^2 z^2]$$

We guess that $J(W) = m \ln W + q$. If the guess is true, we have

$$(p+\theta)(m \ln W + q) = \ln W + \max_{c,z}[\ln c + m\{A + (\bar{r} - A)z - c\} - 0.5m\sigma^2 z^2].$$

The function can be verified by setting $m = \frac{1}{p+\theta}$. The solution (c, z) is $c = p + \theta$ and $z = \frac{\bar{r}-A}{\sigma^2}$. Therefore $C^i = (p+\theta)W^i$. Substitution of these conditions into the Bellman equation yields

$$(p+\theta)q = \ln(p+\theta) + \frac{A}{p+\theta} - 1 + 0.5\frac{(\bar{r}-A)^2}{\sigma^2}$$

This implies $q = \frac{1}{p+\theta}\{\ln(p+\theta) + \frac{A}{p+\theta} - 1 + 0.5\frac{(\bar{r}-A)^2}{\sigma^2}\}$. ■

B Proof of Proposition 2

If we let $f(z)$ denote the right hand side of Eq. (20), it satisfies $f(0 = 0$ and $f(1) = +\infty$. Thus for some z^*, $f(z^*) = 1$. Since $g = A - p - \theta + \sigma^2 z^2$, the balanced growth rate is equal to $G = (A - p - \theta + \sigma^2 z^2) - p(1-\tau)z$. ■

C Proof of Proposition 3

Eq. (20) implies that

$$p\{1 + (1-\tau)z\} + \theta - \sigma^2 z^2 = \frac{1-\alpha}{\alpha} \cdot \frac{z}{1-z}(\sigma^2 z + A - p)$$

Under the inequality assumption $p(1-\tau) > 2\sigma^2$, the left hand side is the decreasing function and the right hand side is the increasing function of z and then if A increases, z decreases. The aggregate growth rate G can be written as $G(z) = A + q(z)$ with $q = -\theta - p\{1+(1-\tau)z\} + \sigma^2 z^2$. Since $q'(z) < 0$ by assumption, $dG/dA = 1 + dq/dz \cdot dz/dA > 0$.
■

D Derivation of Kolmogorov equations

In this section, we summarize the result of BBZ and obtains the stationary equilibrium allocations. In our model, the private bequests are all accidental (i.e., the parameter $\rho = 1$). Moreover, the difference between individual wealth growth rate, and aggregate growth rate (i.e., the value of $g - \tilde{g}$ in BBZ) equals to τ. This greatly simplifies the calculation.

The probability density of X_t, given $X_0 = y$, say $f(x, t; y)$ satisfies

$$f(x,t;y) = (1-p\epsilon)\int_0^\infty f(a, t-\epsilon; y)\,\Pr[X_\epsilon = x|X_0 = a]da + p\epsilon f(x, t-\epsilon; y) + o(\epsilon).$$

if $x > x^*$, and

$$f(x,t;y) = (1-p\epsilon)\int_0^\infty f(a, t-\epsilon; y)\,\Pr[X_\epsilon = x|X_0 = a]da + o(\epsilon).$$

if $x < x^*$. Here $X_\epsilon|X_0$ is log normal and then

$$\Pr[X_\epsilon = x|X_0 = a] = \frac{1}{x\sqrt{2\pi\sigma^2 z^2 \epsilon}} \exp\left(-\frac{1}{2\sigma^2 z^2 \epsilon}\left\{\log x - \log a + (\tau + 0.5\sigma^2 z^2)\epsilon\right\}^2\right).$$

Thus

$$\begin{aligned}
\frac{\partial f}{\partial t} &= (\sigma^2 z^2 + \tau) f + (2\sigma^2 z^2 + \tau) x \frac{\partial f}{\partial x} + 0.5\sigma^2 z^2 x^2 \frac{\partial^2 f}{\partial x^2} \text{ if } x > x^*, \\
\frac{\partial f}{\partial t} &= (\sigma^2 z^2 - p + \tau) f + (2\sigma^2 z^2 + \tau) x \frac{\partial f}{\partial x} + 0.5\sigma^2 z^2 x^2 \frac{\partial^2 f}{\partial x^2} \text{ if } x > x^*
\end{aligned}$$

In the stationary distribution, we can set $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} = 0$. ■

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