Spatial Scale of Agglomeration and Dispersion: Theoretical foundations and empirical implications

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Spatial Scale of Agglomeration and Dispersion:
Theoretical foundations and empirical implications[1]

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Abstract

This paper studies the theoretical properties of existing economic geography models with agglomeration and dispersion forces in a many-region setup, rather than their original two-region space, to investigate the spatial scale—global or local—of agglomeration and dispersion intrinsic to each model. We show that models in the literature reduce to two canonical classes that differ starkly in their engendered spatial patterns and comparative statics. Our formal results offer a consistent explanation for the set of various outcomes from the extant reduced-form regression analyses and also provide qualitative predictions of the treatment effects in the structural model-based studies on regional agglomeration.

Keywords: Agglomeration, Dispersion, Spatial scale, Multiple equilibria, Bifurcation

JEL classification: R12, R13, F15, F22, C62

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1   Introduction

Empirical studies over the past few decades have led to the accumulation of ample evidence that agglomeration externalities are the major source of the lumpy spatial distributions of economic activities (see, e.g., Rosenthal and Strange, 2004, for a survey). A wide variety of formal models have been proposed to investigate the underlying mechanisms (see, e.g., Duranton and Puga, 2004; Behrens and Robert-Nicoud, 2015, for surveys). For analytical tractability, most existing models rely on a location space that abstracts from the diversity of interregional distances inherent in actual regional economies, where a typical approach assumes a location space comprising just two regions.[1] Summarizing the spatial effects in a single interregional distance simplifies the analysis. However, this benefit comes at the cost of losing information on the *spatial scale* of agglomeration and dispersion.[2]

To see this, consider a model with any agglomeration force but without a dispersion force. In such a model, all the mobile agents will concentrate in one region. If some dispersion forces were added to the model, a proportion of mobile agents will deviate from the concentration. In a two-region economy, there is only one alternative region to head for. Hence, there is no variation in the spatial scale of dispersion. However, in a many-region economy in which interregional distances are heterogeneous, the spatial scale of dispersion can vary depending on the nature of the dispersion force. Dispersion may occur *locally* to avoid crowding *inside* the agglomeration as in the case of an urban congestion externality, or it may occur *globally* through attraction from *outside* the agglomeration in the case of a distant, less crowded market.

This study revisits a wide variety of existing economic geography models in *a many-region setup* with diverse interregional distances.[3] By characterizing their bifurcation behaviors behind the spontaneous formation of agglomerations, we show that these models reduce to *two canonical classes*: (*i*) one with a *global dispersion force*[4] and (*ii*) the other with a *local dispersion force*.[5] Formally, these two dispersion forces differ in that the former is dependent,

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[1]Another typical approach allows for the presence of many regions that are equidistant (often zero distance) from one another as in the system of cities model proposed by Henderson (1974). See Tabuchi, Thisse and Zeng (2005) for a recent such application.

[2]Many extant empirical studies have abstracted from the space between locations and focused on the *local interactions* between agglomeration size and location-specific factors (see, e.g., Combes and Gobillon, 2015, for a survey). The empirical studies discussed in Section 6 belong to the other strand of the literature that accounts for global factors (e.g., interregional transport accessibility) on regional agglomerations.

[3]More specifically, we cover static many-region models with a single type of mobile agent (see footnotes 4, 5, and 6). We do not cover models with multiple types of mobile agents (e.g., urban models of Fujita and Ogawa, 1982; Ota and Fujita, 1993; Lucas and Rossi-Hansberg, 2002; Ahlfeldt, Redding, Sturm and Wolf, 2015; Owens, Rossi-Hansberg and Sarte, 2017), or dynamic models (e.g., Desmet and Rossi-Hansberg, 2009, 2014, 2015; Desmet, Nagy and Rossi-Hansberg, 2017; Nagy, 2017).

[4]For example, Krugman (1991); Puga (1999); Ottaviano, Tabuchi and Thisse (2002); Forslid and Ottaviano (2003); Pflüger (2004); Harris and Wilson (1978).

[5]For example, Beckmann (1976); Mossay and Picard (2011); Blanchet, Mossay and Santambrogio (2016);

whereas the latter is independent, on the distance structure of the model. The most realistic formulations incorporate both forces, which we call class (iii).[6] (See **Table 1** in Section 4.3 for the classification of the existing models).

The basic two classes (i) and (ii) exhibit two stark differences. The first difference appears in the response to transport costs. *Global* dispersion (i.e., an increase in the number of agglomerations, a decrease in the spacing of agglomerations, and a decrease in the size of each individual agglomeration) is triggered by *higher* costs. By contrast, *local* dispersion (i.e., a decrease in population density of mobile agents and an enlargement of the spatial extent of an agglomeration) is triggered by *lower* costs. In class (iii) models with both types of dispersion forces, *a decrease in transport costs simultaneously causes both agglomeration at the global scale and dispersion at the local scale*. The second difference shows up in the agglomeration patterns. In the former, multiple and distinct agglomerations emerge; in the latter, the agglomeration *always* results in a *unimodal* regional distribution of mobile agents. The typical location pattern can thus be described as *locally concentrated and globally dispersed* for the former and as *globally concentrated and locally dispersed* for the latter.

The notion of the spatial scale of agglomeration and dispersion is not pervasive in the empirical literature on regional agglomeration. However, it is indispensable to understand the evolution of agglomeration patterns in reality. Consider the case of Japan since 1970. The development of highways and high-speed railway networks in Japan was triggered by the Tokyo Olympics held in 1964. Between 1970 and 2015, the total highway (high-speed railway) length increased from 879 km (515 km) by more than 16 (10) times to 14,146 km (5,350 km). The 302 urban agglomerations that have survived throughout that 45-year period experienced a 21% increase in population size on average (controlling for the national population growth). This means that there was *a selective concentration* from across the country, i.e., *at the global scale*.[7,8] However, this concentration at the global scale was associated with *a dispersion at the local scale*: there was a 94% increase in areal size on average with a 22% decrease in population density for individual agglomerations on average. These seemingly paradoxical evolutions of urban agglomerations in Japan turn out to be a standard outcome of class (iii) models (see Section 5.3).

Accordingly, our results provide novel perspectives for the three major strands of the empirical literature on regional agglomeration. One is on the measure of agglomeration (e.g.,

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Helpman (1998); Redding and Sturm (2008); Murata and Thisse (2005); Allen and Arkolakis (2014).

[6]For example, Tabuchi (1998); Pflüger and Südekum (2008); Takayama and Akamatsu (2011).

[7]Each urban agglomeration is identified as the set of contiguous 1 km-by-1 km cells with a population density of at least $1000/km^2$ and total population of at least 10,000. Population count data are obtained from Statistics Bureau, Ministry of Internal Affairs and Communications of Japan (1970, 2015). The transport network data are obtained from the National Land Numerical Information Download Service of Japan at `http://nlftp.mlit.go.jp/ksj-e/gml/gml_datalist.html`. See Appendix A for more details.

[8]The population size of each agglomeration is computed in terms of its share of the national population, and thus the growth in national population size is controlled for.

Ellison and Glaeser, 1997; Duranton and Overman, 2005; Brülhart and Traeger, 2005; Mori, Nishikimi and Smith, 2005). The other two are on reduced-form regression approaches (see, e.g., Redding and Turner, 2015, §20.4, for a survey) and structural model-based approaches (see, e.g., Redding and Rossi-Hansberg, 2017, for a survey) to evaluate the impacts of exogenous changes, particularly those of interregional transport access on regional agglomeration. Here, we highlight the basic issue in each context.

A scalar index has long been the natural choice for measuring agglomeration, reflecting that abstraction from interregional distances has been the rule in the formal analyses of agglomeration. On the premise of our theoretical results, when the dispersion force is effective at both global and local scales as in reality, agglomeration proceeds at the global scale when dispersion proceeds at the local scale and vice versa. Thus, the meaning of the net effect summarized by a scalar index is unclear. In Section 6.1, we argue for the necessity and utility of more disaggregated measures of agglomeration.

For the reduced-form regression exercises, consider, for example, the contrasting studies of regional agglomeration (i.e., at a global scale) presented by Duranton and Turner (2012) and Faber (2014). The former focused on the growth of large metro areas in the United States, while the latter focused on the growth of peripheral counties in China.[9] The former (latter) revealed a positive (negative) correlation between the size of agglomeration and interregional transport access in a given region. In light of class (i) models, these opposite responses may simply reflect different sides of the same coin. That is, both the results may indicate the tendency of agglomeration at the global scale (toward larger regions) under the treatment, i.e., an improvement in interregional transport access (as in the case of Japan discussed above). Thus, one must carefully interpret the estimated treatment effect, since it is simply an average effect for the set of the selected regions, where the selection often involves some obvious biases, e.g., a larger or smaller subset of all cities. For the excluded but treated regions, the sign of the impacts may well be the opposite. Section 6.2 provides a unified interpretation of a wider variety of empirical evidence on regional agglomeration in terms of our theoretical results.

Finally, regarding structural model-based approaches for regional agglomeration, the two representative models proposed by Redding and Sturm (2008) and Allen and Arkolakis (2014) belong to class (ii), i.e., they cannot explain the endogenous formation of multiple agglomerations by construction. In other words, their basic premise is that the primary source of regional variation in agglomeration size is the heterogeneity in exogenous (or first-nature) regional advantages and that agglomeration externalities play only a secondary role. However, we demonstrate that even in this case, the comparative static outcome is still

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[9]The amount of interregional highway linkages (e.g., number and total length) within a given region is often interpreted as a measure of intra-urban transport infrastructure (e.g., Baum-Snow, 2007; Duranton and Turner, 2012). However, we suggest that it can also be interpreted as a measure of interregional transport infrastructure.

governed by agglomeration externalities and is specific to the model class. In fact, the signs of the treatment effects on agglomeration typically reverse if multiple agglomerations are allowed to form endogenously, i.e., class (i) models are adopted instead.[10]

The remainder of this paper is organized as follows. Section 2 develops a general modeling framework for analyzing agglomeration patterns in a many-region economy and defines the equilibria and their stability. Section 3 characterizes the nature of the dispersion force and provides a formal classification of the spatial patterns of agglomeration in terms of the spatial scale of dispersion forces. Section 4 presents a mapping of existing models of economic geography to the classification. Section 5 outlines the impact of changes in transport costs on the stable equilibrium patterns of agglomeration under representative models. Section 6 discusses the implications of our theoretical results for the empirical literature on regional agglomeration. Finally, Section 7 concludes and discusses future research agendas regarding models with richer and more realistic structures that are not addressed in this study.

2 A general modeling framework for spatial agglomerations

This section introduces a generic format of many-region spatial economic models, which we refer to as *economic geography models*, with agglomeration externalities and the endogenous formation of spatial concentration. As essential preliminaries, the technical aspects (stability and bifurcation of equilibria) and their economic interpretations are discussed.

2.1 Economic geography models

The economy compromises K discrete regions indexed from 0 as $i = 0, 1, \ldots, K-1$, and $\mathcal{K} \equiv \{0, 1, \ldots, K-1\}$ denotes the set of regions. Throughout our analyses, the term "region" indicates a discrete spatial unit in which a mobile agent can locate. Whether the model is interpreted to be intra-urban, interregional, or international is not essential for our results. A "region" may alternatively be termed an urban zone, a municipality, a country, and so forth.[11]

There is a continuum of mobile agents of a single type; an agent chooses a single region in which to locate. We denote the spatial distribution of agents by $\boldsymbol{h} \equiv (h_i)_{i \in \mathcal{K}}$, where its ith element $h_i \geq 0$ is the mass of agents located in region i. The total mass of mobile agents is exogenous constant H, i.e., $\sum_{i \in \mathcal{K}} h_i = H$. Concretely, the set of all possible spatial patterns is given by $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \sum_{k \in \mathcal{K}} h_k = H, h_k \geq 0\}$.

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[10]See the discussions in Section 6.3 and the formal analysis in Appendix D.

[11]On assuming a discrete space, also noted is that there are intrinsic difficulties with employing a continuous space in empirical analyses because of the discrete nature of the data as well as numerical computations.

Given the spatial distribution $\boldsymbol{h}$ of agents, the payoff of choosing each region is determined. The payoff function is denoted by $\boldsymbol{v}(\boldsymbol{h}) \equiv (v_i(\boldsymbol{h}))_{i \in \mathcal{K}}$, where $v_i(\boldsymbol{h})$ denotes the payoff for an agent located in region $i \in \mathcal{K}$. Agents are mobile and are free to choose their locations to possibly improve their own payoffs. Thus, the equilibrium condition for the spatial distribution of agents is formulated as follows: $v^* = v_i(\boldsymbol{h})$ for all regions i such that $h_i > 0$, and $v^* \geq v_i(\boldsymbol{h})$ for any region i such that $h_i = 0$. Here, v^* is the equilibrium payoff level.

Our analysis thus adheres to the most canonical form of economic geography models: static models with a single type of mobile agent. For example, the above description covers models of endogenous city center formation (e.g., Beckmann, 1976). Notably, it also covers single-industry new economic geography (NEG) models; in such models there is only a single type of mobile agent, i.e., the location incentives of firms and workers coincide. We do not consider, however, more involved models with multiple types of mobile agents,[12] sector-wise differentiated spatial frictions,[13] multiple types of increasing returns,[14] and dynamic models[15]. These directions are discussed in Section 7.

The indispensable feature of economic geography models is the presence of *space*: transport costs are incurred by, e.g., the shipment of goods between different regions or social interactions among agents in different locations. Therefore, there is a fundamental trade-off between *transport costs* and *scale economies* associated with the spatial concentration of economic activities (Fujita and Thisse, 2013). The payoff functions of economic geography models include agglomeration and dispersion forces, meaning that spatial equilibria are determined by a tense balance of these two opposing forces that depend on the interregional transport costs. We assume that the spatial friction between regions is summarized by a single *friction matrix* $\boldsymbol{D} = [d_{ij}]$, where $d_{ij} \in [0, 1)$ denotes the freeness of the transport between regions i, j. Also, throughout the paper we focus on a special geographical setup, namely a *racetrack economy*, which we will describe in detail in Section 3.1.

Given the friction matrix $\boldsymbol{D}$ that encapsulates the role of the underlying geography, the microfoundations for the payoff function $\boldsymbol{v}(\boldsymbol{h})$ are typically provided by modeling the *short-run equilibrium* relating to the spatial frictions between locations. Assuming that the relocation of agents is sufficiently slow compared with that through market reactions, the short-run equilibrium conditions (e.g., factor and product markets clearing and trade balance) determine the payoff (utility or profit) in each region as a function of the spatial pattern of agents $\boldsymbol{h}$. We thus assume that the payoff function $\boldsymbol{v}(\boldsymbol{h})$ includes $\boldsymbol{D}$ as a parameter.

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[12]For example, the urban models of Fujita and Ogawa (1982); Ota and Fujita (1993); Lucas and Rossi-Hansberg (2002) as well as their recent applications, e.g., Ahlfeldt et al. (2015); Owens et al. (2017).

[13]For example, Fujita and Krugman (1995) and Mori (1997).

[14]For example, Fujita, Krugman and Mori (1999b); Tabuchi and Thisse (2011), and also Hsu (2012).

[15]Most notably, Desmet and Rossi-Hansberg (2009, 2014, 2015); Desmet et al. (2017); Nagy (2017).

2.2 Stability and bifurcation of the equilibria

Owing to the positive externalities of spatial agglomeration, economic geography models often face a multiplicity of equilibria. A standard approach in the literature is to introduce equilibrium refinement based on *local stability* under myopic evolutionary dynamics, where the rate of change in the number of residents h_i in region i is modeled on the basis of the spatial pattern of agents h and that of payoff $v(h)$.[16] We denote the deterministic dynamic by $\dot{h} = F(h, v(h))$, where the dot over h represents the time derivative. We assume (i) F satisfies differentiability with respect to both arguments in $\mathcal{D}$, (ii) agents relocate in the direction of an increased aggregate payoff under F, and (iii) the total mass of agents is preserved under F.[17] Furthermore, we assume that any spatial equilibrium is a rest point of the dynamic.[18] Given the adjustment dynamic F, the stability of the equilibrium is defined in terms of asymptotic stability under F.

The stability of a given spatial equilibrium is parameter-dependent. As emphasized by the NEG literature, changes in transportation technologies can trigger the endogenous emergence of regional inequality. The basic core–periphery story following Krugman (1991) is as follows: "Consider an economy with two regions that are ex-ante symmetric, where the regions have exactly the same characteristics and mobile agents are uniformly distributed. When interregional transport costs are high, the uniform distribution of mobile agents is a stable equilibrium. If the transportation cost falls below a certain threshold value, the pattern is no longer stable; the agglomeration toward one of the regions occurs, and the core–periphery pattern emerges by self-organization."

Although the intuitive story of the two-region economy backed by the rich interactions of economic forces has its own right, corresponding many-region studies are scarce in the literature. In particular, *which* spatial patterns emerge after an encountered destabilization in a many-region economy is far from obvious. We therefore need better methods to examine the stability of equilibrium patterns in a many-region economy.

Such an abrupt change in spatial patterns due to destabilization is an instance of *bifurcation*. Thus, bifurcation theory in general provides the canonical tools to tackle our problem. This study builds on the following formal facts on the stability and bifurcation of equilibria to examine the formation of spatial patterns in a many-region economy:[19]

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[16] Another approach is *global* stability analysis based on perfect foresight dynamics (Oyama, 2009a,b).

[17] For (i), we assume the differentiability of $F(h, v(h))$ as a whole on the tangent space of $\mathcal{D}$. The second, (ii), is called *positive correlation* (Sandholm, 2010), which is defined by $\sum_{i \in \mathcal{K}} v_i(h) \cdot \dot{h}_i > 0$ for all $h \in \mathcal{D}$. The last, (iii), requires that $F(h, v(h))$ live in the tangent cone of D for all $h \in \mathcal{D}$. Furthermore, although this study focuses on homogeneous payoffs, one can analyze the stability of spatial equilibria with idiosyncratic taste heterogeneity (e.g., Murata, 2003; Redding, 2016; Behrens, Mion, Murata and Südekum, 2017; Monte, Redding and Rossi-Hansberg, 2016) by using *perturbed best response dynamics*.

[18] That is, if h^* is a spatial equilibrium, we have $\dot{h} = F(h^*, v(h^*)) = 0$.

[19] In the rest of the paper, we sacrifice mathematical accuracy to reduce unnecessary burden for general readers. For a rigorous and general textbook treatment of the stability analysis of dynamical systems and

Fact 1. Consider a spatial equilibrium h^*. Let $J \equiv [\partial F_i(h^*, v(h^*))/\partial h_j]$ be the Jacobian matrix of the dynamic F evaluated at h^*. Let the eigenvalues of J be $g = (g_k)_{k \in \mathcal{K}}$.[20] Then, h^* is stable if all the K eigenvalues have strictly negative real parts; it is unstable if any of the eigenvalues has a strictly positive real part.

Fact 2. Let h^* be a stable spatial equilibrium, i.e., an equilibrium at which every eigenvalue of $J(h^*)$ has strictly negative real parts. Suppose that any of the eigenvalues, say g_k, switches its sign because of a change in the value of an underlying model parameter. Then, bifurcation occurs: h^* becomes unstable and the spatial pattern moves in the direction of $\eta_k = (\eta_{k,i})_{i \in \mathcal{K}}$, which is the eigenvector associated with g_k; given a real number ϵ, a pattern that can be expressed as $h^* + \epsilon \eta_k$ emerges.

Note that when we employ Fact 2, we can focus on η_k with $\sum_{i \in \mathcal{K}} \eta_{k,i} = 0$ because we assume that the total number of mobile agents is preserved under F.[21]

The two-region story is related to Facts 1 and 2 in the following way. Consider a two-region economy that comprises two regions 0 and 1 with completely homogeneous characteristics. The uniform pattern $\bar{h} \equiv (h, h)$ is obviously a spatial equilibrium. The (two) eigenvectors of J are given by $\eta_0 = (1, 1)$ and $\eta_1 = (1, -1)$ with the associated eigenvalues g_0 and g_1, respectively. The former, η_0, induces a change in the total mass of mobile agents and is irrelevant in a closed economy. The latter, η_1, expresses the agglomeration of mobile agents toward one of the regions, say 0. The associated eigenvalue, g_1, then coincides with the differential of the payoff difference between the two regions $\Delta v(h) \equiv v_0(h) - v_1(h)$ up to a positive constant. If $g_1 < 0$, then a marginal increase in the population share of region 0 induces a relative *decrease* in the payoff in region 0. Hence, no mobile agent hopes to leave region 1. If a decrease in transport costs changes the sign of g_1 from negative to positive, then relocation becomes strictly beneficial for agents in region 1, i.e., $\bar{h}$ become unstable, and agglomeration emerges.

2.3 Interpreting eigenvalues: Net agglomeration forces

From Facts 1 and 2, by analyzing the eigenpairs (i.e., eigenvalues g and eigenvectors $\{\eta_k\}$) of $J(h^*)$, one can examine *when* the destabilization of a given equilibrium h^* occurs and *which* spatial pattern(s) emerge thereafter. Although seemingly mechanical, as one would expect from the above example of the two-region setup, g and $\{\eta_k\}$ have rich economic meanings.

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bifurcation theory, see, for example, Guckenheimer and Holmes (1983) and Kuznetsov (2004). An earlier attempt to apply bifurcation theory to spatial structural evolution can be found in Wilson (1981).

[20] Allowing notational abuse, $\mathcal{K}$ denotes the K-dimensional index sets for the regions and the eigenvalues and eigenvectors of J.

[21] To be precise, to examine the stability of a given (interior) equilibrium h^*, it suffices to analyze the eigenvalues of the restricted linear map $J(h^*) : T\mathcal{D} \to T\mathcal{D}$, where $T\mathcal{D} \equiv \{\eta \in \mathbb{R}^K \mid \eta \cdot \mathbf{1} = 0\}$ is the tangent space of $\mathcal{D}$ (see Appendix B.4).

The sign of an eigenvalue g_k dictates whether $\boldsymbol{h}^*$ is stable in the direction of the associated eigenvector $\boldsymbol{\eta}_k$. We provide some intuitions. Given an interior equilibrium $\boldsymbol{h}^*$, consider a small variation in the spatial pattern such that $\boldsymbol{h} = \boldsymbol{h}^* + \boldsymbol{\eta}_k$, where $\boldsymbol{\eta}_k \equiv (\eta_{k,i})_{i \in \mathcal{K}}$ is one of the eigenvectors of $\boldsymbol{J}(\boldsymbol{h}^*)$, whose associated eigenvalue is g_k. Then, under our assumptions of $\boldsymbol{F}$, one can show that[22]

$$\text{sgn}[g_k] = \text{sgn}[\delta V(\boldsymbol{\eta}_k)], \tag{2.1}$$

where $\delta V(\boldsymbol{\eta}_k)$ and $\delta V_i(\boldsymbol{\eta}_k)$ are respectively defined by

$$\delta V(\boldsymbol{\eta}_k) \equiv \sum_{i \in \mathcal{K}} \delta V_i(\boldsymbol{\eta}_k) \eta_{k,i} \quad \text{and} \quad \delta V_i(\boldsymbol{\eta}_k) \equiv \sum_{j \in \mathcal{K}} \frac{\partial v_i(\boldsymbol{h}^*)}{\partial h_j} \eta_{k,j}. \tag{2.2}$$

Note that $\eta_{k,i} = h_i - h_i^*$ is either positive or negative. Observe that $\delta V_i(\boldsymbol{\eta}_k)$ is the marginal increase in the payoff in region i when the spatial pattern changes to $\boldsymbol{h} = \boldsymbol{h}^* + \boldsymbol{\eta}_k$. Accordingly, $\delta V(\boldsymbol{\eta}_k)$ is the weighted sum of the marginal increase in the payoffs across the regions.

If g_k is strictly negative (positive), $\delta V(\boldsymbol{\eta}_k)$ is strictly negative (positive). This implies that if $g_k < 0$, the collateral deviation in the $\boldsymbol{\eta}_k$ direction is strictly undesirable for relocated agents. To see this, rewrite $\delta V(\boldsymbol{\eta}_k)$ as follows:

$$\delta V(\boldsymbol{\eta}_k) = \sum_{\eta_{k,i}>0} \delta V_i(\boldsymbol{\eta}_k) \left| \eta_{k,i} \right| - \sum_{\eta_{k,i}<0} \delta V_i(\boldsymbol{\eta}_k) \left| \eta_{k,i} \right|. \tag{2.3}$$

The first (second) term on the right-hand side is the average payoff increase in the destination (origin) regions of migration; thus, the weighted sum $\delta V(\boldsymbol{\eta}_k)$ is the net increase in the payoff experienced by relocated agents. If all $\{g_k\}$ are strictly negative, for any direction there is no incentive to relocate and thus the equilibrium is stable. It is also intuitive to consider a single hypothetical agent who may want to relocate from region i to j; his or her payoff gain is given by $\delta V = \delta V_j - \delta V_i$. If all $\{g_k\}$ are strictly negative, it follows that $\delta V < 0$ and there is no incentive for such a relocation. Conversely, if any of $\{g_k\}$ is positive, a collateral deviation in the $\boldsymbol{\eta}_k$ direction is beneficial for all relocated agents and a snowball effect will kick the spatial pattern out of the equilibrium; that is, the equilibrium is unstable.

In the context of economic geography models, one can interpret each eigenvalue g_k as the *net force* in its associated direction of deviation $\boldsymbol{\eta}_k$ in the sense that g_k reflects the net effect of the agglomeration and dispersion forces at work in the $\boldsymbol{\eta}_k$ direction. Depending on its sign, g_k expresses the net agglomeration force (if positive) or the net dispersion force (if negative). In particular, if only one of them happens to be positive, then the spatial pattern is unstable

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[22]See Appendix B.4. The discussion here assumes that g_k and $\boldsymbol{\eta}_k$ are both real, as this property holds true throughout our analyses below.
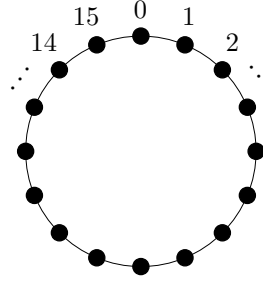
Figure 1: A racetrack economy ($K = 16$)

Note: The thin lines represent the transportation network and the black points represent the discrete regions in which mobile agents can be located. The regions are sequentially numbered.

and agglomeration occurs in the direction of the associated eigenvector.

3 Spatial scale of endogenous agglomeration and dispersion

Although the general facts on the local stability and bifurcation of equilibria are in principle applicable to any situation in general geographical setups (i.e., the assumed structures of D), the analytical results are difficult to obtain; thus, formal implications are limited. This section introduces a minimal and ideal geographical setup, namely a *racetrack economy* that considerably simplifies the local stability analysis of spatial equilibria in general economic geography models. Despite the technical simplification, the setup preserves the heterogeneities in interregional distances—an indispensable feature to express the *spatial scale of agglomeration and dispersion patterns*. By employing the desirable properties of the geographical setup, we reveal the two distinct spatial scales of the dispersion force that determine the spatial pattern of agglomerations. Concrete examples are discussed in Section 4.

3.1 Racetrack economy: Desired testbed

We assume a *racetrack economy* à la Krugman (1993) (**Figure 1**).[23] The K regions are equidistantly spread in a circle and sequentially numbered from zero, with transportation possible only around the circumference. The circumferential length is normalized to unity. Furthermore, we assume that there are no region-fixed advantages in terms of, for instance, local amenities or productivity differences. The geographical setup provides an ideal testbed to analyze the intrinsic properties of a many-region economic geography model for two reasons.

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[23]Our approach to local stability analysis that uses a racetrack economy was developed by Akamatsu, Takayama and Ikeda (2012), and an application can be found in Osawa, Akamatsu and Takayama (2017); see Appendix B for a summary. As the approach focuses on *local bifurcations* from a given equilibrium, group-theoretic bifurcation theory combined with numerical analysis provide complementary insights into the *global bifurcation* behavior of equilibria. See Ikeda, Akamatsu and Kono (2012); Ikeda, Murota, Akamatsu, Kono and Takayama (2014); Ikeda, Murota and Takayama (2017a), as well as Ikeda and Murota (2014).

First, it allows us to isolate the role of endogenous forces accruing from externalities and increasing returns in the determination of spatial patterns of agglomeration. In particular, it abstracts from the location-fixed advantages induced by the *shape* of the underlying transportation network. For instance, in a long narrow economy (e.g., Solow and Vickrey, 1971; Beckmann, 1976), the regions near the boundaries have fewer opportunities to access the other regions; the central portion is advantageous because of the shape of space. In our setup, by contrast, every region has the same level of accessibility to the other regions.[24]

Second, despite its simplicity, the setup incorporates heterogeneities in interregional distances. Let ℓ_{ij} denote the shortest path length from region i to j on the circumference; then, we have for example $\ell_{0,1} = \ell_{1,0} = 1/K$ and $\ell_{K-1,1} = \ell_{1,K-1} = 2/K$.[25] The heterogeneity in interregional distances makes the *relative* location in space matter, which is not the case for the common two-region setup. Furthermore, the symmetric racetrack economy reduces to the two-region setup if $K = 2$; the former is thus a natural generalization of the latter.

In addition, in line with Krugman (1993), we assume that the spatial friction between each pair of regions takes Samuelson's iceberg form, a standard choice for general equilibrium models.[26] In concrete terms, d_{ij} is given by $d_{ij} = \exp[-\tau \ell_{ij}]$ with a transport technology parameter $\tau \in (0, \infty)$. $\boldsymbol{D}$ is thus symmetric because $\ell_{ij} = \ell_{ji}$. Moreover, each d_{ij} is decreasing in τ. When we consider a steady improvement in transportation technology—that is, a continued decrease in τ—the spatial frictions between the regions gradually vanish ($d_{ij} \to 1$ for all i and j as $\tau \to 0$).

3.2 Local and global forces and the basic roles of space

The first virtue of assuming a racetrack structure is that the uniform distribution is always an equilibrium when the payoff function is symmetric across the regions. For this reason, one can follow extant theories that assume the spatial distribution of mobile agents to be initially uniform and study the endogenous formation of spatial patterns due to pure economic forces. We denote the *flat-earth equilibrium* on the racetrack by $\bar{\boldsymbol{h}} \equiv (h, h, \ldots, h)$ with $h \equiv H/K$. Furthermore, it is typical that at the flat-earth equilibrium, $\boldsymbol{J}$ and $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \equiv [\partial v_i(\bar{\boldsymbol{h}})/\partial h_j]$ are closely related. If we let $e_k(\tau)$ be the eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, we often have $g_k(\tau) = c e_k(\tau)$ with a positive constant c.[27] Thus, not only the sign but also the magnitude of $g_k(\tau)$ matters—in

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[24]In this sense, our setup has an intrinsic complementarity with the many-region analyses by Matsuyama (1999), who abstracted from endogenous positive feedbacks and focused on the role of *geography* itself.

[25]In concrete terms, $\ell_{ij} = \min \{|i - j|, K - |i - j|\}$.

[26]Some models, e.g., those by Ottaviano et al. (2002), Tabuchi et al. (2005), and Picard and Tabuchi (2013), have assumed non-iceberg transport technology. In principle, our analytical approach is effective with respect to these models, albeit the analysis is far more tedious compared with the iceberg case; the models can be fit to either class (i) or (ii) [or (iii)] discussed in Section 1 and introduced below. We refrain from analyzing non-iceberg models to simplify our presentation.

[27]For instance, the *replicator dynamic* (Taylor and Jonker, 1978) satisfies $c = h$ (see Appendix B.4).
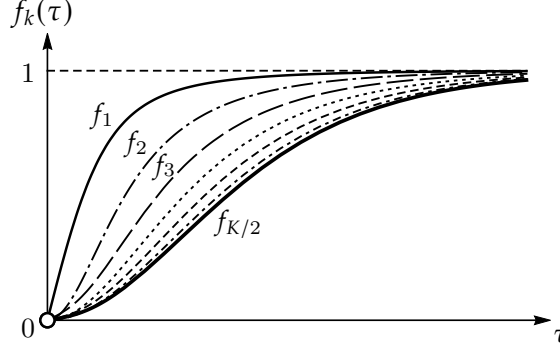
Figure 2: Eigenvalues of the friction matrix D for a racetrack economy with $K = 16$

Note: Every $f_k(\tau)$ for $1 \leq k \leq K$ is an increasing function of τ. Those for $1 \leq k \leq K/2$ are shown in the figure, because we have $f_k(\tau) = f_{K-k}(\tau)$ for $K/2 + 1 \leq k \leq K - 1$. In addition, for each given level of τ, $f_k(\tau)$ is basically decreasing in k for $1 \leq k \leq K/2$ (see Appendix B).

fact, the relative magnitude of $g_k(\tau)$ represents that of the agglomeration and dispersion forces in the $\boldsymbol{\eta}_k$ direction.

The second and most important utility of imposing a racetrack structure is that the role of transport costs in the net agglomeration force becomes transparent. To see this, the notion of the *spatial scales* of agglomeration and dispersion forces is useful. Throughout this paper, we call an agglomeration or dispersion force *global* if it depends on the distance between regions (i.e., the friction structure D), while a force that does not depend on the distance between regions is termed a *local* agglomeration or dispersion force.

Below, we consider a toy model that reveals the intrinsic workings of a global force. Consider the following simplest reduced-form payoff specification that implements only a black-box positive externality of agglomeration but no dispersion force:

$$v(h) = Dh, \tag{3.1}$$

or, in the element-wise form, $v_i(\boldsymbol{h}) = \sum_{j \in \mathcal{K}} d_{ij} h_j$. This simple model is a canonical example of models with a *global agglomeration force*—an agglomeration force that depends on interregional distances. This payoff function implies that each mobile agent wants to be as close as possible to the other agents as in Beckmann (1976), and that there is no counteracting force to prevent them to cluster. A straightforward intuition suggests that all mobile agents agglomerate in one region in equilibrium. This intuition helps us associate the agglomeration/dispersion force with the distance structure of the economy, i.e., the eigenvalues and eigenvectors of D as explained below. It is evident that we have $\nabla v(\boldsymbol{h}) = D$ at the flat-earth equilibrium.

The net agglomeration forces $\{g_k(\tau)\}$, or the eigenvalues of $\boldsymbol{J}$, are thus given by $g_k(\tau) = hd(\tau)f_k(\tau)$ with $\{f_k(\tau)\}$ being the eigenvalues of the row-normalized version of the friction
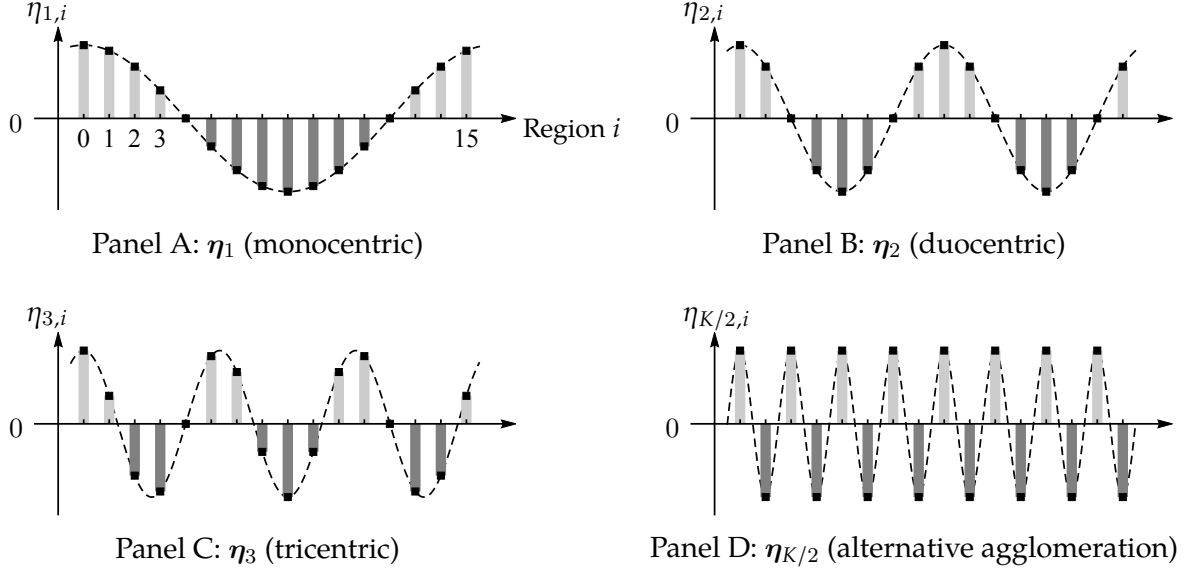
Panel A: $\boldsymbol{\eta}_1$ (monocentric)

Panel B: $\boldsymbol{\eta}_2$ (duocentric)

Panel C: $\boldsymbol{\eta}_3$ (tricentric)

Panel D: $\boldsymbol{\eta}_{K/2}$ (alternative agglomeration)

Figure 3: Illustrations of the eigenvectors $\boldsymbol{\eta}_k$ ($K = 16$; $k = 1, 2, 3, K/2$)

Note: The negative (positive) elements of an eigenvector $\boldsymbol{\eta}_k$ indicate that if the flat-earth pattern is perturbed into the direction, so that the new spatial pattern is $\boldsymbol{h} = \bar{\boldsymbol{h}} + \epsilon \boldsymbol{\eta}_k$ with $\epsilon > 0$, such regions experience a decrease (increase) in their population.

matrix $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d(\tau)$, where $d(\tau) = \sum_{j \in \mathcal{K}} d_{ij}(\tau) > 0$ is the row sum of $\boldsymbol{D}$.[28] In a racetrack economy, we have analytical expressions of the eigenvalues $\{f_k(\tau)\}$ as well as those of their associated eigenvectors $\{\boldsymbol{\eta}_k\}$ (see Appendix B). Consequently, the eigenvectors of $\boldsymbol{J}$ are also given by $\{\boldsymbol{\eta}_k\}$. The eigenvector associated with $g_k(\tau)$ is $\boldsymbol{\eta}_k = (\eta_{k,i}) = (\cos[\theta k i])$ with $\theta \equiv 2\pi/K$; we ignore g_0 in the following because $\boldsymbol{\eta}_0 = (1, 1, \ldots, 1)$ violates the conservation of the total mass of agents.

Figure 2 illustrates $\{f_k(\tau)\}_{k \geq 1}$ for $K = 16$. Each $f_k(\tau)$ ranges from 0 to 1 and decreases if τ decreases. When interregional transport costs decline, the effects of the friction matrix vanish. Thus, we see that $g_k(\tau) > 0$ for all $k \geq 1$, and hence $\bar{\boldsymbol{h}}$ is never stable (Fact 1). Because no dispersion force can stabilize the flat-earth equilibrium, it is also natural that $\bar{\boldsymbol{h}}$ is unstable for any value of τ.

The relative magnitude of the net agglomeration forces $\{g_k(\tau)\}$ is of interest. To this end, for the toy model, one can see that $f_k(\tau)$ determines the relative strength between $\{g_k(\tau)\}$. Note that $f_k(\tau)$ is decreasing in k (see **Figure 2**), with the maximal $f_1(\tau)$ for all τ. Thus, the maximal among $g_k(\tau)$ is also $g_1(\tau)$. But why does this occur?

Looking at the eigenvectors $\{\boldsymbol{\eta}_k\}$ provides intuitions. Some examples of $\boldsymbol{\eta}_k$ with $K = 16$ are illustrated in **Figure 3** for $k = 1, 2, 3, K/2$.[29] The negative (positive) element $\eta_{k,i}$ in $\boldsymbol{\eta}_k$ indicates that if the spatial pattern slightly changed in the $\boldsymbol{\eta}_k$ direction so that $\boldsymbol{h} = \bar{\boldsymbol{h}} + \epsilon \boldsymbol{\eta}_k$

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[28]We assume the replicator dynamic as the underlying dynamic $\boldsymbol{F}$ for illustration purposes (see Example B.3 in Appendix B.4). Note also that the row sum of $\boldsymbol{D}$ is row-independent in a racetrack economy.

[29]To simplify the presentation, we assume that the number of regions K is a multiple of four. Qualitatively, the exact number of regions is inconsequential if it is sufficiently large.

with $\epsilon > 0$, the number of mobile agents decreases (increases) in the region. In a symmetric racetrack economy, the possible directions of change are characterized by the number of *peaks*, k, or, in other words, by the number of population concentrations (i.e., *agglomerations*). η_1 (Panel A of **Figure 3**) is directed to a monopolar pattern with a single peak and hence expresses the emergence of a global concentration of mobile agents; η_2 (Panel B) expresses the emergence of two major concentrations, while η_3 (Panel C) expresses the emergence of three major concentrations; $\eta_{K/2}$ (Panel D) expresses the emergence of the smallest possible agglomerations. In other words, η_1 immediately pushes the flat-earth equilibrium toward a unimodal agglomeration, while η_2, η_3, and $\eta_{K/2}$ (as well as the other η_k except for η_1) pushes the flat-earth equilibrium toward the other multimodal patterns. As we assume a featureless space, the peaks are equidistantly spaced.

Given the knowledge of $\{\eta_k\}$, the maximality of $g_1(\tau)$ now has clear economic meaning. We understand that the associated eigenvector η_1 for $g_1(\tau)$ is a unimodal, monocentric agglomeration (Panel B of **Figure 3**). Since there are no negative effects of agglomeration in the model, a monocentric concentration is the most beneficial outcome for every agent. As the number of peaks in η_k increases, the size of a single agglomeration falls. This obviously reduces the magnitude of the positive externalities and is less favorable. Moreover, $f_k(\tau)$ decreases as τ decreases because when the level of interregional transport costs is low, there is less incentive for agglomeration.

3.3 Endogenous formation of agglomeration out of uniformity

For canonical economic geography models in the literature, at the flat-earth equilibrium, J is related to the row-normalized version of the friction matrix, $\bar{D}(\tau)$, in the following form (see Appendices B and C):[30]

$$J \simeq \nabla v(\bar{h}) \simeq G(\bar{D}(\tau)), \tag{3.2}$$

where the function $G(\bar{D})$ is defined by $G(\bar{D}) \equiv c_0 I + c_1 \bar{D} + c_2 \bar{D}^2$ with model-dependent (positive or negative) coefficients c_0, c_1, and c_2. Consequently, in pallalel with (3.2), the kth ($k \neq 0$) eigenvalue $g_k(\tau)$ of J satisfies (see Appendix B)

$$\mathrm{sgn}[g_k(\tau)] = \mathrm{sgn}[e_k(\tau)] = \mathrm{sgn}\left[G(f_k(\tau))\right], \tag{3.3}$$

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[30]The notation $\simeq$ for the matrices means that the left-hand side coincides with the right-hand side multiplied by some real, symmetric, and circulant matrix J_0, which is positive definite relative to $T\mathcal{D}$. For our purpose in this study (i.e., the local stability analysis of $\bar{h}$), we can practically "ignore" J_0 in our discussion. Also noted is that the convention is just to simplify the presentation.

where $e_k(\tau)$ and $f_k(\tau)$ are the kth eigenvalue of $\nabla v(\bar{h})$ and $\bar{D}(\tau)$, respectively, and the k-independent function $G(f)$ is defined by

$$G(f) = c_0 + c_1 f + c_2 f^2 \tag{3.4}$$

in line with $G(\bar{D})$. The eigenvector associated with each $g_k(\tau)$ is again $\eta_k = (\eta_{k,i}) = (\cos[\theta k i])$ with $\theta \equiv 2\pi/K$ (**Figure 3**). Recall that one can ignore g_0 provided that the underlying dynamic F preserves the total mass of mobile agents.

By employing our definition of local and global forces, we see that c_0 summarizes the *local* agglomeration and dispersion forces in the model and that c_1 and c_2 summarize the *global* ones. Usually, we have $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$. For example, a crowding-out effect inside a region due to congestion or point-wise scarcity of land produces a *local* dispersion force, resulting in a negative constant term ($c_0 < 0$); a global social interaction (e.g., Beckmann, 1976) is suggested by a positive first-order term ($c_1 > 0$); and goods demand from spatially dispersed consumers in other regions (e.g., Krugman, 1991) is indicated by a negative second-order term ($c_2 < 0$).[31]

In the following, we assume the most general case of $G(f)$ in the literature: $G(f)$ is given by $G(f) = c_0 + c_1 f + c_2 f^2$ with $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$, with two roots f^* and f^{**} for $G(f) = 0$ in $(0, 1)$ such that $f^{**} < f^*$. The shape of $G(f)$ under these assumptions is shown in the bottom left panel of **Figure 4**. The functional form of $G(f)$ corresponds to a model with a local dispersion force, global agglomeration force, and global dispersion force.

The properties of $\{f_k(\tau)\}$ are completely *model-independent*; because $\{f_k(\tau)\}$ are merely the eigenvalues of the (normalized version of the) friction matrix $\bar{D}(\tau)$, they are invariant regardless of the economic geography model (i.e., the payoff function $v(h)$) one may assume. Instead, the function $G(f)$ in (3.4), or equivalently the matrix relation (3.2), encapsulates the net effects of the economic interactions in the model and provides insights into the endogenous formation of spatial patterns.

The question posed is as follows: given such $G(f)$, which spatial pattern emerges after an encountered bifurcation? In particular, will it be a *unimodal pattern* or a *multipolar pattern*?

Choose an appropriate value of τ so that $\bar{h}$ is stable; that is, the net agglomeration forces $\{g_k(\tau)\}$ are strictly negative, meaning that any deviation is strictly non-beneficial. Consider a gradual change in τ. When any of the net agglomeration forces becomes positive, the flat-earth equilibrium stops being stable and agglomerations emerge. What one should observe here is the first $g_k(\tau)$ that changes its sign from negative to positive. Let $\tau^\star$ be the critical value at which this occurs. It is evident that $\tau^\star$, or the so-called *break point*, is a solution to the equation $\max_{k \in \mathcal{K}} \{g_k(\tau^\star)\} = 0$. Denote the index of the critical eigenvalue such that

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[31] In Appendix C, we present detailed analyses of how economic geography models are mapped to the coefficients $\{c_i\}$ by taking the models in the literature as concrete examples.
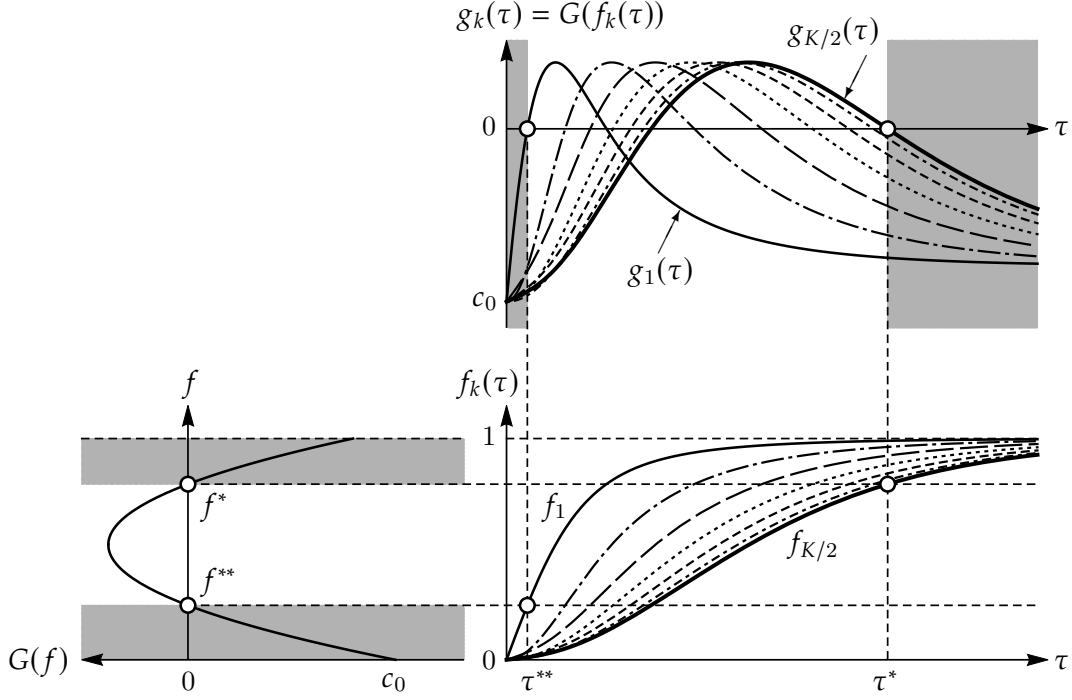
Figure 4: Net agglomeration forces and their model-dependent and -independent components

Note: Top: The net agglomeration forces $\{g_k(\tau)\}$. We consider the simplest case: $g_k(\tau) = G(f_k(\tau))$. Bottom left: An example of the model-dependent function $G(f)$. Bottom right: The eigenvalues $\{f_k(\tau)\}$ of $\bar{D}$, which are model-independent. $\bar{h}$ is stable in the dark gray regions of τ or f.

$g_k(\tau^\star) = \max_k g_k(\tau^\star)$ by $k_\star^{\mathrm{crit}}$. Then, the spatial pattern at $\tau^\star$ is expressed in terms of the $k_\star^{\mathrm{crit}}$th eigenvector as $h = \bar{h} + \epsilon \eta_{k_\star^{\mathrm{crit}}}$, where ϵ is a real number. Under our assumption of $G(f)$, the curves of $\{g_k(\tau)\}$ behave as in the top panel of **Figure 4**; the upper envelope of the curves represents $\max_{k \in \mathcal{K}} \{g_k(\tau^\star)\}$, and the critical points are found where the curve crosses the horizontal axis. There are two solutions, τ^* and τ^{**}, and we have $k_*^{\mathrm{crit}} = K/2$ and $k_{**}^{\mathrm{crit}} = 1$. See **Figure 5** for the spatial patterns that emerge at τ^* (Panel A) and τ^{**} (Panel B).

The stability of the flat-earth equilibrium for the *higher* level of τ is attributed to the *global* dispersion force, while that for the *lower* level of τ is attributed to the *local* dispersion force. As transport costs decline from a high level, the flat-earth equilibrium collapses at τ^* because the global dispersion force declines (recall that $f_k(\tau)$ decreases as τ decreases). When τ decreases below another threshold, τ^{**}, it brings about a situation where the flat-earth equilibrium becomes stable again because the local dispersion force, which always exists regardless of τ, overcomes the agglomeration force.

3.4 Rethinking redispersion

Panels A and B of **Figure 5** illustrate the two mutually distinct spatial patterns that emerge at τ^* and τ^{**}, respectively. Panel A illustrates the spatial pattern that emerges at τ^*, which is

Panel A: $\bar{h} + \epsilon \eta_{K/2}$　　　　Panel B: $\bar{h} + \epsilon \eta_1$
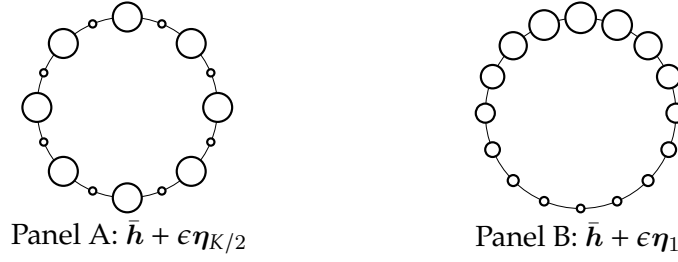
Figure 5: Schematic illustrations of the spatial patterns at τ^* and τ^{**} ($K = 16$)

Note: The size of the small white circle represents the number of mobile agents in the region. Panel A: bifurcation at τ^* (a locally concentrated and globally dispersed pattern); Panel B: bifurcation at τ^{**} (a globally concentrated and locally dispersed pattern).

interpreted as a *locally concentrated and globally dispersed pattern*. This is characterized by the formation of many small agglomerations spread around the circumference. In the pattern, mobile agents are locally concentrated, whereas the agglomerations are equidistantly spaced or globally dispersed.[32] Panel B illustrates the pattern at τ^{**}, which is interpreted as a *globally concentrated and locally dispersed pattern*. In this pattern, agents are globally concentrated to shape a unimodal distribution (a single agglomeration with a large spatial extent).

The two critical points τ^* and τ^{**} are customarily termed in the literature the "emergence of core and periphery" and "redispersion (revival of the periphery)," respectively, and the process as a whole is denoted "bell-shaped development" (Fujita and Thisse, 2013). When transport costs are very high ($\tau > \tau^*$), the symmetric equilibrium is stable. In the first stage of the decline in transport costs, the destabilization of the symmetric equilibrium results in spatial inequality. In the later stage, once established, agglomeration is no longer sustainable and the symmetric configuration is stable again ($\tau < \tau^{**}$).

The redispersion process is simply considered to be the reverse process of agglomeration. For any model with a single type of mobile agent, it is supposed that there is no essential difference in the spatial patterns in the two stages (around τ^* and τ^{**}).[33] Indeed, this is true in the two-region setup where the two relevant eigenvectors coincide: $\eta_{K/2} = \eta_1 = (1, -1)$. However, our analysis so far has shown that it is *not* the case in a many-region economy. The two bifurcations at τ^* and τ^{**} are of a distinct nature: each represents the emergence of mutually distinct spatial patterns and is attributed to dispersion forces at different spatial scales (i.e., global and local).

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[32]Observe that the spatial pattern resembles those obtained by the numerical simulations presented by Krugman (1993) for $K = 12$. The spatial pattern is also similar to the pre-assumed spatial patterns in Tabuchi and Thisse (2011).

[33]Takatsuka and Zeng (2009) analyzed redispersion behavior in a two-region economy, NEG model with *multiple industries* and distinct returns to scale, and found asymmetry in the two processes, namely that the industrial composition in each region is different in the redispersion phase.
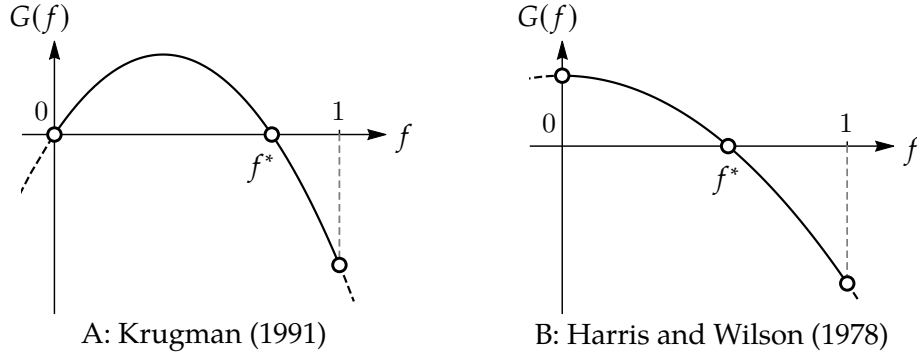
Figure 6: $G(f)$ for the models proposed by Krugman (1991) and Harris and Wilson (1978)

4 Classification of models by the spatial scale of dispersion

The distinction between global and local dispersion forces allows us to reduce economic geography models to two canonical classes: (i) those with only a global dispersion force and (ii) those with only a local dispersion force. This section provides concrete examples of global and local dispersion forces by employing selected models in the literature. For every model discussed in this section, J is shown to have up to the second-order term of $\bar{D}$ as in (3.2), meaning that $G(f)$ is (at most) a quadratic of f as in (3.4). **Table 1** is the resultant classification. Detailed analyses of the models in the table are relegated to Appendix C.

4.1 Class (i): Models with a global dispersion force

Global dispersion forces are those that arise *outside* a given agglomeration, typically implemented as spatially dispersed demand. A global dispersion force usually appears in J as a negative (second-order) term with respect to D. For instance, the NEG models proposed by Krugman (1991), Puga (1999), Forslid and Ottaviano (2003), and Pflüger (2004) satisfy $c_0 = 0$ and we have $G(f) = c_1 f + c_2 f^2$ with $c_1 > 0$ and $c_2 < 0$. Furthermore, the model proposed by Harris and Wilson (1978), a classical model of spatial self-organization proposed in the field of geography, satisfies $G(f) = c_0 + c_2 f^2$ with $c_0 > 0$ and $c_2 < 0$.[34]

Figure 6 illustrates $G(f)$ for Krugman (1991) and Harris and Wilson (1978). Because $G(f)$ is a concave quadratic with $G(0) \geq 0$, $G(f)$ has at most a single solution f^* in $(0, 1)$; this implies that a single critical value (break point) of transport costs τ^* can exist.[35] As discussed in the previous section, at τ^*, the emergent pattern is *locally concentrated and globally dispersed* (or a *multimodal* pattern) in which multiple distinct agglomerations are endogenously formed

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[34]For the drawn cases, the underlying parameters satisfy the so-called "no-blackhole condition" that ensures the stability of the flat-earth pattern in the higher extreme of τ. Otherwise, we have $G(1) > 0$ and the flat-earth pattern is always unstable.

[35]We can show that the influential model of Ottaviano et al. (2002) also endogenously produces globally dispersed patterns; hence, this is a class (i) model. As the model assumes non-iceberg transportation technology, we do not discuss it here to simplify our presentation (see also Footnote 26).

(Panel A of **Figure 5**).

The seminal model of Krugman (1991) is an example. Appendix C provides the omitted derivations of the indirect utility function and other formulae of this model as well as the results under the other models discussed below. The payoff function (i.e., indirect utility function of mobile workers) is given by

$$v_i(\boldsymbol{h}) = w_i P_i^{-\mu} \tag{4.1}$$

where w_i denotes the nominal wage of mobile workers and

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} h_j w_j^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)} \tag{4.2}$$

denotes the price index in region i. The parameters μ and σ are the expenditure share of the manufactured good and elasticity of substitution between the varieties, respectively. The wage is obtained as the (unique) solution to the so-called wage equation that reflects the short-run utility maximization of consumers, trade balance, and zero-profit condition for firms. In each region, there is an exogenous endowment of immobile workers.

For the model, one has

$$\boldsymbol{J} \simeq \mu \left(\frac{1}{\sigma - 1} + \frac{1}{\sigma} \right) \bar{\boldsymbol{D}} - \left(\frac{\mu^2}{\sigma - 1} + \frac{1}{\sigma} \right) \bar{\boldsymbol{D}}^2. \tag{4.3}$$

The exact mappings to the coefficients $c_1 > 0$ and $c_2 < 0$ are thus given by $c_1 = \mu(\bar{\kappa} + \kappa)$ and $c_2 = -(\mu^2 \bar{\kappa} + \kappa)$, where we let $\bar{\kappa} \equiv 1/(\sigma - 1)$ and $\kappa \equiv 1/\sigma$. The coefficients c_1 and c_2 capture the *net* effects of the agglomeration and dispersion forces in the Krugman (1991) model, respectively. In particular, $\mu\bar{\kappa}$ in c_1 represents the so-called price-index effect, whereas $\mu\kappa$ represents a home-market effect. On the contrary, c_2 is the market-crowding effect: $\mu^2\bar{\kappa}$ in c_2 is due to firms' competition over demand from mobile agents and in κ is due to that from immobile agents. For more detailed discussions on the interpretations of the coefficients, see Remark C.2 in Appendix C.1.

4.2   Class (ii): Models with a local dispersion force

A local dispersion force acts *inside* each region and does not explicitly depend on the spatial distribution of mobile agents. The urban costs induced within each region (e.g., housing costs, congestion externalities) are typical. Examples include the frameworks of Helpman (1998), Redding and Sturm (2008), and Murata and Thisse (2005) as well as the perfectly
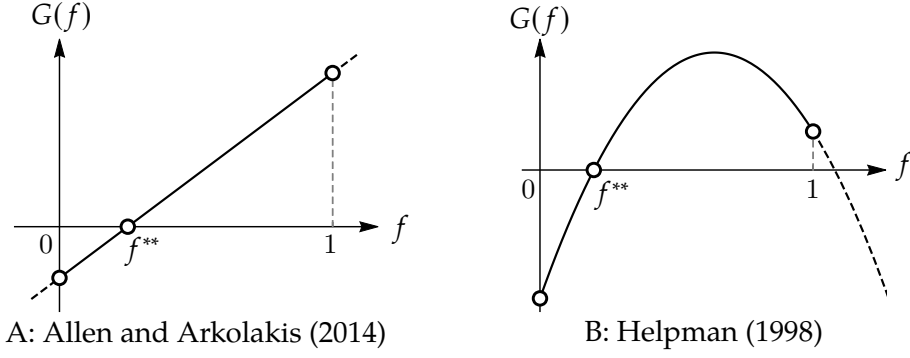
A: Allen and Arkolakis (2014)   B: Helpman (1998)

Figure 7: $G(f)$ for the models proposed by Allen and Arkolakis (2014) and Helpman (1998)

competitive framework of Allen and Arkolakis (2014).[36] Furthermore, the model proposed by Beckmann (1976) focusing on the internal structure of cities (Mossay and Picard, 2011; Blanchet et al., 2016) is another representative example.[37] At the flat-earth pattern, a local dispersion force appears in J as a negative *constant* term with respect to D (i.e., $c_0 < 0$).

Figure 7 illustrates $G(f)$ for the models proposed by Allen and Arkolakis (2014) and Helpman (1998). For these models, there exists at most a single critical point of f^{**}. If the model parameters are set such that there is an endogenous formation of agglomeration, the flat-earth equilibrium is stable for the lower level of transport costs. At the only bifurcation point τ^{**}, a *globally concentrated and locally dispersed pattern* (or a *unimodal* pattern) with a single agglomeration is endogenously formed (Panel B of **Figure 5**). In this class of models, without location-fixed factors, the only possible spatial pattern associated with agglomeration is a globally concentrated and locally dispersed pattern.

The model proposed by Allen and Arkolakis (2014) is a recent example. The indirect utility function of mobile workers is given by

$$v_i(\boldsymbol{h}) = h_i^{\beta} w_i P_i^{-1}, \tag{4.4}$$

where P_i denotes the price index for the model

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} h_j^{\alpha(\sigma-1)} w_j^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)}. \tag{4.5}$$

The parameters $\alpha > 0$ and $\beta < 0$ are the exponents of a reduced-form Marshallian externality

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[36]Without exogenous location-fixed factors, the model of Redding and Rossi-Hansberg (2017) (§3) is equivalent to that of Redding and Sturm (2008). The model of Monte et al. (2016) also belongs to class (ii), albeit it adds an extra urban cost as well as taste heterogeneity; we note that an idiosyncratic utility shock is equivalent to a local dispersion force (see Appendix B for a brief discussion). It is also evident that Picard and Tabuchi (2013) is a class (ii) model.

[37]We here consider a discrete-space version of the Beckmann model as formulated in Akamatsu, Fujishima and Takayama (2017). Akamatsu et al. (2017) showed that a discrete-space version of the model asymptotically converges to the continuous variant as the number of regions increases.

and a local congestion externality in amenities, respectively, and $w_i(\boldsymbol{h})$ is the market wage in region i. For this model, the source of agglomeration is the reduced-form local positive externality represented by the parameter α. One has

$$\boldsymbol{J} \simeq -(\alpha + \beta - \gamma_0)\boldsymbol{I} + (\alpha + \beta + \gamma_1)\bar{\boldsymbol{D}}, \tag{4.6}$$

with $\gamma_0 \equiv (1 + \alpha)/\sigma$ and $\gamma_1 \equiv (1 - \beta)/\sigma$. We thus have $G(f) = c_0 + c_1 f$ with $c_0 = -(\alpha + \beta - \gamma_0)$ and $c_1 = \alpha + \beta + \gamma_1$. If $\alpha + \beta \leq 0$, there is *no agglomeration force* and the flat-earth equilibrium is stable for any value of τ. If $\alpha + \beta > 0$, there is a local positive agglomeration force; we have $c_0 < 0$ and $c_1 > 0$ as well as $G(1) > 0$. If the agglomeration force is strong ($0 < \alpha + \beta$), the model can express endogenous agglomeration. In the net form, as indicated in (4.6), the model does not have any global dispersion force. Thus, we conclude that the model produces only unimodal patterns. In fact, Figure VIII in Allen and Arkolakis (2014) confirms this result. In other contexts, the model presented by Beckmann (1976) (Mossay and Picard, 2011; Blanchet et al., 2016; Akamatsu et al., 2017) yields a similar linear functional form of $G(f)$ since it incorporates a first-order global agglomeration force and a local dispersion force.

As discussed by Allen and Arkolakis (2014), their model is isomorphic to the Helpman (1998) model with local landownership (i.e., Redding and Sturm, 2008). One can show that the assumptions concerning landownership do not alter the above conclusion. For Helpman's original model, with public landownership, under appropriate normalizations, one obtains $\boldsymbol{J} \simeq c_0 \boldsymbol{I} + c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$, so that $G(f) = c_0 + c_1 f + c_2 f^2$ with $c_0 = -(1 - \mu)$, $c_1 = \mu(\bar{\kappa} + \kappa)$, and $c_2 = -(\kappa + \mu^2 \bar{\kappa}) + (1 - \mu)$. Again, $\bar{\kappa} = 1/(\sigma - 1)$ and $\kappa = 1/\sigma$, where μ is the expenditure share of manufactured goods and σ is the elasticity of substitution between manufactured goods. We have $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$; for the model, the agglomeration force is derived from the second term in $\boldsymbol{J}$, whereas dispersion forces are derived from the others. Observe that c_1 is as per the model proposed by Krugman (1991), meaning that the agglomeration force of the latter is isomorphic to that of the former. Panel B of **Figure 7** illustrates the shape of $G(f)$ for the model. It follows that whenever there is an endogenous agglomeration, we have $G(1) > 0$; thus, f^* does not exist. Although $c_2 < 0$ and there seemingly exists a global dispersion force, it is not effective. The main dispersion force of the model is therefore derived from the consumption of non-tradable housing stocks that produces a negative pecuniary externality through the local housing market.

4.3 Classification of representative economic geography models

Table 1 classifies representative economic geography models in the literature according to the nature of their dispersion forces and resulting stable spatial patterns (including our toy

model discussed in Section 3.2). The exact mapping to the coefficients of $G(f)$ is provided by **Table 2** at the end of Appendix C. As discussed, at the flat-earth equilibrium of a given model, one can characterize the fundamental tradeoff between the centripetal and centrifugal forces by the coefficients $\{c_i\}$ or the shape of $G(f)$. In particular, one can clearly distinguish the spatial scale of the model's effective dispersion force.

There are two canonical model classes, (i) and (ii). The former includes models with only a global dispersion force, while the latter includes models with only a local dispersion force. The second column of **Table 1** summarizes the characteristic spatial patterns for both of these model classes. Although our analysis concerns the endogenous formation of spatial patterns under a multiplicity of equilibria, class (i) and (ii) models have qualitatively different behavior and can yield mutually contradicting implications when employed for counterfactual exercises. This point is discussed in Section 6.3, and a formal analysis is provided in Appendix D.

In addition, a few models in the literature have the two dispersion forces effectively at work. For instance, Tabuchi (1998), Pflüger and Südekum (2008), and Takayama and Akamatsu (2011) presented both local and global dispersion forces. We refer to these as *class (iii)* models. Such models produce spatial patterns with mixed characteristics of global and local dispersion, which we discuss below by employing a numerical example. For this model class, $G(f)$ is a concave quadratic that has two roots in the $(0, 1)$ interval as in the bottom left panel of **Figure 4**. As discussed in Section 3, the flat-earth equilibrium is stable for both high and low transport costs.

Notably, our classification seems to be backed by a more general principle. A large body of studies outside economics focus on the formation of spatial patterns, typically on the basis of reaction–diffusion systems (Kondo and Miura, 2010). In that literature, it is now widely accepted that the basic requirement to form multiple peaks in stationary spatial patterns (i.e., in our context, stable locally concentrated and globally dispersed patterns) is *a short-range positive feedback combined with a long-range negative feedback* with respect to a concentration of mobile factors (Meinhardt and Gierer, 2000). Note that the negative term (global dispersion force) of $\bar{D}$ in J can be interpreted as a long-range negative feedback.

5 Numerical examples

In the many-region setup, the first bifurcation, or the emergence of agglomeration, may not be the end of the story. The overall evolutionary path of the spatial structure in line with a monotonic change (e.g., decline) in transport costs is of interest. Fortunately, the intrinsic properties of the whole evolutionary process of the agglomeration patterns can be qualitatively predicted by the above results on the stability of the flat-earth pattern. This

Table 1: Classification of economic geography models in the literature

Dispersion force	Spatial patterns	Economic geography models
None	Concentration to a *single* region (unimodal patterns)	The toy model defined by (3.1)
Only global [class (i)]	Locally concentrated and globally dispersed (multimodal patterns)	Krugman (1991) Puga (1999) Forslid and Ottaviano (2003) Pflüger (2004) Harris and Wilson (1978)
Only local [class (ii)]	Globally concentrated and locally dispersed (unimodal patterns)	Helpman (1998) Murata and Thisse (2005) Redding and Sturm (2008) Allen and Arkolakis (2014) Redding and Rossi-Hansberg (2017) (§3) Beckmann (1976) Mossay and Picard (2011) Blanchet et al. (2016)
Both [class (iii)]	Mixed characteristics of the classes (i) and (ii) (multimodal patterns)	Tabuchi (1998) Pflüger and Südekum (2008) Takayama and Akamatsu (2011)

Note: Appendix C provides detailed analyses of the models, with **Table 2** summarizing the exact mappings of each model to the coefficients of the corresponding model-dependent function $G(f) = c_0 + c_1 f + c_2 f^2$.

section provides some numerical illustrations. The models of Krugman (1991), Helpman (1998), and Pflüger and Südekum (2008) are chosen as representative examples of models with only a global dispersion force [class (i)], those with only a local dispersion force [class (ii)], and those with both dispersion forces [class (iii)], respectively. The numerical examples in this section are conducted in an eight-region ($K = 8$) symmetric racetrack economy. Following the literature, the replicator dynamic (Taylor and Jonker, 1978) is employed as the underlying dynamic $\boldsymbol{F}$. The chosen parameters are described in Appendix C.

5.1 Class (i): Models with a global dispersion force

Figure 8 reports the evolutionary path of stable equilibrium patterns in the course of decreasing τ for the Krugman (1991) model. The black solid (dashed) curves depict the stable (unstable) equilibrium values of population share $\boldsymbol{\lambda} = (\lambda_i)$ at each τ, where $\lambda_i \equiv h_i/H$. Consider a gradual decrease in τ from a sufficiently high level at which the flat-earth equilibrium is stable. The uniform distribution with no agglomerations is initially stable until τ reaches the break point τ^*. As discussed in the previous section, the bifurcation at τ^* pushes the spatial pattern in the direction of $\boldsymbol{\eta}_{K/2} = (1, -1, 1, -1, 1, -1, 1, -1)$. This results in the formation of a globally dispersed pattern with four disjoint and equidistantly separated
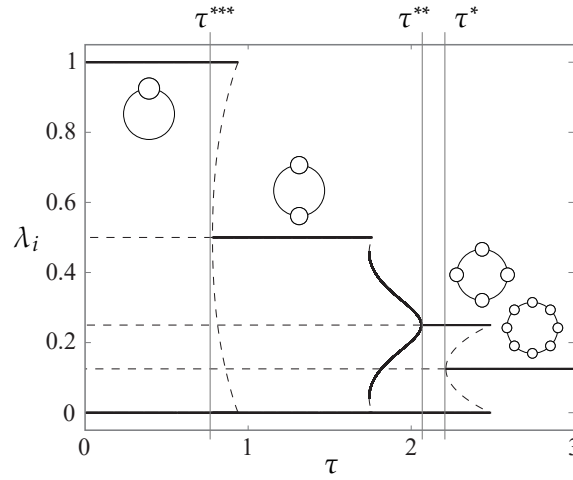
Figure 8: Bifurcation diagram of a class (i) model

Note: Krugman (1991) is taken as the underlying example model.

point-wise agglomerations.

A further decrease in τ triggers the second and third bifurcations at τ^{**} and τ^{***}, respectively.[38] Observe that the bifurcations at τ^{**} and τ^{***} sequentially double the spacing between agglomerated regions, reducing their number as $4 \to 2 \to 1$. At the lower extreme of τ, a monopolar pattern emerges. Note that each agglomeration has no spatial extent at any level of transportation cost, since the local dispersion force is absent. In the model, better interregional access (a smaller τ) enlarges the size of each agglomeration. Such an effect is, however, limited to the selected regions. Depending on the stage of the spatial structural evolution, the impact of an improvement in transportation on the size of *each* agglomeration can be either positive (for the selected regions) or negative (for the others). In this sense, there are no monotonic relationships between the level of transport costs and size of *each* agglomeration. In fact, this point is already apparent in two-region models that explicitly incorporate agglomeration economies combined with interregional distance.

In our many-region setup, however, there comes another indeterminacy. As the spatial structure evolves, *selected regions may decline* to form the hinterland of the currently selected ones—the *agglomeration shadow*.[39] Consider, as an example, the fourth region at the six o'clock position in **Figure 8**. The region is selected at the transitions at τ^* and τ^{**}, meaning that the impact of a decrease in τ is positive. After τ^{***} is encountered, however, it immediately loses its population. For the region, a monotonic decrease in τ implies a win situation followed by a lose situation. This indicates that if the empirical realities resemble this class of model, whether the impact of a further decline in transport costs on a specific region is positive or negative is indeterminate *even when a monotonic relation for the region is supported by historical*

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[38]In fact, one can analytically derive these critical values at τ^{**} and τ^{***} (see Akamatsu et al., 2012; Osawa et al., 2017) and characterize the spatial patterns that emerge at these points.

[39]The concept of the agglomeration shadow was first introduced by Arthur (1994) and it is formalized in the context of the general equilibrium model of Fujita and Krugman (1995).
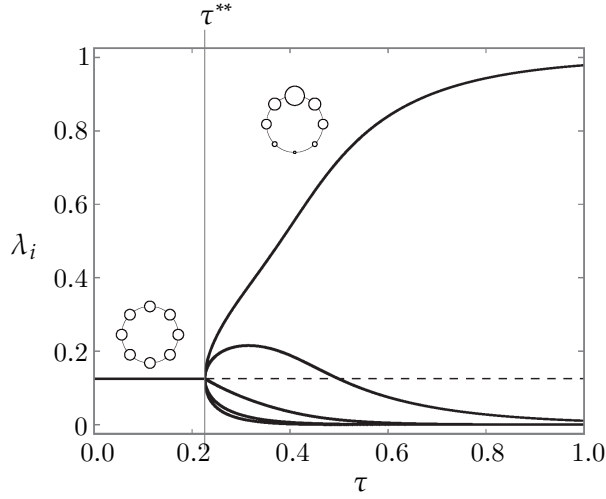
Figure 9: Bifurcation diagram of a class (ii) model

Note: Allen and Arkolakis (2014) is taken as the underlying example model.

data. At least in a symmetric racetrack economy, we do not have any clear implication for models in this class regarding the impact of a uniform reduction in transport costs on the population (or output) size of an *individual region* since whether the population share of a region grows or declines is *in principle* indeterminate a priori. Instead, possible predictions focus on the *global spatial distribution* of agglomerations, namely the number of concentrations and spacing between them, which monotonically decreases and increases, respectively.

5.2   Class (ii): Models with a local dispersion force

Figure 9 is similar to **Figure 8** for the model proposed by Allen and Arkolakis (2014). This model incorporates only a local dispersion force; the flat-earth equilibrium is stable for lower values of τ. At τ^{**} in **Figure 7**, bifurcation in the direction of η_1 leads to the emergence of a unimodal pattern. This is *the* bifurcation in the model; after the emergence of the unimodal pattern at τ^{**}, when τ increases further, the spatial pattern monotonically and smoothly converges to a monopolar pattern (i.e., the complete concentration of mobile agents at a single region) as τ approaches infinity. Thus, if we define the number of agglomerations for the model by that of the peaks (i.e., local maxima) in h, it is at most one. The model does not allow locally concentrated and globally dispersed patterns to emerge; such models would be interpreted as expressing the evolution of the *spatial extent of a single agglomeration*.

Quantitative spatial models that employ class (ii) models (e.g., Redding and Sturm, 2008; Allen and Arkolakis, 2014) emphasize the uniqueness of the equilibrium, through which calibrations and counterfactual analyses have determinate implications. These studies are conducted under parameter settings that ensure the uniqueness of the equilibrium *regardless* of the level of interregional transport costs (Redding and Rossi-Hansberg, 2017). This is made possible because the local dispersion force in class (ii) models does *not* depend on the
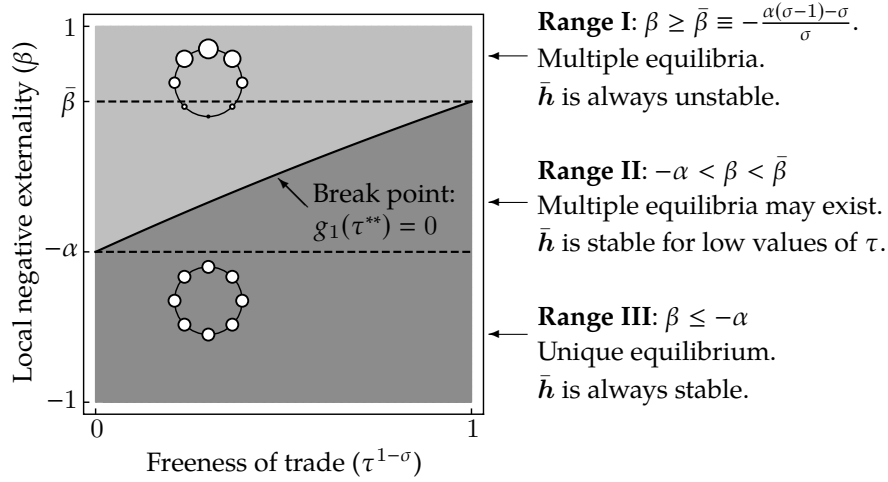
Figure 10: Classification of the uniqueness and stability of the equilibria for the Allen and Arkolakis (2014)

Note: We let $\alpha = 0.6$ and $\sigma = 4$ for illustration purposes. $\bar{h}$ is stable in the dark gray region, while not in the light gray region. At τ^{**} such that $g_1(\tau^{**}) = 0$, a unimodal pattern emerges. $\alpha + \beta \leq 0$ is a sufficient condition for the stability of $\bar{h}$ regardless of τ. A comparison with Figure I of Allen and Arkolakis (2014), where their classification corresponds to Ranges I, II, and III above, would be interesting.

level of accessibility to the other regions; consequently, if a sufficiently strong local dispersion force is imposed, no endogenous agglomeration is caused under any level of transport costs. Notably, in our setup, since the uniform distribution of mobile agents across regions is always an equilibrium on a symmetric racetrack, the uniqueness of the equilibrium automatically implies that the flat-earth pattern $\bar{h}$ is the only equilibrium and is stable. **Figure 10** indicates our classification of possible spatial patterns and their stability for the model proposed by Allen and Arkolakis (2014) in a racetrack economy with arbitrary K. Their uniqueness condition is $\beta \leq -\alpha$ (i.e., Range III in the figure).[40] This uniqueness directly implies that the uniform distribution is stable regardless of the level of τ.

5.3 Class (iii): Models with both dispersion forces

Tabuchi (1998), Pflüger and Südekum (2008), and Takayama and Akamatsu (2011) considered both local and global dispersion forces. In effect, these models exhibit the rich and meaningful interplay between the *number and spacing of agglomerations* and *spatial extent of each agglomeration* without any location-fixed factors but only with pure economic forces.

In these models, the evolutionary process of spatial agglomeration patterns in the course of a monotonic change in τ is expected to be a combination of the two examples presented above. This is indeed the case. **Figure 11** depicts the evolution of the number of agglomerations in the course of decreasing τ for the model proposed by Pflüger and Südekum (2008) in

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[40]For the model of Helpman (1998), the condition for the uniqueness of the equilibrium is given by $(1-\mu)\sigma > 1$.
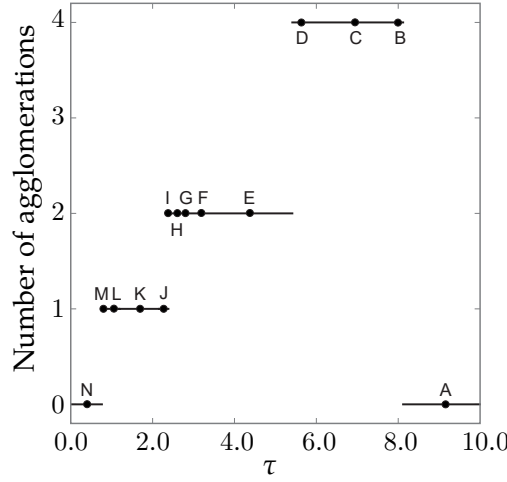
Figure 11: Evolution of the number of agglomerations in a class (iii) model

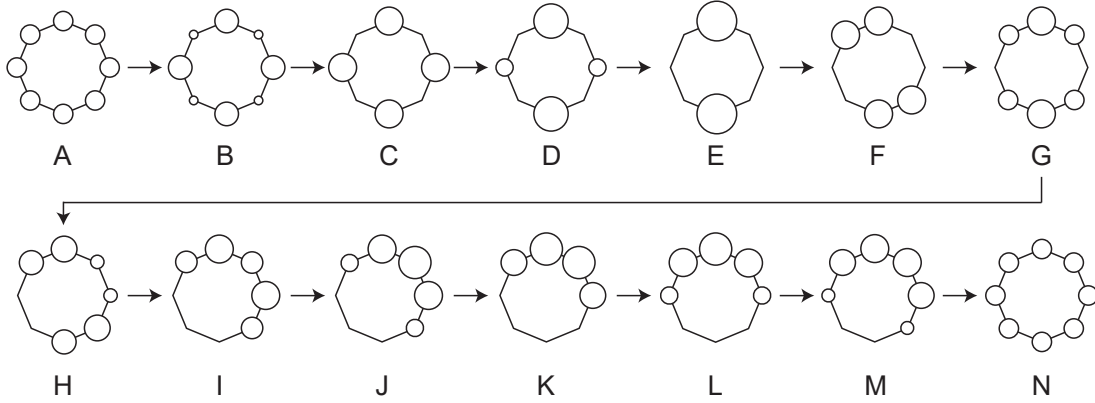*Note:* Pflüger and Südekum (2008) is taken as the underlying example model.

Figure 12: Evolution of the spatial pattern in **Figure 11**

Note: The alphabets below the spatial patterns correspond to those in **Figure 11**.

a symmetric eight-region racetrack economy. We define the number of agglomerations in a spatial distribution of mobile agents, h, by that of the local maxima therein.[41] By comparing **Figure 11** with **Figure 8** and **Figure 9**, we observe that the former is basically a combination of the latter two, as expected. When τ gradually decreases from a very high level, the number of agglomerations evolves as $0 \to 4 \to 2 \to 1$ as in the class (i) models (**Figure 8**), while in the later stage $1 \to 0$ as per the class (ii) models (**Figure 9**).[42] The initial stage is governed

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[41] For example, for Pattern I in **Figure 12**, we evenly split the population of the region in the middle of the two peaks. Further, if two consecutive regions have the same population as in Pattern K in **Figure 12**, this is counted as a single agglomeration.

[42] At this point, there is good reason to suspect that although seemingly identical, the flat-earth patterns at the higher and lower levels of τ are distinct in nature. Specifically, one would argue that the number of agglomerations must be eight (one), instead of zero, at large (low) τ. We refrain from these arguments, however, because the two stages of dispersion are indistinguishable by the mere observation of h.

by a decline in the global dispersion force, while the later stage is governed by a relative rise in the local dispersion force. As the importance of distance declines given improvements in transportation access, local congestion overcomes the agglomeration force and so-called redispersion occurs.

The evolution of the spatial patterns provides richer intuitions. **Figure 12** illustrates the spatial patterns associated with **Figure 11** (see also Section 3.2 to understand the figure). The flat-earth pattern is initially stable (Pattern A); the first bifurcation leads to a four-centric global dispersion (C), whereas the dispersion associated with the second bifurcation is two-centric (E). These transitions are wholly in line with those of Krugman (1991); they are governed by the gradual decline in the global dispersion force in the model. After these transitions, the evolutionary behavior becomes more interesting; the decline in the global dispersion force increases the relative importance of the local dispersion force. The two-centric agglomerations formed in (E) gradually increase their spatial extent (F, G) because of the local dispersion effects combined with the relative decline in the global agglomeration force. A further decline in τ implies a triumph of the global agglomeration force against the global dispersion force since the latter declines faster than the former. Consequently, the two agglomerations gradually merge (H, I) to form a monopolar agglomeration (J, K), while maintaining their spatial extent due to the strong local dispersion force. As the relative importance of the local dispersion force increases further, a gradual expansion of the single agglomeration occurs (L, M) followed by complete redispersion (N). This rich and intuitive interplay of the global and local scales of agglomeration and dispersion can be studied only in many-region setups.

6 Implications for empirical studies

In the previous sections, we argued that the consideration of the diversity in interregional distances is the key factor to explaining the actual spatial pattern of agglomeration. The theoretical understanding of agglomeration and dispersion mechanisms at different spatial scales helps us find an appropriate way to quantify the spatial patterns of agglomeration, which in turn allows us to properly formulate regression as well as structural models to identify the causal mechanisms of regional agglomeration. This section highlights these points by reviewing selected empirical studies of the relation between transport costs and regional agglomerations.

6.1 Measures of agglomeration

One strand of the literature concerns the measurement of industrial agglomeration. Unlike population agglomerations that have been identified in terms of distinct metropolitan areas or

population clusters (e.g., Baum-Snow, 2007; Duranton and Turner, 2012; Rozenfeld, Rybski, Gabaix and Makse, 2011), industrial agglomerations have typically been measured by using an aggregated scalar index (e.g., Ellison and Glaeser, 1997; Duranton and Overman, 2005; Brülhart and Traeger, 2005; Mori et al., 2005).

Of the two pioneering indices of industrial agglomeration, that proposed by Ellison and Glaeser (1997) intended to control for the spatial concentration of employment that accrued from the distribution of employment among establishments, while the other presented by Duranton and Overman (2005) aimed to resolve the spatial aggregation biases that arose from regional data by using geo-coded microdata on establishments.

While these refinements may be reasonable in certain contexts, a major reservation about these scalar indices is that by construction they do not distinguish spatial scales of agglomeration and dispersion. With respect to cross-sectional comparisons among industries, this means that there is no way to distinguish spatial scales at which variations in the index value arise, even though the underlying agglomeration mechanisms qualitatively differ at each scale.[43] Another consequence of the abstraction from spatial scales is that these indices inevitably neutralize the opposing responses of agglomeration at global and local scales to a given change in transport costs.

Distinguishing individual agglomerations on a map as in the case of population agglomerations is necessary to separate the effects at different spatial scales. Kerr and Kominers (2015) and Mori and Smith (2014) proposed clustering methods designed for economic agglomerations. Pelleg and Moore (2000), Ishioka (2000), and Brendan and Dueck (2007) proposed heuristic clustering techniques for general purposes. Our theory suggests that agglomeration at the global scale is reflected in a smaller number of agglomerations (as well as a larger spacing between them), whereas that at the local scale is reflected in a smaller spatial extent of each individual agglomeration. These spatial properties of agglomeration can be quantified by using the identified clusters.

An advantage of such a clustering approach for industrial agglomerations is that unlike the case of population agglomeration, one can obtain variations in agglomeration patterns across industries. By using the clustering method proposed by Mori and Smith (2014), Mori and Smith (2015) indicated a wide variation in the degrees of agglomeration at both global and local scales across three-digit manufacturing industries in Japan. The variations across industries in turn can be used to test the theoretical implications of the spatial patterns of agglomeration, for example, the causal relation among the number, size, and spatial extent of agglomerations and transport costs. One such application by Mori, Mun and Sakaguchi

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[43]Duranton and Overman (2005) distinguish distances between establishments; however, they do not distinguish between intra- and inter-agglomeration distances. Thus, it is not clear whether these bilateral distances represent distances between agglomerations or those within an agglomeration (see Mori and Smith, 2015, §6 and Appendix B).

(2017) is discussed in the next section.

6.2 Reduced-form regression approaches

We have thus far shown that endogenous agglomeration mechanisms generally do not isolate which existing agglomerations grow or decline given an improvement in interregional transport access. This indeterminacy is due to the underlying second-nature advantage. Nonetheless, the theory offers a clear prediction of the global and local spatial pattern of agglomerations. The former prediction is that *there is agglomeration at the global scale*: the number of agglomerations decreases, the distance between neighboring agglomerations increases (reflecting the growing agglomeration shadow), and the sizes of the surviving individual agglomerations increase (see Sections 5.1 and 5.3). The latter prediction is that *there is dispersion at the local scale*: The spatial extent of each individual agglomeration increases, for example, in the form of suburbanization (see Sections 5.2 and 5.3).

In Sections 6.2.1 and 6.2.2, we argue that these theoretical predictions are useful to understand the diverse results from extant empirical studies using reduced-form regressions to explain global and local patterns of agglomeration, respectively. Further, in Section 6.2.3, we discuss the context in which these predictions can be actually tested.

6.2.1 On the size of an agglomeration

A typical regression model used to evaluate the impact of a new transport network on regional growth has the following form (see, e.g., Redding and Turner, 2015, §20.4, for a survey):

$$\text{SIZE}_{it} = C_0 + C_1 \text{ACCESS}_{it} + C_2 x_i + \gamma_{it} + \eta_t + \epsilon_{it} \tag{6.1}$$

where SIZE_{it} and ACCESS_{it} measure agglomeration size and interregional transport access, respectively, in region i in year t; x_i denotes the region-specific and year-invariant covariates, γ_{it} denotes the region- and year-specific unobserved effects, η_t denotes the year-specific unobserved effect, ϵ_{it} denotes the region- and year-specific errors, and C_0, C_1 and C_2 are the coefficients to be estimated, where C_1 is of interest here.

The existing literature on the relation between agglomeration size and interregional transport access in a region shows mixed results. We start from two studies drawing contrasting conclusions. Faber (2014) investigated the impact of the construction of the national highway network in China on the agglomerations in peripheral counties during 1997–2006. Duranton and Turner (2012) studied a similar situation in the United States during 1983–2003; however, they focused on the impact on agglomerations in relatively large metro areas instead of peripheral alternatives. SIZE_{it} represents the changes in output measures such as the gross domestic product and gross value added as well as that of population size in a county

in the former, while it is the change in metro-area population or employment in the latter. $ACCESS_{it}$ represents the change in interregional highway accessibility in both cases.[44] Their results exhibited a stark difference: the former (latter) generally found a significantly negative (positive) estimate of C_1 in (6.1).

Yet, other studies report indefinite results. For Chinese data similar to those used by Faber (2014), Baum-Snow, Henderson, Turner, Zhang and Brandt (2016, Tables 4 & 5) found insignificant estimates of C_1 when both large and small regions along the network were included in the regression. For US data similar to those used by Duranton and Turner (2012), Baum-Snow (2017, Table 5) ran a variant of (6.1) to estimate the impact of interregional transport access on industry-specific employment in a metro area. However, the estimated coefficient of C_1 was insignificant for all but manufacturing employment, for which it was negatively significant.[45] Thus, the estimated impacts of interregional transport access on the size of an individual agglomeration vary widely, and there is no consensus, even on the sign of the impacts.

From the knowledge on endogenous agglomeration mechanisms obtained in this study, behind the incoherent regression results, we suspect the ignorance of the effects of interregional transport costs on the spatial distribution of agglomerations. Recall that, in class (i) models, a uniform improvement in interregional accessibility at a given location does not necessarily result in a growth or decline in agglomeration size at that location (see Section 5.1). It is thus natural to obtain an insignificant average effect of improved transport access along the new transport network as in Baum-Snow et al. (2016, Tables 4 & 5).[46] What happened appears to be an agglomeration at the global scale toward a smaller number of larger regions. In Faber (2014), the decline in peripheral regions is a mirror image of the growth in core regions excluded from his regression.[47] It is similarly expected that in Duranton and

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[44]In the baseline specification of (6.1) in Faber (2014), $ACCESS_{it}$ represents a binary variable that takes the value 1 if a given region i is connected by the newly constructed highway at time t, while it is set to the initial sum of the interstate highway length within a metro area (i.e., in 1983) in Duranton and Turner (2012). In particular, Duranton and Turner (2012) considered $ACCESS_{it}$ to be the level of intra-urban (rather than inter-urban) transport infrastructure. However, we believe that the stock of interstate highways within a given metro area also reflects the level of inter-urban connectivity.

[45]Similar studies by Storeygard (2016) and Yamasaki (2017) have established a positive relation between interregional transport access and regional agglomeration in the case of Sub-Saharan Africa for 2002–2008 and Japan for 1885–1920, respectively. Since the focus of these studies is the early stage of economic development, their results may not be directly comparable with those of Faber (2014) and Duranton and Turner (2012) as well as with our theoretical results. For example, the latter study investigated the situation in which industrialization took place along with the introduction of railways in response to the spread of steam power in Japan. However, the decomposition of the causal relationship among industrialization, improvement in interregional transport access, and urbanization is not obvious.

[46]The distance effects vary across different types of economic activities such as industries, which further obscures their aggregate effects on population agglomeration. See Head and Mayer (2004, §7.2) for a related discussion based on market potential.

[47]It is also pointed out by Baum-Snow et al. (2016, Tables 4 & 5) that there was a significant increase in agglomeration size in large regions in the China case.

Turner (2012), the growth in large metro areas is a mirror image of the decline in the peripheral areas excluded from their regression, although there is no explicit discussion on this aspect in their study.

Moreover, both Faber (2014, Table 6) and Duranton and Turner (2012, Table E2) found evidence of the agglomeration shadow, namely that a larger distance from the nearest major agglomeration tends to promote the growth of a region, which further suggests the relevance of class (i) mechanisms.[48]

One way to more clearly distinguish differential impacts of transport network development on individual regions is to introduce complementary variations in the measures of interregional transport accessibility. Our theory suggested on the one hand that a general improvement on transport accessibility induces global agglomeration, and hence, its average effect is ambiguous as suggested by the extant studies. On the other hand, in reality, the development of transport network often results in substantially heterogeneous accessibilities across nodes in the network. For example, the network nodes at junctions and terminals of highways and high-speed railways typically are more likely on the shortest routes between many pairs of locations than other nodes in the network, i.e., the *betweenness* is relatively higher in these locations. A higher betweenness at a given region still does not necessarily guarantee the growth of agglomeration there in the process of global agglomeration. However, after controlling for the effect of general improvement in transport accessibility, it should be more likely that a major agglomeration takes place at one of these nodes with high betweenness by the same reason that the exogenous regional advantage induces agglomeration in Class (i) models (see Appendix D). See Mori and Takeda (2017) for a recent attempt in this direction.

6.2.2 On the spatial extent of an agglomeration

Using the same specification (6.1) makes perfect sense when it comes to evaluating local dispersion. Baum-Snow (2007) and Baum-Snow, Brandt, Henderson, Turner and Zhang (2017) presented evidence of local dispersion as a consequence of improved interregional transport access in the cases of US metro areas for 1950–1990 and Chinese prefectures during 1990–2010, respectively.[49] In these studies, $SIZE_{it}$ denotes the change in the population/production size of the central area within a larger region i (the metro area for the United States and the province for China). They both reported a significantly negative estimated coefficient of C_1 given an improvement in interregional access after controlling for the growth of each

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[48]In Duranton and Turner (2012), the monotonic relationship between the size of a metro area and level of inter-urban transport infrastructure in (6.1) is rationalized by assuming an open-city specification in the underlying theoretical model. However, the significant urban shadow effect among the included metro areas casts doubt on this justification.

[49]Garcia-López (2012) and Garcia-López, Holl and Viladecans-Marsal (2015) conducted similar studies by using Spanish data.

region.[50,51] Their findings are consistent with our results on local dispersion (see Sections 5.2 and 5.3).[52]

Recall the population agglomeration at the global scale and dispersion at the local scale in response to the development of the national highway and high-speed railway networks in Japan after 1970 discussed in Section 1 (and Appendix A). Empirical evidence suggests that China and the United States experienced essentially the same phenomena.

6.2.3 On the spatial patterns of agglomeration

Finally, we explore the possibility of testing our theoretical predictions on the spatial patterns of agglomerations mentioned above rather than testing hypotheses about an individual agglomeration by using regression models of type (6.1). For population agglomerations, we typically have only a single set of agglomerations at a given point in time, which makes the hypotheses concerning their spatial distribution untestable. However, such tests become possible by considering an individual industry as a unit of observation. If a distinct set of agglomerations can be identified for each industry, we have variations in the spatial patterns of agglomerations across industries at a given point in time. As a recent attempt, Mori et al. (2017) adopted the clustering framework developed by Mori and Smith (2014) and used data on three-digit manufacturing industries in Japan during 1995–2015. They showed that the number of agglomerations decreases (i.e., agglomeration at the global scale proceeds) and the average areal size of individual agglomerations increases (i.e., dispersion at the local scale proceeds) in response to a decrease in industry-specific sensitivity to transport costs: the transport cost per unit distance and unit value of output.[53] In their study, by setting a unit of observation to an individual industry, the variation across industries made it possible to directly test the two predictions from our theoretical results mentioned above on the spatial patterns of agglomerations.

To sum up, knowledge on the behavior of general economic geography models brings together seemingly unrelated pieces of empirical evidence on agglomeration patterns. Our interpretation of the results from the literature suggests the strong relevance of endogenous agglomeration mechanisms to the observed regional variations in agglomeration size.

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[50] As discussed in these studies, the results for local dispersion can also be interpreted as suburbanization in response to improved intra-urban transport infrastructure in classical urban economic theory (e.g., Alonso, 1964).

[51] Faber (2014, Table 5 and Figure 4) showed related evidence that agglomeration relatively proceeds in regions at a certain distance (around 100–150 km) from highways rather than those along highways.

[52] Baum-Snow (2017) extended the work of Baum-Snow (2007) by replacing the outcome variable in (6.1) with the local dispersion (or suburbanization) of employment in each industry instead of population and showed variations in the extent of the local dispersion across industries.

[53] This definition of the sensitivity to transport costs is an empirical counterpart of the iceberg transport costs in our theoretical models.

6.3 Structural model-based approaches

We now turn to the structural model-based approach used to evaluate the causal effects of regional agglomerations summarized by Redding and Rossi-Hansberg (2017, §3). In this perhaps one of the most popular approaches in quantitative spatial economics, the basic premise is that the primary source of the regional variation in agglomeration size is the heterogeneity in exogenous (or first-nature) regional advantages rather than endogenous (or second-nature) advantages considered in this study. Thus, given the exogenous productivity or residential amenity difference across regions, a larger population of a given region is always associated with higher exogenous productivity or amenity in that region.

A remarkable feature of these models (e.g., Redding and Sturm, 2008; Allen and Arkolakis, 2014) is that they not only rely on exogenous advantages to explain agglomeration patterns but also incorporate agglomeration externalities to the extent that the unique equilibrium is guaranteed, thereby preserving the tractability of the model.[54,55] This subtle situation has been realized by adopting class (ii) models. As shown in Section 5.2, these models have a parameter range in which agglomeration diseconomies dominate agglomeration economies independently of the level of transport costs. This special property of class (ii) models is due to the independence of local dispersion forces on the distance structure of the model. In this context, the model parameters are calibrated to replicate the relevant regional variations (such as regional population sizes) in the absence (or presence) of a given treatment such as transport development; then the counterfactual regional variations are derived in the presence (or absence) of the treatment given these calibrated parameter values.

There are two caveats in understanding the implications obtained from the results of this approach.

First, although agglomeration externalities account for some of the regional variations, most variations appear to be absorbed by the structural residuals. Consider the study by Redding and Sturm (2008), for example. By using a many-region extension of the Helpman (1998) model, they quantified the impact of the change in market accessibility in Germany before and after the division/reunification of the country after the war. To determine the time-invariant set of parameter values, their model was calibrated to fit the city size distributions in prewar Germany. If the log of actual city size is regressed on the log of unobserved amenity (i.e., the structural residuals), one can immediately find that most of the variation in city size is explained by these residuals, as shown in Figure 13, in which the

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[54] Redding (2016) and Monte et al. (2016) extended the work of Redding and Sturm (2008) by adding different sources of exogenous location-fixed factors.

[55] The majority of structural model-based studies of a regional economy involve no agglomeration externalities (see, e.g., Donaldson and Hornbeck, 2016; Baum-Snow et al., 2016; Alder, 2016; Caliendo, Parro, Rossi-Hansberg and Sarte, 2016).
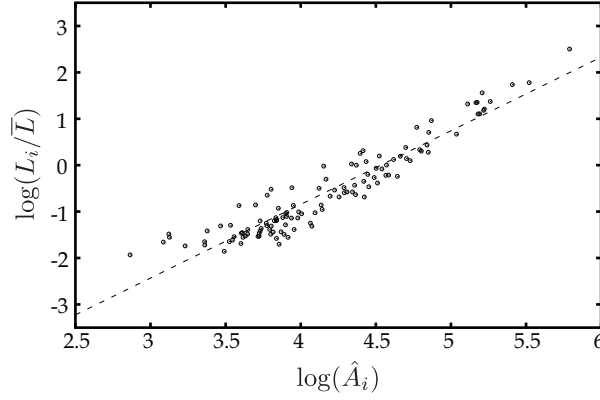
Figure 13: The relationship between city sizes and the estimated unobserved amenities in Redding and Sturm (2008)

dashed line indicates the fitted line by OLS:

$$\log(L_i/\overline{L}) = \underset{(0.210)}{-7.191} + \underset{(0.050)}{1.587}\log(\hat{A}_i)\,, \quad \text{adj.}\,R^2 = 0.896 \tag{6.2}$$

where L_i and $\overline{L}$ denote the population size of city i and average city size in prewar Germany, respectively, and $\hat{A}_i$ denotes the estimated unobserved amenity (i.e., the variation unexplained by the model) in city i. Standard errors are in parentheses.[56] Thus, 90% of the variation in city size is unexplained by the model and fixed exogenously in their counterfactual exercises. While this is not surprising given that there is no possibility of endogenous agglomeration in this setup, the result indicates that the structure of their model plays only a small role (10% at most) in explaining the treatment effect.

Second, their predictions of treatment effects crucially depend on a specific feature of the underlying economic geography models. In particular, in class (ii) models, the decline in interregional transport costs reduces each regional agglomeration caused by the first-nature advantage of the region and promotes local dispersion. This causal relation was demonstrated by a numerical counterfactual exercise in Redding and Rossi-Hansberg (2017, §3.9). In fact, it can be formally shown (see Appendix D) that this scale-down effect is a general property of class (ii) models, although there are certain cases in which local landownership mitigates the effect by counteracting congestion externalities.

However, if class (i) models are used instead, even in the parameter range in which no endogenous agglomeration occurs (as in Redding and Rossi-Hansberg, 2017, §3), the sign of the treatment effect reverses, namely agglomeration externalities would scale up the first-nature advantage (see Appendix D for a formal proof). The resulting implications thus become the opposite depending on the choice of dispersion force to be included in the

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[56]The data for the regression are available from the online appendix of Redding and Sturm (2008).

model.[57]

Thus, even in the context in which first-nature heterogeneity plays a primary role, knowledge of endogenous agglomeration and dispersion mechanisms is crucial to understand the logic and direction of the treatment effect. For that purpose, our stylized analytical framework appears to be useful.

More recent structural approaches have shifted to accommodate multiple equilibria with endogenous agglomerations (e.g., Ahlfeldt et al., 2015; Owens et al., 2017; Nagy, 2017). In that case, class (ii) models are no longer useful, as they can endogenously generate at most a unimodal agglomeration. To explain both multiple agglomerations and local dispersion by using endogenous mechanisms, one needs a model of either class (iii) or of a more general class not addressed in this study. We briefly touch on these issues in Section 7 when delineating future research directions.

7   Concluding remarks

In this study, we formally classified economic geography models in terms of their model-specific spatial patterns of agglomeration. By allowing the presence of many regions, the spatial scale of agglomeration and dispersion is made explicit unlike the two-region setup or many-region setup without an interregional space. In particular, the two dispersions at high and low transport costs that look identical in a two-region setup turned out to differ and take place at global and local spatial scales, respectively. In fact, when dispersion proceeds at the local scale, agglomeration typically proceeds at the same time but at the global scale and vice versa.

Our theoretical results indicated a new direction for future empirical research based on endogenous agglomerations. First, the contrasting agglomeration and dispersion behaviors at different spatial scales suggest the need to distinguish individual agglomerations rather than measuring agglomeration by using a scalar index. Second, endogenous agglomeration and dispersion mechanisms generally do not isolate the growth and decline in individual agglomerations and can only provide insights into their spatial patterns. Our new results on the impact of transport development on the spatial patterns of agglomeration could thus pro-

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[57]Fajgelbaum and Schaal (2017) used a specification similar to those of class (ii) models to study the impact of transport network development in European countries. In their model, the variation in regional advantage is exogenous, while there are congestion externalities in interregional transportation. Since the externalities are effective only locally, they act as a local dispersion force in class (ii) models when agents are mobile. Like Redding and Rossi-Hansberg (2017, §3.9), under this model, they observe (Figure 5 in their paper) that an improvement in interregional transport access generally mitigates the variation in regional advantages, which disperses population from the originally larger (more advantageous) regions to the originally smaller regions. Although the average impact of improved transport access on regional population growth is found to be insignificant (Table 1 in their paper), one must bear in mind that this result reflects the fact that the signs of the impact of improved transport access are the *opposite* in the originally larger and smaller regions.

vide a unified interpretation of the variety of existing results from reduced-form regressions on regional agglomerations. Furthermore, we showed that our analytical framework is useful for obtaining formal predictions of treatment effects in structural model-based analyses that aim to explain agglomeration patterns.

However, the relatively simple classification of the spatial scale of agglomeration and dispersion in our study is owing to the simple structure of the models considered. Below, we discuss possible future directions of theoretical research to account for richer, more realistic variations in agglomeration and dispersion across different spatial scales.

At least three major research directions could be pursued. The first possible extension is to distinguish location incentives between firms and consumers/workers as is traditionally done in the urban economics literature (e.g., Fujita and Ogawa, 1982; Ota and Fujita, 1993; Lucas and Rossi-Hansberg, 2002). In all the models considered in this study, the location incentive of firms and that of consumers/workers coincide. This simplification may be justifiable when the global pattern of agglomeration (in particular, sizes and locations of cities) is the subject of the study. However, their distinction becomes crucial for explaining the location patterns within a city. There are recent attempts, for example, by Ahlfeldt et al. (2015) and Owens et al. (2017) in this direction. These models typically abstract from the intercity/regional interactions in an *open-city* setup. Nonetheless, the possible equilibrium patterns and their stability properties in this class of models are still not well known and this could thus represent a fruitful avenue for future research.

The second possibility is to consider different transport cost structures by industry. For instance, Fujita and Krugman (1995) introduced transport costs for land-intensive rural (agricultural) goods along with those of urban goods. In the presence of rural goods that are costly to transport, the delivered price for such goods is lower in regions farther away from the agglomerations, which generates a dispersion force. This is similar to the local dispersion force in that even a small deviation from an urban agglomeration will decrease the price of rural goods and increase the payoff of the deviant. However, the advantage of dispersion persists outside the agglomeration, i.e., it depends on the distance structure of the model. This type of dispersion force leads to the formation of an *industrial belt*, a continuum of agglomeration associated with multiple atoms of agglomeration as demonstrated by the simulations in Mori (1997) and Ikeda, Murota, Akamatsu and Takayama (2017b). The formal characterization of industrial belts remains to be carried out.

The final possible extension is to incorporate diversity in increasing returns, leading to diversity in the spatial scale of agglomeration and dispersion as well as diversity in agglomeration size. In all the models with a global dispersion force considered in this study, the sizes of agglomerations in the equilibrium are basically the same (see the simulation exercises in Section 5.1) since each model has only one type of increasing return. This

is counterfactual as actual city size distributions are diverse and well known to roughly follow the power law.[58] Initial attempts to account for the diversity in increasing returns by introducing multiple increasing returns industries have been made in the context of the NEG models proposed by, e.g., Fujita et al. (1999b), Tabuchi and Thisse (2011), and Hsu (2012)'s spatial competition model. Alternatively, Desmet and Rossi-Hansberg (2009), Desmet and Rossi-Hansberg (2014), Desmet and Rossi-Hansberg (2015), Desmet et al. (2017), and Nagy (2017) incorporated dynamic externalities through endogenous innovation and spillover effects.

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[58]The models considered in this study are consistent with heterogeneously sized agglomerations in the equilibrium; however, it is not possible to replicate the high diversity of city sizes seen in reality.

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A Development of the transport network and urban agglomeration (UA) patterns in Japan, 1970–2015

To compare population agglomeration patterns in Japan, we define an *urban agglomeration* (*UA*) as the set of contiguous 1 km-by-1 km cells with a population density of at least $1000/km^2$ and total population of at least 10,000.[59] (The basic results below remain the same for alternative threshold densities and populations.) For the part of Japan contiguous by roads to at least one of the four major islands (Hokkaido, Honshu, Shikoku, and Kyushu), 503 and 450 UAs are identified, as depicted in Panels (a) and (b) of Figure 14 for 1970 and 2015, respectively, where the warmer color indicates a larger population. These together account for 64% and 78% of the total population in 1970 and 2015, respectively. Thus, there is a substantial 18% increase in the urban share over these 40 years. Of the 503 UAs that existed in 1970, 302 survived, while 201 either disappeared or integrated with other UAs by 2015. Of the 450 UAs that existed in 2015, 148 were newly formed after 1970 (including those split from existing UAs).[60]

Panels (c) and (d) of Figure 14 show the highway and high-speed railway networks in use in 1970 and 2015, respectively. The comparison of these panels indicates an obvious substantial expansion of these networks during these 45 years, as mentioned in the text.

Panels (a), (b), and (c) of Figure 15 show the distributions of the growth rates of population share (in the national population), the areal size and population density of individual UAs for the set of the 302 UAs that survived throughout the 45-year period. A UA experienced an average growth rate of 21% (75%) of population share, 94% (105%) of areal size, and -22% (22%) of population density (per km^2), respectively, where the numbers in parentheses are the standard deviations.

As a larger population share was concentrated in a smaller number of UAs in 2015 than in 1970, the spatial size of an individual UA almost doubled on average. However, these spatial expansions are not simply due to the shortage of available land in UAs. Note that population density decreased by 22% on average. We take this as evidence of global concentration with local dispersion under the improvement in interregional transport access.

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[59] Population count data are obtained from Statistics Bureau, Ministry of Internal Affairs and Communications of Japan (1970, 2015).

[60] UA i in year s is said to be *associated with* UA j in year t ($\neq s$) if the intersection of the spatial coverage of i and that of j accounts for the largest population of i among all the UAs in year t. For years $s < t$, if i and j are associated with each other, they are considered to be *the same* UA. If i is associated with j but not vice versa, then i is considered to have been *absorbed* into j, while if j is associated with i but not vice versa, then j is considered to have *separated* from i. If i is not associated with any UA in year t, then i is considered to have *disappeared* by year t, while if j is not associated with any UA in year s, then j is considered to have newly *emerged* by year t.

(a) Urban agglomrations in 1970

(b) Urban agglomrations in 2015

(c) Highway and high-speed railway network in 1970

(d) Highway and high-speed railway network in 2015
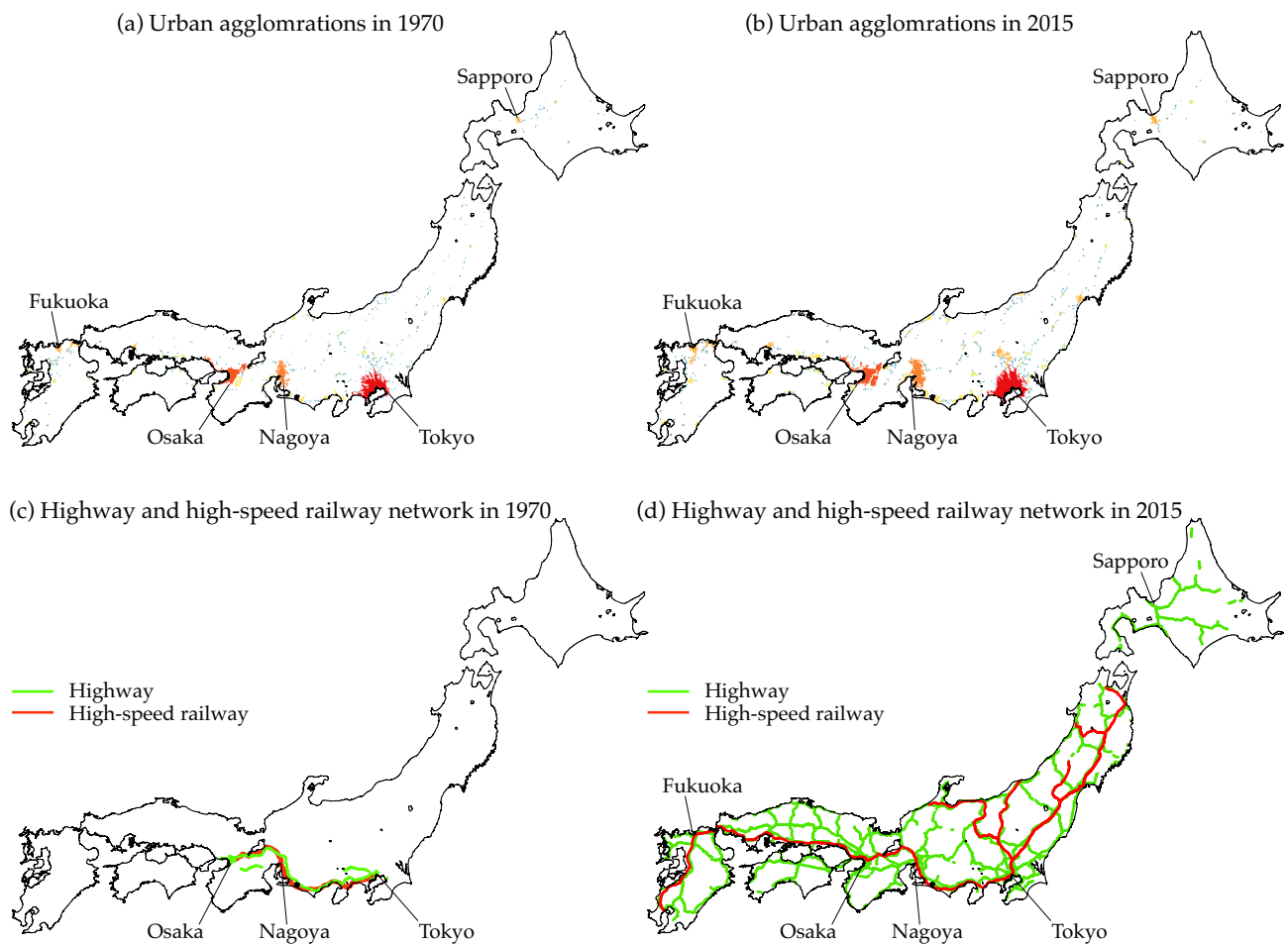
Figure 14: UAs and transport network in Japan

(a) Population share growth

Average = 0.21
Standard deviation= 0.75

(b) Area growth

Average = 0.94
Standard dveiation = 1.05

(c) Population density growth

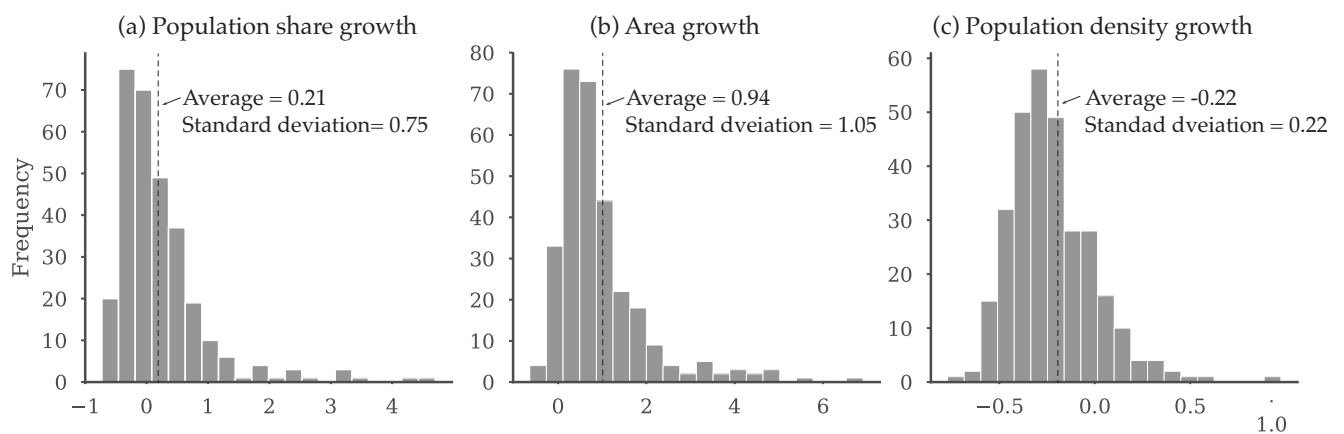Average = -0.22
Standad dveiation = 0.22

Figure 15: Growth rates of the sizes of UAs in Japan

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B   Racetrack economy and simplification of the stability analysis

The friction matrix D for a racetrack economy is a *circulant matrix*. In this appendix, we first review some useful properties of circulant matrices. Then, we see how the stability analysis of the flat-earth pattern in a symmetric racetrack economy is simplified by these properties.

B.1   Facts on the properties of circulant matrices

A *circulant matrix C* of dimension K is defined as a K-by-K square matrix of the form

$$
C \equiv \begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} \\
c_{K-1} & c_0 & c_1 & c_2 & \cdots & c_{K-2} \\
c_{K-2} & \ddots & \ddots & \ddots & \ddots & \\
 & \ddots & \ddots & \ddots & \ddots & c_2 \\
c_2 & \cdots & c_{K-2} & c_{K-1} & c_0 & c_1 \\
c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} & c_0
\end{bmatrix}.
\tag{B.1.1}
$$

The elements of each row of C are identical to those of the previous row, but are moved one position to the right and wrapped around. The whole circulant is evidently determined by the first row vector $c = (c_i)_{i=0}^{K-1}$. Circulant matrices are known to satisfy the following properties (see, e.g., Horn and Johnson, 2012):

Lemma B.1. Let C_1 and C_2 be circulant matrices. Then, their sum $C_1 + C_2$ and product $C_1 C_2$ also are circulants. They are also commutative, i.e., $C_1 C_2 = C_2 C_1$. Let C_3 be a non-singular circulant matrix. Then, its inverse C_3^{-1} is also a circulant.

Lemma B.2. Let C be a K-by-K circulant matrix. Let $Z = [z_{jk}]$ be the discrete Fourier transformation (DFT) matrix, where $z_{jk} = K^{-1/2} \exp[i\theta jk]$ with $\theta \equiv 2\pi/K$ and $i \equiv \sqrt{-1}$. Then, C is diagonalized by the similarity transformation by Z as $Z^* C Z = \mathrm{diag}[\lambda]$, where * denotes the conjugate transpose.

$\lambda = (\lambda_i)$ are the eigenvalues of C. The kth eigenvalue and associated eigenvector of C are λ_k and the kth row of the DFT matrix Z, respectively. Furthermore, λ is given by the DFT of the first row vector c of C as $\lambda = K^{1/2} Z c^\top$.

Remark B.1. It follows that all K-by-K circulant matrices share the same eigenvectors.

Consider a matrix A defined by a matrix polynomial of a non-singular circulant matrix C:

$$
A = c_0 I + c_1 C + c_2 C^2 + \cdots .
\tag{B.1.2}
$$

Thanks to Lemma B.1 and B.2, one has that (i) A is a circulant matrix (note that I is also a circulant) and thus that (ii) its eigenvalues $\mu \equiv (\mu_i)$ are given by those of C, λ, by the relation

$$
\mathrm{diag}[\mu] = Z^* A Z = Z^* (c_0 I + c_1 C + c_2 C^2 + \cdots) Z = c_0 I + c_1 \mathrm{diag}[\lambda] + c_2 \mathrm{diag}[\lambda]^2 + \cdots ,
\tag{B.1.3}
$$

or, in the element-wise manner,

$$\mu_k = c_0 + c_1 \lambda_k + c_2 \lambda_k^2 + \cdots . \tag{B.1.4}$$

Remark B.2. If C is also symmetric, A is symmetric. This implies that the eigenvalues $\boldsymbol{\lambda} = (\lambda_k)$ and $\boldsymbol{\mu} = (\mu_k)$ as well as their associated eigenvectors are all real.

B.2 Eigenvalues and eigenvectors of the friction matrix

We derive the eigenvalues and eigenvectors of the friction matrix for later use. To simplify the notation, we define $r \in (0, 1)$ to represent the *freeness of transport* between two consecutive regions on the racetrack: $r(\tau) \equiv \exp[-\tau/K]$, where we rescale τ so that the circumferential length of the economy is fixed. From the definition of r, it is a monotonically decreasing function of the transportation cost (technology) parameter τ and hence r and τ are mutually interchangeable. We use r as the transport technology parameter in the present appendix. By using r, one has $d_{ij} = r^{\ell_{ij}}$.

To analyze specific models, it is useful to derive the eigenvalues of the *row-normalized* friction matrix $\bar{D} \equiv D/d$ with $d \equiv \sum_{j \in \mathcal{K}} d_{0,j}$. We note that every row has the same row sum because D is circulant. It turns out that in a racetrack economy, D is a circulant matrix since $d_{ij} = r^{\ell_{ij}} = r^{\ell_{i+1,j+1}} = d_{i+1,j+1}$ for all $i, j \pmod{K}$ for indices). Furthermore, D is symmetric and real and hence all the eigenvalues and eigenvectors are real. The analytical expressions of the eigenvalues and eigenvectors of $\bar{D}$ are available (Akamatsu et al., 2012):

Lemma B.3. Let $f_k(r)$ be the kth eigenvalue of the row-normalized friction matrix $\bar{D}$ for a racetrack economy with K regions. Assume that K is a multiple of four. Define $\Psi_k(r) > 0$ and $\bar{\Psi}(r) > 0$ by

$$\Psi_k(r) \equiv \frac{1 - r^2}{1 - 2\{\cos[\theta k]\}r + r^2}, \quad \bar{\Psi}(r) \equiv \frac{1 + r^{K/2}}{1 - r^{K/2}} \tag{B.2.1}$$

with $\theta = 2\pi/K$. Then, $\{f_k(r)\}$ is given by

$$f_k(r) = \begin{cases} \Psi_k(r)\Psi_{K/2}(r) & (k: \text{even}) \\ \Psi_k(r)\Psi_{K/2}(r)\bar{\Psi}(r) & (k: \text{odd}) \end{cases} \quad k = 0, 1, 2, \dots, K/2, \tag{B.2.2}$$

where $k = 1, 2, \dots, K/2 - 1$ are of multiplicity two. The associated eigenvectors are

$$\boldsymbol{\eta}_0 = (1, 1, \dots, 1), \tag{B.2.3}$$

$$\boldsymbol{\eta}_k^+ = (\cos[\theta k i]), \quad \boldsymbol{\eta}_k^- = (\sin[\theta k i]) \quad k = 1, 2, \dots, K/2 - 1, \tag{B.2.4}$$

$$\boldsymbol{\eta}_{K/2} = (1, -1, 1, -1, \dots, 1, -1) \tag{B.2.5}$$

where $\boldsymbol{\eta}_k^+$ and $\boldsymbol{\eta}_k^-$ are the two eigenvectors associated with $f_k(r)$ ($k = 1, 2, \dots, K/2 - 1$).

Remark B.3. For each k with $1 \leq k \leq K/2 - 1$, we may focus on a single eigenvector of the form $\boldsymbol{\eta}_k = \boldsymbol{\eta}_k^+ = (\cos[\theta k i])$ since we do not distinguish rotationally symmetric patterns; any linear combination

of η_k^+ and η_k^- reduces to a single trigonometric curve with the same wavelength as them.

See **Figure 2** and **Figure 3** for an illustration. In particular, $f_0(r) = 1$, $f_1(r) = (1-r)/(1+r)$, and $f_{K/2}(r) = \{(1-r)/(1+r)\}^2$. Furthermore, by employing the analytical expression of $\{f_k(r)\}$ in Lemma B.3, one shows

Corollary B.1. $\{f_k(r)\}$ satisfy the following properties if K is a multiple of four.

1. Every $f_k(r)$ is a monotonically decreasing function of r except for $f_0(r) = 1$.

2. For all r, $\{f_k(r)\}$ with $k = 0, 1, 2, \ldots, K/2$ are ordered as

$$\begin{cases} 1 = f_0 > f_2 > \cdots > f_{2k} > \cdots > f_{K/2}, \\ 1 > f_1 > f_3 > \cdots > f_{2k+1} > \cdots > f_{K/2-1}. \end{cases} \tag{B.2.6}$$

with $f_1(r) > f_2(r)$ and $f_{K/2-1}(r) > f_{K/2}(r)$.

The second property yields that $\min_k\{f_k(r)\} = f_{K/2}(r)$ and $\max_{k \geq 1}\{f_k(r)\} = f_1(r)$ for all r. We note that every $f_k(r(\tau))$ ($k \geq 1$) as a function of τ is monotonically increasing.

Example B.1. In a racetrack economy with $K = 4$, the friction matrix D is given by

$$D = \begin{bmatrix} 1 & r & r^2 & r \\ & 1 & r & r^2 \\ & & 1 & r \\ Sym. & & & 1 \end{bmatrix}. \tag{B.2.7}$$

Its row sum is $d = 1 + r + r^2 + r = (1+r)^2$ and thus $\bar{D} = D/(1+r)^2$. The eigenvalues of $\bar{D}$ are given by

$$f_0 = 1, \quad f_1 = \frac{1-r}{1+r}, \quad f_2 = \left(\frac{1-r}{1+r}\right)^2. \tag{B.2.8}$$

The associated eigenvectors are $\eta_0 = (1,1,1,1)$, $\eta_1^+ = (1,0,-1,0)$, $\eta_1^- = (0,1,0,-1)$, and $\eta_2 = (1,-1,1,-1)$.

B.3   Representing the eigenvalues of $\nabla v(\bar{h})$ by those of $\bar{D}$

We assume that the payoff function v is differentiable. Let $\mathcal{D} \equiv \{h \in \mathbb{R}^K \mid h \cdot \mathbf{1} = H, \ h_i \geq 0\}$ denote the set of possible spatial patterns. For simplicity, we assume that v is defined for the non-negative orthant $\mathbb{R}_+^K$.

Assumption B.1. The payoff function $v : \mathbb{R}_+^K \to \mathbb{R}^K$ is continuously differentiable.

Given v, we define a spatial equilibrium by the following *variational inequality problem* (VIP):

$$[\text{VIP}] \quad \text{Find } h^* \in \mathcal{D} \text{ such that } v(h^*) \cdot (h - h^*) \leq 0 \text{ for all } h \in \mathcal{D}. \tag{B.3.1}$$

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Definition B.1. A spatial equilibrium is a solution to [VIP].

An alternative equivalent definition of the long-run equilibria is found in the main text.

The flat-earth equilibrium $\bar{h} \equiv (h, h, \ldots, h)$ with $h \equiv H/K$ is obviously a spatial equilibrium; because $v(\bar{h}) = \bar{v}\mathbf{1}$ with the uniform level of payoff $\bar{v}$, we have $v(\bar{h})\cdot(h-\bar{h}) = \bar{v}\mathbf{1}\cdot(h-\bar{h}) = \bar{v}(H-H) = 0$ for all $h \in \mathcal{D}$. In preparation for Appendix B.4 below, we discuss the eigenvalues of the Jacobian matrix of the payoff function at the flat-earth equilibrium. Appendix C demonstrates that at the flat-earth equilibrium in a racetrack economy, we can express the Jacobian matrix of the payoff function in the following way:

$$\nabla v(\bar{h}) = G_0(\bar{D})G(\bar{D}), \tag{B.3.2}$$

$$G(\bar{D}) \equiv c_0 I + c_1 \bar{D} + c_2 \bar{D}^2, \tag{B.3.3}$$

where $G_0(\cdot)$ and $G(\cdot)$ are interpreted as matrix polynomials. $G_0(\bar{D})$ is a positive definite matrix defined by $\bar{D}$. Since $\bar{D}$ is circulant, $\nabla v(\bar{h})$ is also circulant. Thus, we can express the kth eigenvalues of $\nabla v(\bar{h})$, e_k, by that of $\bar{D}$, f_k, in terms of the model-dependent functions $G_0(f)$ and $G(f)$:

$$e_k = G_0(f_k)G(f_k) \tag{B.3.4}$$

where $G_0(f_k)$ and $G(f_k)$ are the kth eigenvalues of $G_0(\bar{D})$ and $G(\bar{D})$, respectively. The associated eigenvectors are the same as those of $\bar{D}$. As we have $G_0(f_k) > 0$, to examine the sign of e_k, we only need to check that of $G(f_k)$.

B.4 Stability analysis of the flat-earth equilibrium

By employing the above results, this section derives the results presented in Sections 2.3 and 3.

Notations. In relation to $\mathcal{D}$, let $T\mathcal{D}(h) \equiv \{z \in \mathbb{R}^K \mid z = \alpha(y - h) \text{ for some } y \in \mathcal{D} \text{ and } \alpha \geq 0\}$ denote the tangent cone of $\mathcal{D}$ at $h \in \mathcal{D}$ and $T\mathcal{D} \equiv \{z \in \mathbb{R}^K \mid z \cdot \mathbf{1} = 0\}$ denote the tangent space of $\mathcal{D}$. Note that for any $h \in \text{int } \mathcal{D}$, we have $T\mathcal{D}(h) = T\mathcal{D}$ because $\mathcal{D}$ is a convex subset of a hyperplane.

B.4.1 Derivations for Section 2.3

We summarize our assumptions on the dynamic F as follows, where with notational abuse we let $F(h) = F(h, v(h))$. We assume that F is defined for the non-negative orthant $\mathbb{R}^K_+$.

Assumption B.2. The dynamic $F : \mathbb{R}^K_+ \to \mathbb{R}^K$ satisfies the following properties:

1. (Conservation) the total mass of mobile agents is invariant, i.e., $F(h) \in T\mathcal{D}(h)$ for all $h \in \mathcal{D}$.

2. (Differentiability) $F(h)$ is continuously differentiable with respect to h and $v(h)$ in $\mathcal{D}$.

3. (Stationarity at spatial equilibria) if h^* is a spatial equilibrium, then $F(h^*) = 0$.

4. (Positive correlation) $v(h) \cdot F(h) > 0$ for all $h \in \mathcal{D}$ such that $F(h) \neq 0$.

Example B.2. The set of dynamics that satisfies Assumption B.2 includes the *replicator dynamic* (Taylor and Jonker, 1978), which is the leading instance of the general class of *imitative dynamics*, the Brown–von Neumann–Nash dynamic (Brown and von Neumann, 1950; Nash, 1951), which is an instance of *excess payoff dynamics*, and, for interior equilibria, the *projection dynamic* (Dupuis and Nagurney, 1993). For more examples, see Sandholm (2010).

Consider a small deviation $\eta \in T\mathcal{D}(h^*) = T\mathcal{D}$ at an interior equilibrium $h^* \in \text{int } \mathcal{D}$. By conservation, we must have $F(h^* + \eta) \in T\mathcal{D}$ for such η; it follows that

$$F(h^* + \eta) = F(h^*) + \nabla F(h^*)\eta + o(\|\eta\|) = \nabla F(h^*)\eta + o(\|\eta\|) \in T\mathcal{D}. \tag{B.4.1}$$

Since $J \equiv \nabla F(h^*)$ maps all $\eta \in T\mathcal{D}$ into $T\mathcal{D}$, J defines a linear map from $T\mathcal{D}$ to $T\mathcal{D}$. Thus, the stability analysis of an interior equilibrium h^* reduces to examining the eigenvalues of the restricted linear map $J : T\mathcal{D} \to T\mathcal{D}$. We thus focus on the deviations η that live in $T\mathcal{D}$ (i.e., η such that $\eta \cdot \mathbf{1} = 0$). In effect, we can ignore g_0, which is the associated eigenvalue for $\eta_0 \equiv (1, 1, \ldots, 1)$ because η_0 is the basis for $T\mathcal{D}^\perp$ (the orthogonal space of $T\mathcal{D}$, which is one-dimensional).

For general isolated interior equilibria $h^* \in \text{int } \mathcal{D}$, we have $v(h^*) = \bar{v}\mathbf{1}$ with the uniform level of payoff $\bar{v}$ and $F(h^*) = \mathbf{0}$, implying that $v(h^*) \cdot F(h^*) = 0$. Because h^* is an isolated interior equilibrium, the positive correlation property of F requires that there is a neighborhood $O \subset \mathcal{D}$ of h^* such that $v(h) \cdot F(h) > 0$ for all $h \in O \setminus \{h^*\}$. Moreover, from the differentiability of v and F in $\text{int } \mathcal{D}$, we can expand v and F in the vicinity of the equilibrium; that is, for a sufficiently small η such that $h^* + \eta \in \mathcal{D}$ (i.e., $\eta \in T\mathcal{D}$), the positive correlation property is equivalent to the condition:

$$(\bar{v}\mathbf{1} + \nabla v(h^*)\eta) \cdot (F(h^*) + J\eta) = (\nabla v(h^*)\eta) \cdot (J\eta) > 0. \tag{B.4.2}$$

Note that $(J\eta) \cdot \mathbf{1} = 0$ because $J\eta \in T\mathcal{D}$ for all $\eta \in T\mathcal{D}$.

In (B.4.2), suppose that $\eta = \eta_k$, where η_k ($k \geq 1$) is the kth eigenvector of the restricted linear map J with the associated eigenvalue being g_k. Then, with $h = h^* + \eta_k$, we have

$$(J\eta_k) \cdot (\nabla v(h^*)\eta_k) = (g_k\eta_k) \cdot (\nabla v(h^*)\eta_k) = g_k(\eta_k^\top \nabla v(h^*)\eta_k) > 0, \tag{B.4.3}$$

which shows that, as in (2.1), if g_k and η_k are real (in particular, if J is symmetric)

$$\text{sgn}[g_k] = \text{sgn}[\eta_k^\top \nabla v(h^*)\eta_k] = \text{sgn}\left[\sum_{i \in \mathcal{K}} \delta V_i \eta_{k,i}\right] \quad \text{where} \quad \delta V_i \equiv \sum_{j \in \mathcal{K}} \frac{\partial v_i(h^*)}{\partial h_j} \eta_{k,i}. \tag{B.4.4}$$

B.4.2 Derivations for Section 3

At the flat-earth equilibrium in a racetrack economy, we have stronger results. First, because J and $\nabla v(\bar{h})$ are both symmetric and circulant, the eigenvectors for the two matrices are both real and the same (see Appendix B.1). Thus, by letting e_k be the kth eigenvalue of $\nabla v(\bar{h})$, which is associated with

the eigenvector $\boldsymbol{\eta}_k = (\eta_{k,i}) = (\cos[\theta k i])$ (see Lemma B.3), (B.4.3) further implies that

$$g_k \left(\boldsymbol{\eta}_k^\top \nabla v(\boldsymbol{h}^*) \boldsymbol{\eta}_k \right) = g_k \left(e_k \boldsymbol{\eta}_k^\top \boldsymbol{\eta}_k \right) = g_k e_k \|\boldsymbol{\eta}_k\|^2 > 0. \tag{B.4.5}$$

By noting that g_k and e_k are real, at the flat-earth equilibrium in a racetrack economy, we have

$$\mathrm{sgn}[g_k] = \mathrm{sgn}[e_k] \tag{B.4.6}$$

for all $k \geq 1$. For convenience, we introduce a notation to describe the above situation.

Definition B.2. Let A and B be two K-by-K symmetric circulant matrices. By $A \simeq B$, we denote that $A = CB$ with a symmetric circulant matrix C that is positive definite relative to $T\mathcal{D}$.

Observe that if we have $\boldsymbol{J} \simeq \boldsymbol{B}$ for some symmetric circulant matrix $\boldsymbol{B}$, we may study the eigenvalues of $\boldsymbol{B}$ instead of those of $\boldsymbol{J}$ to examine the stability of $\bar{\boldsymbol{h}}$. Because $\boldsymbol{J} = \boldsymbol{CB}$ and $\boldsymbol{J}, \boldsymbol{C}$, and $\boldsymbol{B}$ are circulant, by employing the properties of circulant matrices, we have $g_k = c_k b_k$ with c_k and b_k being the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{B}$, respectively; moreover, because $\boldsymbol{J}, \boldsymbol{B}$, and $\boldsymbol{C}$ are symmetric, g_k, b_k, and c_k are real (Appendix B.1). Since $\boldsymbol{C}$ is symmetric, circulant, and positive definite relative to $T\mathcal{D}$, we have $c_k > 0$ for $k \geq 1$. In sum, it follows that $\mathrm{sgn}[g_k] = \mathrm{sgn}[b_k]$ for $k \geq 1$. We summarize this as follows.

Lemma B.4. Assume that the dynamic $\boldsymbol{F}$ satisfies Assumption B.2 and consider the flat-earth equilibrium $\bar{\boldsymbol{h}}$ in a racetrack economy. Then, $\boldsymbol{J} \equiv \nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ and $\nabla v(\bar{\boldsymbol{h}})$ are both symmetric and circulant. Furthermore, we have $\boldsymbol{J} \simeq \nabla v(\bar{\boldsymbol{h}})$.

Thus, for the stability analysis of $\bar{\boldsymbol{h}}$, we may study the signs of the eigenvalues e_k ($k \geq 1$) of $\nabla v(\bar{\boldsymbol{h}})$ because we have $\mathrm{sgn}[g_k] = \mathrm{sgn}[e_k]$. In particular, by using our notation, $\boldsymbol{J}$ satisfies

$$\boldsymbol{J} \simeq c_0 \boldsymbol{I} + c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2 \tag{B.4.7}$$

at the flat-earth equilibrium (see Appendix B.3). Thus, the stability of $\bar{\boldsymbol{h}}$ is governed by the model-dependent function $G(\cdot)$ in (B.3.3) because (B.4.7) implies

$$\mathrm{sgn}[g_k] = \mathrm{sgn}[G(f_k)], \tag{B.4.8}$$

$$G(f_k) \equiv c_0 + c_1 f_k + c_2 f_k^2. \tag{B.4.9}$$

Furthermore, not only the signs but also the magnitudes of the eigenvalues $\{g_k\}$ and $\{e_k\}$ of $\boldsymbol{J}$ and $\nabla v(\bar{\boldsymbol{h}})$ are often related in a much stronger way.

Observation B.1. For canonical evolutionary dynamics in the literature, it often follows that $g_k = \bar{c} e_k$ for $k \geq 1$ with a common, positive constant $\bar{c}$.

Example B.3. The *replicator dynamic*, which is the de facto standard dynamic in the NEG literature, is defined by $F_i(\boldsymbol{h}) \equiv h_i \{v_i(\boldsymbol{h}) - \bar{v}(\boldsymbol{h})\}$ where $\bar{v}(\boldsymbol{h}) \equiv (1/H) \sum_{j \in \mathcal{K}} v_j(\boldsymbol{h}) h_j$ is the average payoff across

regions. One has

$$\nabla F(h) = \psi_0(h) + \psi_1(h)\nabla v(h) \tag{B.4.10}$$

with $\psi_0(h)$ and $\psi_1(h)$ defined by $\psi_0(h) \equiv \mathrm{diag}[v(h) - \bar{v}(h)\mathbf{1}] - (1/H)hv(h)^\top$ and $\psi_1(h) \equiv \mathrm{diag}[h](I - (1/H)\mathbf{1}h^\top)$, respectively. It follows that, at the flat-earth equilibrium, $\psi_0(\bar{h}) = -\bar{v}E$ and $\psi_1(\bar{h}) = h(I - E)$, where $E \equiv (1/K)\mathbf{1}\mathbf{1}^\top$ is a K-by-K matrix whose elements are all $1/K$. This implies that

$$g_k = \begin{cases} -\bar{v} < 0 & \text{if } k = 0, \\ he_k & \text{if } 1 \leq k \leq K - 1, \end{cases} \tag{B.4.11}$$

where $\{e_k\}$ are the eigenvalues of $\nabla v(\bar{h})$. Therefore, $J \simeq \nabla v(\bar{h})$ as well as $\bar{c} = h$.

B.4.3 Extension: Taste heterogeneities

The local stability of the equilibria in models with idiosyncratic taste heterogeneity á la Murata (2003) and Redding (2016) can be analyzed by employing the associated *perturbed best response dynamics* as is conducted by Akamatsu et al. (2012) for the logit equilibrium under the logit dynamic. To be precise, for models with a randomized preference $\tilde{v}(h)$ and the associated perturbed best response dynamic $\tilde{F}$, we have $J = \nabla \tilde{F}(\bar{h}, \tilde{v}(\bar{h})) \simeq \Phi\nabla v(\bar{h}) - \eta I$, where η is a positive constant that reflects the magnitude of the heterogeneity and Φ is the projection matrix onto $T\mathcal{D}$. $v(h)$ is interpreted as the homogeneous part of the underlying payoff function $\tilde{v}(h)$ (see Sandholm, 2010). Assuming idiosyncratic taste heterogeneity is thus mathematically equivalent to incorporating an *extra local dispersion force*.[61]

Example B.4 (Logit equilibrium). Consider a logit equilibrium (an equivalent of an equilibrium under an idiosyncratic multiplicative Fréchet shock in the payoff function) with the noise parameter η. It is standard that the equilibrium condition is

$$h_i = P_i(h)H, \quad \text{where } P_i(h) \equiv \frac{\exp[v_i(h)/\eta]}{\sum_{j\in\mathcal{K}} \exp[v_j(h)/\eta]}. \tag{B.4.12}$$

The logit dynamic is defiled by $F_i(h) = HP_i(h) - h_i$. At $\bar{h}$, for the finite values of η, we have

$$J = \eta^{-1}\Phi\nabla v(\bar{h}) - I \simeq \Phi\nabla v(\bar{h}) - \eta I. \tag{B.4.13}$$

Observe that $\eta \to \infty$ implies $J = -I$, which indicates that $\bar{h}$ is always stable; it is intuitive that under sufficient heterogeneity on the side of the preferences of mobile agents, the equilibrium is unique.

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[61]Interested readers should consult Chapter 8 of Sandholm (2010) for local stability analysis via the linearization of evolutionary dynamics in population games as well as the consequences of assuming random utility models on the Jacobian matrix of the dynamic J at an equilibrium.

C Analyses of economic geography models

In this appendix, we derive the Jacobian matrix of the payoff function at the flat-earth equilibrium, $\nabla v(\bar{h})$, for the models included in **Table 1**. As discussed in the main text and as in Appendix B above, this suffices for our purpose. **Table 2** at the end of this appendix summarizes the exact mappings from each model to the coefficients of a model-dependent function $G(f) = c_0 + c_1 f + c_2 f^2$. We note that as soon as one has an analytical expression of $G(f)$, one can derive the break points with respect to the relevant parameters and study the implications of the model.

C.1 Krugman (1991) (Km) model

Following Fujita, Krugman and Venables (1999a), this section introduces a many-region version of Krugman (1991)'s seminal model in line with our context.

Assumptions. There are K discrete regions whose set is denoted by $\mathcal{K}$. There are two types of workers: unskilled and skilled. Each worker inelastically supplies one unit of labor. The total endowments of skilled and unskilled workers are H and L, respectively. Skilled workers are mobile across regions; $h_i \geq 0$ denotes their population in region i, where $\boldsymbol{h} \equiv (h_i)_{i \in \mathcal{K}}$ is their spatial pattern across regions. Throughout Appendix C, $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \boldsymbol{h} \cdot \mathbf{1} = H, \ h_i \geq 0\}$ denotes the set of all possible spatial distributions of mobile (skilled) workers. Unskilled workers are immobile; their population in region i is denoted by l_i.

There are two industrial sectors: agriculture (abbreviated as A) and manufacturing (abbreviated as M). The A-sector is perfectly competitive and a unit input of unskilled labor is required to produce one unit of goods. We choose A-sector goods as the numéraire. The M-sector is modeled by Dixit–Stiglitz monopolistic competition. M-sector goods are horizontally differentiated and produced under increasing returns to scale using skilled labor as the input.

The goods of both sectors are transported. The transportation of A-sector goods is frictionless, While the transportation of M-sector goods is of an iceberg form. For each unit of M-sector goods transported from region i to j, only the proportion $1/\tau_{ij}$ arrives, where $\tau_{ij} > 1$ for $i \neq j$ and $\tau_{ii} = 1$.

Preference. All workers share an identical preference for both M- and A-sector goods. The utility function U of a worker in region i is given by a two-tier form. The upper tier is the following Cobb–Douglas function:

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1 - \mu) \ln C_i^{\mathrm{A}} \quad (0 < \mu < 1), \tag{C.1.1}$$

where C_i^{A} is the consumption of A-sector goods in region i, C_i^{M} the lower-tier manufacturing aggregate in region i, and μ the constant expenditure share of manufactured goods. The lower tier, C_i^{M}, is defined by the following constant elasticity of substitution (CES) aggregate:

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j \in \mathcal{K}} \int_0^{n_j} q_{ji}(\xi)^{(\sigma-1)/\sigma} \mathrm{d}\xi \right)^{\sigma/(\sigma-1)}, \tag{C.1.2}$$

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where n_j is the number of varieties produced in region j, z $q_{ji}(\xi)$ is the consumption of variety $\xi \in [0, n_j]$, and σ is the CES between any two varieties. As we take A-sector goods as the numéraire, the budget constraint of a worker in region i is given by

$$C_i^A + \sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi) q_{ji}(\xi) d\xi = y_i, \tag{C.1.3}$$

where $p_{ji}(\xi)$ denotes the delivered price in region i of the M-sector goods produced in region j and y_i denotes the income of the worker. The incomes (wages) of skilled and unskilled workers are represented by w_i and w_i^u, respectively.

Demand. Utility maximization yields the following demand:

$$C_i^M = \mu \frac{y_i}{P_i}, \quad C_i^A = (1 - \mu) y_i, \quad q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}} C_i^M, \tag{C.1.4}$$

where P_i denotes the price index of the differentiated product in region i:

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi)^{1-\sigma} d\xi \right)^{1/(1-\sigma)}. \tag{C.1.5}$$

Since the total income in region i is given by $Y_i = w_i h_i + w_i^u l_i$, we have the following total demand $Q_{ji}(\xi)$ for the variety ξ produced in j:

$$Q_{ji}(\xi) = \frac{\mu \{p_{ji}(\xi)\}^{-\sigma}}{P_i^{1-\sigma}} (w_i h_i + w_i^u l_i). \tag{C.1.6}$$

The total supply $x_i(\xi)$ of the differentiated variety ξ in region i should meet the total demand from all regions including the transport costs incurred by shipments:

$$x_i(\xi) = \sum_{j \in \mathcal{K}} \tau_{ij} Q_{ij}(\xi). \tag{C.1.7}$$

Firm behavior. With free trade in the A-sector, the wage of the unskilled worker w_i^u is equalized. As A-sector goods are the numéraire, we have $w_i^u = 1$. In the M-sector, to produce x_i units of the differentiated product, a firm requires $\alpha + \beta x_i$ units of skilled labor. With increasing returns, every firm specializes in a single variety. The cost function of a firm in region i producing variety ξ is thus given by

$$C_i(x_i(\xi)) \equiv w_i \{\alpha + \beta x_i(\xi)\}. \tag{C.1.8}$$

Therefore, an M-sector firm located in region i specializing in variety ξ faces the following profit:

$$\Pi_i(\xi) = \sum_{j \in \mathcal{K}} p_{ij}(\xi) Q_{ij}(\xi) - C_i(x_i(\xi)). \tag{C.1.9}$$

Since we have a continuum of firms, each is negligible in the sense that its action has no impact on the market (i.e., the price indices). It is standard that the profit maximization of firms yields

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma-1} w_i \tau_{ij} \tag{C.1.10}$$

and that $p_{ij}(\xi)$ is independent of ξ. This fact in turn implies that $Q_{ij}(\xi)$ and $x_i(\xi)$ also do not depend on ξ. We thus omit ξ in the following.

Short-run equilibrium. In the short run, the spatial distribution $\boldsymbol{h} = (h_i)_{i\in\mathcal{K}}$ of skilled workers is fixed. Given $\boldsymbol{h}$, we determine the short-run equilibrium wage $\boldsymbol{w} \equiv (w_i)_{i\in\mathcal{K}}$ by the M-sector product market-clearing condition (PMCC), zero-profit condition (ZPC), and skilled labor market-clearing condition (LMCC). First, the ZPC for every M-sector firm dictates that $x_i = \alpha(\sigma-1)/\beta$, meaning that the required skilled labor input is $\alpha\sigma$. Then, skilled labor-market clearing yields $\alpha\sigma n_i = h_i$. By using $n_i = h_i/(\alpha\sigma)$, we have

$$P_i = \frac{\sigma\beta}{\sigma-1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{K}} h_j(w_j\tau_{ji})^{1-\sigma}\right)^{1/(1-\sigma)}, \tag{C.1.11}$$

with $d_{ij} \equiv \tau_{ij}^{1-\sigma}$; $\boldsymbol{D} = [d_{ij}] = [\tau_{ij}^{1-\sigma}]$ is the friction matrix. By employing the formula up to here, the M-sector PMCC (C.1.7) implies that

$$w_i h_i = \mu \sum_{j\in\mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{kj}}(w_j h_j + l_j), \tag{C.1.12}$$

which is the so-called *wage equation*. By adding up (C.1.12), we obtain

$$\sum_{i\in\mathcal{K}} w_i h_i = \frac{\mu}{1-\mu}L, \tag{C.1.13}$$

which constrains $\boldsymbol{w}$ at any configuration $\boldsymbol{h}$. The existence and uniqueness of the solution for the wage equation under a fixed $\boldsymbol{h}$ and normalization constraint (C.1.13) follow from standard non-linear complementarity problem arguments (Facchinei and Pang, 2007) and thus we omit them. Given the solution $\boldsymbol{w}(\boldsymbol{h})$ of (C.1.12), we have the following indirect utility function of skilled workers:

$$v_i(\boldsymbol{h}) = \bar{\kappa}\ln[\Delta_i] + \ln[w_i], \tag{C.1.14}$$

where $\bar{\kappa} \equiv \mu/(\sigma-1)$ and $\Delta_i \equiv \sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{ki}$. Note that we omit the constant terms as they do not affect the properties of the equilibrium spatial patterns. We follow this convention for the rest of Appendix C. The long-run equilibria are defined by the VIP in Appendix B.3 based on the payoff function (C.1.14).

Jacobian matrix at the flat-earth equilibrium. Assume a racetrack economy (i.e., $d_{ij} = \tau_{ij}^{1-\sigma} = \exp[-\tau\ell_{ij}]$ with $\tau > 0$; see Section 3.1) with a uniform unskilled labor endowment (i.e., $l_i = l \equiv L/K$ for all $i \in \mathcal{K}$). Then, it is trivial that the flat-earth pattern is a long-run equilibrium. As we must evaluate $\nabla\boldsymbol{v}(\bar{\boldsymbol{h}})$, we

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first derive $\nabla v(h) = [\partial v_i(h)/\partial h_j]$ at any interior solution h. We have

$$\frac{\partial v_i(h)}{\partial h_j} = \frac{\bar{\kappa}}{\Delta_i}\left(\frac{\partial \Delta_i}{\partial h_j} + \sum_{k \in \mathcal{K}} \frac{\partial \Delta_i}{\partial w_k}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j} \tag{C.1.15}$$

$$= \bar{\kappa}\left(\frac{1}{\Delta_i}w_j^{1-\sigma}d_{ji} + (1-\sigma)\sum_{k \in \mathcal{K}}\frac{1}{\Delta_i}h_k w_k^{-\sigma}d_{ki}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j} \tag{C.1.16}$$

$$= \bar{\kappa}\left(\frac{1}{h_j}m_{ji} + (1-\sigma)\sum_{k \in \mathcal{K}}m_{ki}\frac{1}{w_k}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j}, \tag{C.1.17}$$

where $m_{ij} \equiv h_i w_i^{1-\sigma}d_{ij}/\Delta_j$, or in vector–matrix form $M = [m_{ij}] = \mathrm{diag}[w^{1-\sigma} \circ h]D(\mathrm{diag}[\Delta])^{-1}$ with $\Delta = [\Delta_i] = D^\top \mathrm{diag}[w]^{1-\sigma}h$. We let $x^a \equiv [x_i^a]$ and $x \circ y \equiv [x_i y_i]$. Noting that $\bar{\kappa}(1-\sigma) = -\mu$, we have

$$\nabla v(h) = \bar{\kappa}M^\top \mathrm{diag}[h]^{-1} - \mu M^\top \mathrm{diag}[w]^{-1}\nabla w(h) + \mathrm{diag}[w]^{-1}\nabla w(h) \tag{C.1.18}$$

$$= \bar{\kappa}M^\top \mathrm{diag}[h]^{-1} + (I - \mu M^\top)\mathrm{diag}[w]^{-1}\nabla w(h), \tag{C.1.19}$$

where $\nabla w(h) \equiv [\partial w_i(h)/\partial h_j]$ is yet to be known. By letting

$$W_i(h, w) \equiv w_i h_i - \mu \sum_{k \in \mathcal{K}} m_{ik}(w_k h_k + l), \tag{C.1.20}$$

the wage equation is equivalent to $W(h, w) = 0$. Thanks to the implicit function theorem, it can be shown that $\nabla w(h) = -(\nabla_w W)^{-1}(\nabla W)$, where $\nabla_w W \equiv [\partial W_i/\partial w_j]$ and $\nabla W \equiv [\partial W_i/\partial h_j]$ are given by

$$\frac{\partial W_i}{\partial w_j} = \delta_{ij}h_i - \mu \sum_{k \in \mathcal{K}}\frac{\partial m_{ik}}{\partial w_j}(w_k h_k + l) - \mu m_{ij}h_j \tag{C.1.21}$$

$$= \delta_{ij}h_i - \mu(1-\sigma)\frac{1}{w_j}\left(\delta_{ij}\sum_{k \in \mathcal{K}}m_{ik}(w_k h_k + l) - \sum_{k \in \mathcal{K}}m_{ik}m_{jk}(w_k h_k + l)\right) - \mu m_{ij}h_j, \tag{C.1.22}$$

$$\frac{\partial W_i}{\partial h_j} = \delta_{ij}w_i - \mu \sum_{k \in \mathcal{K}}\frac{\partial m_{ik}}{\partial h_j}(w_k h_k + l) - \mu m_{ij}w_j \tag{C.1.23}$$

$$= \delta_{ij}w_i - \mu\frac{1}{h_j}\left(\delta_{ij}\sum_{k \in \mathcal{K}}m_{ik}(w_j h_j + l) - \sum_{k \in \mathcal{K}}m_{ik}m_{jk}(w_k h_k + l)\right) - \mu m_{ij}w_j, \tag{C.1.24}$$

with δ_{ij} being Kronecker's delta. In vector–matrix form, we have

$$\nabla_w W = \mathrm{diag}[h] - \mu(1-\sigma)(\mathrm{diag}[MY] - M\mathrm{diag}[Y]M^\top)\mathrm{diag}[w]^{-1} - \mu M\mathrm{diag}[h] \tag{C.1.25}$$

$$\nabla W = \mathrm{diag}[w] - \mu(\mathrm{diag}[MY] - MYM^\top)\mathrm{diag}[h]^{-1} - \mu M\mathrm{diag}[w], \tag{C.1.26}$$

where $Y = [Y_i] \equiv [w_i h_i + l]$ is the vector of regional income.

Assume the flat-earth equilibrium, where $h = h\mathbf{1}$ with $h \equiv H/K$. Then, we know that the (uniform level of the) equilibrium wage is given by $\bar{w} \equiv \mu/(1-\mu) \cdot L/H$ and the (uniform level of the) total income of a region is $\bar{Y} \equiv l/(1-\mu) = 1/(1-\mu) \cdot L/K$, where $\bar{Y}/\bar{w} = h/\mu$ and $\bar{Y}/h = \bar{w}/\mu$. We also have

$M = \bar{D} = D/d$, where d is the row sum of D. It then follows that

$$\nabla w(\bar{h}) = \frac{\bar{w}}{h}\left[\sigma I - \mu \bar{D} - (\sigma - 1)\bar{D}^2\right]^{-1}\bar{D}\left(\mu I - \bar{D}\right) \tag{C.1.27}$$

and thus that

$$\nabla v(\bar{h}) = \frac{1}{h}\left[I - \kappa \bar{D} - \rho \bar{D}^2\right]^{-1}[(\kappa + \bar{\kappa})\bar{D} - (\mu\bar{\kappa} + \sigma^{-1})\bar{D}^2], \tag{C.1.28}$$

where $\kappa \equiv \mu/\sigma$ and $\rho \equiv (\sigma - 1)/\sigma \in (0, 1)$. We recall that circulant matrices commute (Lemma B.1). It thus follows that for the Km model, we have $\nabla v(\bar{h}) \simeq c_1 \bar{D} + c_2 \bar{D}^2$ with $c_1 = \kappa + \bar{\kappa}$ and $c_2 = -(\mu\bar{\kappa} + \sigma^{-1})$.

Remark C.1. A comparison with the literature would be useful for providing some insights. By letting $\{e_k\}$ be the eigenvalues of $\nabla v(\bar{h})$, we have

$$e_k = \frac{1}{h}f_k\frac{(\kappa + \bar{\kappa}) - (\mu\bar{\kappa} + \sigma^{-1})f_k}{1 - \kappa f_k - \rho f_k^2} = \frac{K}{H}\left(\frac{1 - \rho}{\rho}\right)f_k\left[\frac{\mu(1 + \rho) - (\mu^2 + \rho)f_k}{1 - \mu(1 - \rho)f_k - \rho f_k^2}\right]. \tag{C.1.29}$$

However, this expression is a generalized version of equation (5.27) in Fujita et al. (1999a) for the Km model in the symmetric two-region setting (with a rearrangement):

$$\frac{1}{P_0^{-\mu}}\frac{d\omega}{d\lambda} = \frac{2}{\lambda + (1 - \lambda)}\left(\frac{1 - \rho}{\rho}\right)Z\left[\frac{\mu(1 + \rho) - (\mu^2 + \rho)Z}{1 - \mu(1 - \rho)Z - \rho Z^2}\right], \tag{C.1.30}$$

which expresses the change in the real wage $\omega \equiv w_0 P_0^{-\mu}$ in region 0 when its share of skilled workers λ slightly increases. Here, Z is "an index of trade barriers" defined by equation (5.25), ibid:

$$Z \equiv \frac{1 - T^{1-\sigma}}{1 + T^{1-\sigma}}, \tag{C.1.31}$$

where $T > 1$ is the iceberg transport cost parameter between regions. We thus see that *the "real wage differential" exercise often conducted in the literature is a special case of our approach.* In fact, if we assume $K = 2$, the only possible deviation direction is $\eta_1 = (1, -1)$, which corresponds to agglomeration toward one of the regions. Given the freeness of transport r between the two regions, its associated eigenvalue of $\bar{D}$ is given by

$$f_1 = \frac{1 - r}{1 + r}, \tag{C.1.32}$$

which precisely coincides with the above Z—since $r = T^{1-\sigma}$ for this case. In the two-region economy, there is only a single possible deviation direction: agglomeration. We thus only have to investigate the sign of $d\omega/d\lambda$. In a many-region racetrack economy, however, there are multiple possible deviation directions and thus the stability of the flat-earth pattern depends on the signs of all $(e_k)_{k=1}^{K-1}$.

Remark C.2. As emphasized by the literature, the coefficients of $G(f)$ have clear economic interpretations. The first, c_1, represents the demand externality through a price index ($\bar{\kappa}$) and a home market effect (κ). For the former, $\bar{\kappa}$, observes that $1/(\sigma - 1)$ in $\bar{\kappa}$ is the markup of firms or the magnitude

of product differentiation; an agglomeration, by improving the proximity of mobile agents to the production locations of firms, increases the payoff of agglomerated regions. The latter, κ, is a home market effect. Note that $1/\sigma$ in κ is the share of the fixed cost (the wage of a mobile agent required to operate) in a firm's production cost. The second, c_2, on the contrary, represents the dispersion force. The centrifugal force of the model is due to the increased market competition caused by the concentration of firms (the so-called "market-crowding effect"). Since there is spatially dispersed demand (immobile agents), firms in a region of agglomeration may hope to relocate to other, less crowded regions (σ^{-1} in c_2). In addition, the price-index effect by reducing a firm's market share and hence the wage of mobile agents produces another global dispersion force ($\mu\bar{\kappa}$ in c_2). This effect produces a dispersion force from *outside* a region.

Numerical simulation. **Figure 8** assumes the Km model. The parameters are set as $\mu = 0.4$, $\sigma = 10$, $L = 8$, and $H = 1$.

C.2   Forslid and Ottaviano (2003) (FO) model

The FO model is a slightly simplified version of Krugman (1991)'s NEG model. The model is sometimes called the *footloose-entrepreneur model*, since a unit of skilled (mobile) labor is required as the fixed input of a manufacturing firm. The only difference is that the variable input of M-sector firms in the Km model is now replaced by unskilled labor. Specifically, to produce $x_i(\xi)$ units of product ξ, an M-sector firm now requires α units of skilled labor and $\beta x_i(\xi)$ units of unskilled labor. Therefore, for the FO model, the total cost of production for a firm in region i is

$$C_i(x(\xi)) = \alpha w_i + \beta x_i(\xi)w_i^{\mathrm{u}}. \tag{C.2.1}$$

The wage equalization of the A-sector ($w_i^{\mathrm{u}} = 1$ for all $i \in \mathcal{K}$) then implies that

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1}\tau_{ij} \tag{C.2.2}$$

provided that A-sector goods are produced in every region (we assume $\beta x_i n_i < l_i$ for all $i \in \mathcal{K}$). Again, we drop ξ in what follows.

Short-run equilibrium. The short-run equilibrium conditions are again the PMCC, LMCC, and ZPC. First, since a firm requires α units of skilled labor, the LMCC implies that $\alpha n_i = h_i$, which in turn yields the price index P_i for the FO model:

$$P_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_{j\in\mathcal{K}} h_j d_{ji}\right)^{1/(1-\sigma)}, \tag{C.2.3}$$

where $d_{ji} \equiv \tau_{ji}^{1-\sigma}$ is the trade friction between regions i and j. Note that unlike the Km model, P_i does not depend on the wage $w = (w_i)_{i\in\mathcal{K}}$. The ZPC implies that the operating profit of a firm is entirely

absorbed by the wage bills:

$$w_i = \left(\sum_{j \in \mathcal{K}} p_{ij} Q_{ij} - \beta x_i \right). \tag{C.2.4}$$

Together with the PMCC, we have the following wage equation for the model:

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\sum_{k \in \mathcal{K}} d_{kj} h_k} (w_j h_j + l_j). \tag{C.2.5}$$

This equation is analytically solvable. Specifically, in vector–matrix form, we have

$$\boldsymbol{w} = \kappa \left[\boldsymbol{I} - \kappa \boldsymbol{M} \operatorname{diag}[\boldsymbol{h}] \right]^{-1} \boldsymbol{M} \boldsymbol{l}, \tag{C.2.6}$$

where $\kappa \equiv \mu/\sigma$, $\boldsymbol{l} \equiv (l_i)$, and $\boldsymbol{M} \equiv [m_{ij}] = [d_{ij}/\Delta_j] = \boldsymbol{D} \{ \operatorname{diag}[\boldsymbol{\Delta}] \}^{-1}$ with $\Delta_i = \sum_{j \in \mathcal{K}} d_{ji} h_j$, meaning that $\boldsymbol{\Delta} = [\Delta_i] = \boldsymbol{D}^\top \boldsymbol{h}$. The indirect utility $\boldsymbol{v}(\boldsymbol{h})$ of each of the many-region FO models is expressed as

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + \ln[w_i], \tag{C.2.7}$$

where $\bar{\kappa} \equiv \mu/(\sigma - 1)$. We again ignore the constant terms. The long-run equilibria are defined by (B.3.1) with respect to the above (C.2.7).

Jacobian matrix at the flat-earth equilibrium. In a racetrack economy, by following the same line of logic as in the Km model, we obtain

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h} \left[\boldsymbol{I} - \kappa \bar{\boldsymbol{D}} \right]^{-1} \left[(\bar{\kappa} + \kappa) \bar{\boldsymbol{D}} - (\bar{\kappa} \kappa + 1) \bar{\boldsymbol{D}}^2 \right], \tag{C.2.8}$$

where $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$. We thus conclude that $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$ with $c_1 = \bar{\kappa} + \kappa$ and $c_2 = -(\bar{\kappa} \kappa + 1)$.

C.3 Pflüger (2004) (Pf) model

The Pf model is a further simplified version of the FO model (and hence the Km model) in which we assume a quasi-linear utility function for the upper tier as follows:

$$U(C_i^M, C_i^A) = C_i^A + \mu \ln C_i^M. \tag{C.3.1}$$

Taking A-sector goods as the numéraire, it is standard that utility maximization yields the following demand, where the income effect in C_i^M is lost compared with the Km and FO models:

$$C_i^A = y_i - \mu, \quad C_i^M = \mu \frac{1}{P_i}, \tag{C.3.2}$$

where the price index P_i is the same as that in the FO model. Thus, by replacing the total income of a region $Y_i = w_i h_i + l_i$ in (C.2.5) with the total number of workers $h_i + l_i$, we obtain the following "wage

equation":

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\sum_{k \in \mathcal{K}} d_{kj} h_k} (h_j + l_j), \tag{C.3.3}$$

which has already been solved. Indirect utility is given by

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + w_i, \tag{C.3.4}$$

where $\Delta_i \equiv \sum_{j \in \mathcal{K}} d_{ji} h_j$. The long-run equilibria are defined by (B.3.1) with respect to the above (C.2.7). *Jacobian matrix at the flat-earth equilibrium.* We show

$$\nabla \boldsymbol{v}(\boldsymbol{h}) = \bar{\kappa} \boldsymbol{M}^\top + \kappa \left(\boldsymbol{M} - \boldsymbol{M} \operatorname{diag}[\boldsymbol{H}] \boldsymbol{M}^\top \right) \tag{C.3.5}$$

with $\boldsymbol{H} = [H_i] \equiv [h_i + l_i]$ and $\boldsymbol{M} = [m_{ij}] \equiv [d_{ij}/\Delta_j]$ and thus that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h} \left[(\bar{\kappa} + \kappa) \bar{\boldsymbol{D}} - \kappa(1 + \epsilon) \bar{\boldsymbol{D}}^2 \right] \tag{C.3.6}$$

with $\epsilon = L/H$ being the ratio of the number of unskilled to skilled workers. We thus see that $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$ with $c_1 = \bar{\kappa} + \kappa$ and $c_2 = -\kappa(1 + \epsilon)$.

C.4 Helpman (1998) (Hm) model

Helpman (1998) removed the A-sector in Krugman (1991) and thereby assumed that all workers are mobile. Instead of the A-sector, the Hm model introduces the housing (abbreviated as H) sector and each region i is endowed with a fixed stock A_i of housing.

Preference. The utility function of a worker in region i is given by

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{H}}) = \mu \ln C_i^{\mathrm{M}} + \gamma \ln C_i^{\mathrm{H}}, \tag{C.4.1}$$

where C_i^{H} is the consumption of H-sector goods in region i and γ is its constant expenditure share ($\gamma + \mu = 1$). The budget constraint of a worker located in region i is represented by

$$p_i^{\mathrm{H}} C_i^{\mathrm{H}} + \sum_j \int_0^{n_j} p_{ji}(\xi) q_{ji}(\xi) \mathrm{d}\xi = y_i, \tag{C.4.2}$$

where p_i^{H} is the price of H-sector goods in region i. Utility maximization leads to the following demand for H-sector goods:

$$C_i^{\mathrm{H}} = \gamma \frac{y_i}{p_i^{\mathrm{H}}}. \tag{C.4.3}$$

Housing market clearing. In the H-sector, total demand $h_i C_i^{\mathrm{H}}$ in region i cannot be greater than maximum supply A_i. If demand in region i is less than supply, the price p_i^H should be the lower

boundary (i.e., zero); Otherwise, it is positive. Thus, we have the following housing market-clearing condition:

$$\begin{cases} h_i C_i^H = A_i & \text{if} \quad p_i^H > 0, \\ h_i C_i^H \le A_i & \text{if} \quad p_i^H = 0, \end{cases} \quad \forall i. \tag{C.4.4}$$

From (C.4.3), $p_i^H \ne 0$ for any long-run equilibria; because $C_i^A \to \infty$ and thus $U \to \infty$ as $p_i^H \to 0$, such a spatial pattern is never sustainable. We thus conclude that

$$C_i^H = \frac{A_i}{h_i}, \quad p_i^H = \gamma \frac{y_i h_i}{A_i} \tag{C.4.5}$$

and that $h_i > 0$ at any long-run equilibrium.

Landownership. We here consider two types of assumptions on landownership: *public landownership* (abbreviated as PL) and *local landownership* (LL). In the original formulation, housing stocks are equally owned by all workers (i.e., PL). In this way, the income of a worker in region i is the sum of the wage and dividend of rental revenue, $y_i = w_i + \bar{w}^H$, where

$$\bar{w}^H = \frac{1}{H} \sum_{i \in \mathcal{K}} p_i^H C_i^H h_i = \frac{\gamma}{H} \sum_{i \in \mathcal{K}} y_i h_i, \tag{C.4.6}$$

meaning that rearrangement yields

$$\bar{w}^H = \frac{\gamma}{(1-\gamma)H} \sum_{i \in \mathcal{K}} w_i h_i. \tag{C.4.7}$$

We set $\bar{w}^H = 1$ to normalize w_i to satisfy $\sum_{i \in \mathcal{K}} w_i h_i = (\mu/\gamma)H$. On the contrary, Ottaviano et al. (2002), Murata and Thisse (2005), and Redding and Sturm (2008) assumed that housing stocks are locally owned (i.e., LL). Hence, $y_i = w_i + w_i^H$, where $\bar{w}_i = p_i^H C_i^H = \gamma y_i$, which in turn yields $y_i = w_i/\mu$. Also for this case, analogous to the PL case, we constrain w by using the condition $\sum_{i \in \mathcal{K}} w_i h_i = (\mu/\gamma)H$ for normalization purposes.

Short-run equilibrium. Regarding the short-run equilibrium conditions, the only difference from the Km model is the total expenditure in each region, which is now

$$Y_i = \begin{cases} (w_i + 1)h_i, & \text{(for PL)}, \\ w_i h_i / \mu, & \text{(for LL)}. \end{cases} \tag{C.4.8}$$

The short-run equilibrium wage equation is thus given by

$$[\text{PL}] \quad w_i h_i = \mu \sum_{j \in \mathcal{K}} \frac{d_{ij} w_i^{1-\sigma} h_i}{\sum_{k \in \mathcal{K}} d_{kj} w_k^{1-\sigma} h_k} (w_j + 1)h_j, \tag{C.4.9}$$

$$[\text{LL}] \quad w_i h_i = \sum_{j \in \mathcal{K}} \frac{d_{ij} w_i^{1-\sigma} h_i}{\sum_{k \in \mathcal{K}} d_{kj} w_k^{1-\sigma} h_k} w_j h_j. \tag{C.4.10}$$

Given the solution w for (C.4.9) or (C.4.9), indirect utility $v(h)$ is expressed as

$$v_i(h) = \begin{cases} \bar{\kappa} \ln[\Delta_i] + \mu \ln[w_i + 1] - \gamma (\ln[h_i] - \ln[A_i]), & \text{(for PL)}, \\ \bar{\kappa} \ln[\Delta_i] + \mu \ln[w_i] - \gamma (\ln[h_i] - \ln[A_i]), & \text{(for LL)} \end{cases}, \qquad \text{(C.4.11)}$$

where $\Delta_i \equiv \sum_{j \in \mathcal{K}} h_j w_j^{1-\sigma} d_{ji}$.

Jacobian matrix at the flat-earth equilibrium. Let $A_i = A$ for all regions to abstract from the location-fixed exogenous effects. For the PL case, we can show that

$$\nabla v(\bar{h}) = \frac{1}{h} \left\{ \bar{\kappa} \bar{D} + \mu(\mu I - \bar{D}) \left[\sigma I - \mu \bar{D} - (\sigma - 1)\bar{D}^2 \right]^{-1} \bar{D}(I - \bar{D}) - \gamma I \right\} \qquad \text{(C.4.12)}$$

$$= \frac{\sigma}{h} \left[\sigma I - \mu \bar{D} - (\sigma - 1)\bar{D}^2 \right]^{-1} \left\{ -\gamma I + (\bar{\kappa} + \kappa)\bar{D} + \left\{ \gamma - \left(\mu \bar{\kappa} + \frac{1}{\sigma} \right) \right\} \bar{D}^2 \right\}, \qquad \text{(C.4.13)}$$

meaning that $\nabla v(\bar{h}) \simeq c_0 I + c_1 \bar{D} + c_2 \bar{D}^2$ with $c_0 = -\gamma$, $c_1 = \mu \left(\frac{1}{\sigma-1} + \frac{1}{\sigma} \right)$, and $c_2 = \gamma - \frac{1}{\sigma} - \frac{\mu^2}{\sigma-1}$. Recall that γ is the expenditure share of housing goods. Hence, the dispersion force expressed by $c_0 < 0$ solely arises from local housing. For the LL case, we can show that

$$\nabla v(\bar{h}) = \frac{1}{h} \left\{ \bar{\kappa} \bar{D} + \mu(I - \bar{D}) \left[\sigma I + (\sigma - 1)\bar{D} \right]^{-1} \bar{D} - \gamma I \right\} \qquad \text{(C.4.14)}$$

$$= \frac{\sigma}{h} \left[\sigma I + (\sigma - 1)\bar{D} \right]^{-1} \left\{ -\gamma I + \left(\frac{\mu}{\sigma-1} + \frac{\mu}{\sigma} - \gamma \frac{\sigma-1}{\sigma} \right) \bar{D} \right\}. \qquad \text{(C.4.15)}$$

From this, we conclude that for the LL case $\nabla v(\bar{h}) \simeq c_0 I + c_1 \bar{D}$ with $c_0 = -\gamma$ and $c_1 = \frac{\mu}{\sigma-1} + \frac{\mu}{\sigma} - \gamma \frac{\sigma-1}{\sigma}$.

Remark C.3. For the model, the condition for the uniqueness of the equilibrium is given by $\gamma \sigma = (1 - \mu)\sigma > 1$ (Redding and Sturm, 2008). If $\gamma \sigma > 1$ is satisfied, regardless of the assumption on landownership, the flat-earth equilibrium is stable.

Remark C.4. The regional model formulated in §3 of Redding and Rossi-Hansberg (2017) is an enhanced version of the Hm model with LL, in which the variable input of skilled labor is allowed to depend on region i (i.e., productivity differs across regions). That is, the cost function of firms in region i becomes

$$C_i(x_i(\xi)) = w_i(\alpha + \beta_i x_i(\xi)). \qquad \text{(C.4.16)}$$

This then implies that the short-run equilibrium price and price index in region i become

$$p_{ij}(\xi) = \frac{\sigma \beta_i}{\sigma - 1} \tau_{ij} w_i, \qquad \text{(C.4.17)}$$

$$P_i = \frac{\sigma}{\sigma - 1} \left(\frac{1}{\alpha \sigma} \sum_{j \in \mathcal{K}} h_j (\beta_j w_j \tau_{ji})^{1-\sigma} \right)^{1/(1-\sigma)}, \qquad \text{(C.4.18)}$$

respectively. As the model assumes LL, the wage equation for the model is

$$w_i h_i = \sum_{j \in \mathcal{K}} \frac{h_i A_i w_i^{1-\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} h_k A_k w_k^{1-\sigma} d_{kj}} w_j h_j, \tag{C.4.19}$$

where $A_i \equiv \beta_i^{1-\sigma}$. Thus, by abstracting from first natures by setting $A_i = \bar{A}$, the model reduces to the Hm model under LL.

C.5 Puga (1999) (Pg) model

Puga (1999) generalized the Km model in two directions, namely (i) the inter-sector mobility of workers between the A-sector and the M-sector (without immobile workers but land) and (ii) intermediate inputs in the M-sector, both as in Krugman and Venables (1995).

Assumptions. There is only a mass H of mobile workers, with h_i denoting the number of workers in region i. We denote by h_i^{M} and h_i^{A} the numbers of workers engaged in the M- and A-sectors, respectively ($h_i = h_i^{\mathrm{M}} + h_i^{\mathrm{A}}$). The homogeneous preference of consumers is the same as in the Km model, with the expenditure share of the M-sector good μ and elasticity of substitution between manufactured varieties σ. Each region is endowed with A_i units of land owned by immobile landlords that have the same preference as the workers. We assume that if a worker relocates, then he or she first enters the M-sector of the destination region. The stability of the spatial pattern h is then reduced to the study of $h^{\mathrm{M}} \equiv [h_i^{\mathrm{M}}]$.

A-sector. The A-sector is perfectly competitive and produces a homogeneous output by using labor and land under constant returns to scale. A-sector goods are costless to trade and set as the numéraire. Let X_i^{A} be the gross regional product of the A-sector. In line with the original study, we specify a Cobb–Douglas production function with labor share $\bar{\mu}$; in concrete terms, we have $X_i^{\mathrm{A}} = (h_i^{\mathrm{A}})^{\bar{\mu}} A_i^{1-\bar{\mu}}$. This implies that the total labor costs of A-sector firms are given by $\bar{\mu} X_i^{\mathrm{A}} = w_i h_i^{\mathrm{A}}$, while their land costs (= the total rental revenue of landlords) are $(1 - \bar{\mu}) X_i^{\mathrm{A}} = \frac{1-\bar{\mu}}{\bar{\mu}} w_i h_i^{\mathrm{A}}$. In particular, labor demand in this sector is given by a function of the wage $h_i^{\mathrm{A}} = A_i (w_i/\bar{\mu})^{1/(\bar{\mu}-1)}$, because $w_i = \bar{\mu}(h_i^{\mathrm{A}}/A_i)^{\bar{\mu}-1}$. Let $h_i^{\mathrm{A}} = \epsilon_i h_i^{\mathrm{M}}$, meaning that $h_i = (1 + \epsilon_i) h_i^{\mathrm{M}}$; we here consider the case $h_i^{\mathrm{M}} \neq 0$, because we are interested in the stability of complete dispersion. We also have $\epsilon_i \equiv (A_i/h_i^{\mathrm{M}})(w_i/\bar{\mu})^{1/(\bar{\mu}-1)}$. The regional rental revenue from land, R_i, in terms of h_i^{M} is

$$R_i \equiv \frac{1-\bar{\mu}}{\bar{\mu}} \epsilon_i w_i h_i^{\mathrm{M}}. \tag{C.5.1}$$

By employing the above formulae, the elasticity ν_i of a region's labor supply to the M-sector with respect to wage is

$$\nu_i \equiv \frac{w_i}{h_i^{\mathrm{M}}} \frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = \frac{h_i^{\mathrm{A}}}{h_i^{\mathrm{M}}} \frac{1}{1-\bar{\mu}} = \epsilon_i \frac{1}{1-\bar{\mu}}. \tag{C.5.2}$$

Further, if $\bar{\mu} = 0$, $X_i^{\mathrm{A}} = A_i$ as well as $R_i = A_i$ and $\epsilon_i = 0$.

M-sector. By considering the simplest possible model of intermediate inputs as in Krugman and Venables (1995), the minimum cost function of the M-sector is replaced by

$$C(x_i(\xi)) = P_i^{\hat{\mu}} w_i^{1-\hat{\mu}}(\alpha + \beta x_i(\xi)), \tag{C.5.3}$$

where P_i is the price index of M-sector goods in region i and $\hat{\mu}$ the share of intermediates in firms' costs. The profit-maximizing price is given by

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1} P_i^{\hat{\mu}} w_i^{1-\hat{\mu}} \tau_{ij}, \tag{C.5.4}$$

which, together with the definition of P_i, implies that we should solve a system of non-linear equations to obtain P_i. In concrete terms, the price indices $\{P_i\}$ should satisfy

$$P_i = \frac{\sigma\beta}{\sigma - 1} \left(\sum_{j \in \mathcal{K}} n_j \left(P_j^{\hat{\mu}} w_j^{1-\hat{\mu}} \right)^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)}, \tag{C.5.5}$$

where $d_{ij} \equiv \tau_{ij}^{1-\sigma}$. We must solve (C.5.5) along with the wage equation to be defined below.

In line with the Km model, the ZPC of firms implies $x_i(\xi) = \alpha(\sigma - 1)/\beta$. Firms' minimized production cost in region i is then given by $C_i = (\alpha\sigma)P_i^{\hat{\mu}} w_i^{1-\hat{\mu}}$, meaning that labor demand in the M-sector of region i is

$$h_i^{\mathrm{M}} = (1 - \hat{\mu})\frac{C_i}{w_i} n_i = \alpha\sigma(1 - \hat{\mu})P_i^{\hat{\mu}} w_i^{-\hat{\mu}} n_i. \tag{C.5.6}$$

The mass of varieties produced in region i is thus given as follows:

$$n_i = \frac{1}{\alpha\sigma(1 - \hat{\mu})} P_i^{-\hat{\mu}} w_i^{\hat{\mu}} h_i^{\mathrm{M}}. \tag{C.5.7}$$

For simplicity, in the following, as in the original study, we normalize the constants such that $\alpha = 1/\sigma$ and $\beta = (\sigma - 1)/\sigma$. Then, by plugging (C.5.7) to (C.5.5), we have

$$P_i^{1-\sigma} = \frac{1}{1 - \hat{\mu}} \sum_{j \in \mathcal{K}} h_j^{\mathrm{M}} P_j^{-\hat{\mu}\sigma} w_j^{1-\sigma+\hat{\mu}\sigma} d_{ji}. \tag{C.5.8}$$

Land is locally owned by immobile landlords that share the same preference as mobile workers; their regional expenditure on M-sector goods is given by μR_i. In addition, the regional expenditure of firms on intermediates is given by

$$\hat{\mu} C_i n_i = \frac{\hat{\mu}}{1 - \hat{\mu}} w_i h_i^{\mathrm{M}}. \tag{C.5.9}$$

Total expenditure in region i on M-sector goods is $Y_i = \mu w_i h_i + \mu R_i + \hat{\mu} C_i n_i$. By using (C.5.1) as well

as $h_i = (1 + \epsilon_i)h_i^{\mathrm{M}}$, this is simplified to

$$Y_i = \mu(1 + \epsilon_i)w_i h_i^{\mathrm{M}} + \mu R_i + \frac{\hat{\mu}}{1 - \hat{\mu}} w_i h_i^{\mathrm{M}} = \left[\mu\left(1 + \frac{\epsilon_i}{\hat{\mu}}\right) + \frac{\hat{\mu}}{1 - \hat{\mu}}\right]w_i h_i^{\mathrm{M}}. \tag{C.5.10}$$

From the ZPC of firms, the wage equation for the model is given by[62]

$$\underbrace{\frac{1}{1 - \hat{\mu}} w_i h_i^{\mathrm{M}}}_{\text{M-sector firms' total cost}} = \underbrace{\sum_{j \in \mathcal{K}} \frac{h_i^{\mathrm{M}} P_i^{-\hat{\mu}\sigma} w_i^{1-\sigma+\hat{\mu}\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} h_k^{\mathrm{M}} P_k^{-\hat{\mu}\sigma} w_k^{1-\sigma+\hat{\mu}\sigma} d_{kj}} \left[\mu\left(1 + \frac{\epsilon_j}{\hat{\mu}}\right) + \frac{\hat{\mu}}{1 - \hat{\mu}}\right]w_j h_j^{\mathrm{M}}}_{\text{M-sector firms' total revenue}} .. \tag{C.5.11}$$

The short-run wage $w = (w_i)$ and price index $P = (P_i)$ are obtained as the solution for the system of non-linear equations (C.5.8) and (C.5.11). We require $\hat{\mu} < \frac{\sigma-1}{\sigma}$, meaning that P and w are uniquely determined for any transportation cost.

Given P and w, the indirect utility function is

$$v_i(h) = \frac{\mu}{\sigma - 1} \ln[\Delta_i] + \ln[w_i] \tag{C.5.12}$$

with $\Delta_i = \sum_{j \in \mathcal{K}} h_j^{\mathrm{M}} P_j^{-\hat{\mu}\sigma} w_j^{1-\sigma+\hat{\mu}\sigma} d_{ji}$.

Jacobian matrix of the payoff function at the flat-earth equilibrium. Let $A_i = A$ for all i and consider the flat-earth equilibrium. Let $h \equiv H/K$ be the uniform number of mobile agents; let also h^{M} and h^{A} be the number of mobile agents engaged in the M- and A-sectors, respectively. Further, let $\bar{Y}, \bar{P}, \bar{w}, \bar{\Omega}$, and $\bar{\epsilon}$ be the uniform level of regional expenditure, price index, wage, Ω_i, and ratio ϵ_i of h_i^{A} to h_i^{M} at the flat-earth equilibrium, respectively. By adding up the wage equations (C.5.11) at the flat-earth equilibrium, we show

$$\bar{\epsilon} = \frac{h^{\mathrm{A}}}{h^{\mathrm{M}}} = \bar{\mu}\frac{1 - \mu}{\mu}. \tag{C.5.13}$$

A larger $\bar{\mu}$ (μ) implies a larger (smaller) $\bar{\epsilon} = h^{\mathrm{A}}/h^{\mathrm{M}}$, which is intuitive. The explicit formula of $\bar{\epsilon}$ yields

$$\bar{Y} = \frac{\bar{w}}{1 - \hat{\mu}} \frac{\bar{\epsilon}}{1 + \bar{\epsilon}} h, \quad \bar{w} = \bar{\mu}\left(\frac{h}{A}\frac{\bar{\epsilon}}{1 + \bar{\epsilon}}\right)^{\bar{\mu}-1}. \tag{C.5.14}$$

Together with the fact that $\bar{\epsilon}/\bar{\mu} = \frac{\mu}{1-\mu}$, $\bar{\mu}$ does not affect the stability of $\bar{h}$ but rather scales total global income. We also have $\bar{P} = \rho\bar{w}$, where $\rho \equiv \{(h^{\mathrm{M}}d)(1 - \hat{\mu})\}^{1/(1-\sigma+\hat{\mu}\sigma)}$ with d being the row sum of D and $\Delta_i = \bar{\Delta} \equiv (1 - \hat{\mu})\bar{P}^{1-\sigma}$. Note that at the equilibrium, we have

$$\frac{1}{\bar{\mu}}\frac{\partial \epsilon_i}{\partial h_i^{\mathrm{M}}} = -\frac{1}{\bar{\mu}}\frac{\bar{\epsilon}}{h^{\mathrm{M}}} = -\frac{1}{h^{\mathrm{M}}}\frac{1 - \mu}{\mu}. \tag{C.5.15}$$

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[62]The original analyses in Puga (1999) allow a positive profit of firms. In this appendix, we adhere to the ZPC so that comparisons with the other models are possible.

In addition, with $\bar{\nu} \equiv \frac{\bar{\mu}}{1-\bar{\mu}} \frac{1-\mu}{\mu}$ being the elasticity of labor supply from the A-sector to the M-sector with respect to w_i,

$$\frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = \bar{\nu} \frac{h^{\mathrm{M}}}{w_i}, \quad \frac{1}{\bar{\mu}} \frac{\partial \epsilon_i}{\partial w_i} = \frac{1}{\bar{\mu}} \frac{\partial \epsilon_i}{\partial h^{\mathrm{M}}} \frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = -\bar{\nu} \frac{1}{w_i} \frac{1-\mu}{\mu}. \tag{C.5.16}$$

We also assume that $\partial h_i / \partial h_i^{\mathrm{M}} = \partial h_i^{\mathrm{M}} / \partial h_i = 1$ as discussed. It follows that, at $\bar{h}$,

$$\frac{\partial Y_i}{\partial h_i^{\mathrm{M}}} = \left(\frac{\hat{\mu}}{1-\hat{\mu}} + \mu \right) \bar{w}, \quad \frac{\partial Y_i}{\partial w_i} = \left(\frac{1}{1-\hat{\mu}} + (1-\mu)\bar{\nu} \right) h^{\mathrm{M}}. \tag{C.5.17}$$

The Jacobian matrix of the payoff function is computed as

$$\frac{\partial v_i}{\partial h_j} = \frac{\mu}{\sigma - 1} \frac{1}{\Delta_i} \left(\frac{\partial \Delta_i}{\partial h_j} + \sum_{k \in \mathcal{K}} \frac{\partial \Delta_i}{\partial P_k} \frac{\partial P_k}{\partial h_j} + \sum_{k \in \mathcal{K}} \frac{\partial \Delta_i}{\partial w_k} \frac{\partial w_k}{\partial h_j} \right) + \delta_{i,j} \frac{1}{w_i} \frac{\partial w_i}{\partial h_j}, \tag{C.5.18}$$

where $\delta_{i,j}$ is Kronecker's delta; below, we evaluate $\nabla \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial h_j]$, $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial P_j]$, $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial w_j]$, $\nabla \boldsymbol{P} \equiv [\partial P_i / \partial h_j]$, and $\nabla \boldsymbol{w} \equiv [\partial w_i / \partial h_j]$.

For $\nabla \boldsymbol{\Delta}$, $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta}$, and $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta}$, we compute as follows: $\nabla \boldsymbol{\Delta} = \bar{\Delta}(h^{\mathrm{M}})^{-1} \bar{\boldsymbol{D}}$, $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta} = \bar{\Delta} \bar{w}^{-1} a \bar{\boldsymbol{D}}$, as well as $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta} = \bar{\Delta} \bar{P}^{-1} b \bar{\boldsymbol{D}}$, with $a \equiv 1 - \sigma + \hat{\mu}\sigma$ and $b \equiv -\hat{\mu}\sigma$. Thus, at the flat-earth pattern, $\nabla v(\bar{h})$ is

$$\nabla v(\bar{h}) = \frac{\mu}{\sigma - 1} \left(\frac{1}{h^{\mathrm{M}}} \bar{\boldsymbol{D}} + \frac{1}{\bar{P}} b \bar{\boldsymbol{D}} \nabla \boldsymbol{P} \right) + \frac{1}{\bar{w}} \left(\boldsymbol{I} + \frac{\mu a}{\sigma - 1} \bar{\boldsymbol{D}} \right) \nabla \boldsymbol{w}. \tag{C.5.19}$$

The remaining task is to evaluate $\nabla \boldsymbol{P}$ and $\nabla \boldsymbol{w}$. First, by totally differentiating the definition of the price index (C.5.8), we have $\nabla \boldsymbol{Q} \mathrm{d} h^{\mathrm{M}} + \nabla_{\boldsymbol{w}} \boldsymbol{Q} \mathrm{d} w + \nabla_{\boldsymbol{P}} \boldsymbol{Q} \mathrm{d} \boldsymbol{P} = \boldsymbol{0}$ with

$$\nabla_{\boldsymbol{P}} \boldsymbol{Q} \equiv [(\sigma - 1)\boldsymbol{I} + b\bar{\boldsymbol{D}}], \quad \nabla_h \boldsymbol{Q} \equiv \frac{\bar{P}}{h^{\mathrm{M}}} \bar{\boldsymbol{D}}, \quad \nabla_{\boldsymbol{w}} \boldsymbol{Q} \equiv a \frac{\bar{P}}{\bar{w}} \bar{\boldsymbol{D}}. \tag{C.5.20}$$

In addition, the total differentiation of the wage equation implies $\nabla \boldsymbol{W} \mathrm{d} h^{\mathrm{M}} + \nabla_{\boldsymbol{w}} \boldsymbol{W} \mathrm{d} w + \nabla_{\boldsymbol{P}} \boldsymbol{W} \mathrm{d} \boldsymbol{P} = \boldsymbol{0}$ with

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} \equiv h^{\mathrm{M}} \boldsymbol{I} - (1 - \hat{\mu}) \left[a \frac{1}{\bar{w}} \bar{Y}(\boldsymbol{I} - \bar{\boldsymbol{D}}^2) + \frac{\partial \bar{Y}}{\partial w} \bar{\boldsymbol{D}} \right], \tag{C.5.21}$$

$$\nabla_h \boldsymbol{W} \equiv \bar{w} \boldsymbol{I} - (1 - \hat{\mu}) \left[\frac{1}{h^{\mathrm{M}}} \bar{Y}(\boldsymbol{I} - \bar{\boldsymbol{D}}^2) + \frac{\partial \bar{Y}}{\partial h^{\mathrm{M}}} \bar{\boldsymbol{D}} \right], \tag{C.5.22}$$

$$\nabla_{\boldsymbol{P}} \boldsymbol{W} \equiv -(1 - \hat{\mu}) b \frac{1}{\bar{P}} \bar{Y}(\boldsymbol{I} - \bar{\boldsymbol{D}}^2) \nabla \boldsymbol{P}. \tag{C.5.23}$$

We have already computed $\bar{Y}$, $\partial \bar{Y} / \partial w$, and $\partial \bar{Y} / \partial h^{\mathrm{M}}$. These relations yield the analytical expressions for the Jacobian matrices $\nabla \boldsymbol{P}$ and $\nabla \boldsymbol{w}$:

$$\nabla \boldsymbol{P} = -[\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W} - \nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}]^{-1} [\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_h \boldsymbol{W} - \nabla_h \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}], \tag{C.5.24}$$

$$\nabla \boldsymbol{w} = [\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W} - \nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}]^{-1} [\nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_h \boldsymbol{W} - \nabla_h \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W}]. \tag{C.5.25}$$

Summing up the computations up to here, patient computation yields

$$\nabla v(\bar{h}) = J_0^{-1}\left[\check{\mu}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)\bar{D} - \left(\frac{\check{\mu}^2}{\sigma-1}+\frac{1}{\sigma}+\omega\right)\bar{D}^2\right], \tag{C.5.26}$$

where J_0 is a positive definite matrix defined by $\bar{D}$, $\check{\mu} \equiv \hat{\mu} + \mu(1-\hat{\mu})$, which is loosely interpreted as the aggregate expenditure share of M-sector goods, and $\omega \equiv \frac{\mu(1-\hat{\mu})}{\sigma(\sigma-1)}(1-\bar{v})$ is a constant that summarizes the effects of labor mobility between the A- and M-sectors at $\bar{h}$. Thus, $\nabla v(\bar{h}) \simeq c_1\bar{D} + c_2\bar{D}^2$ with

$$c_1 = \check{\mu}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right) > 0, \tag{C.5.27}$$

$$c_2 = -\left(\frac{\check{\mu}^2}{\sigma-1}+\frac{1}{\sigma}+\omega\right) < 0. \tag{C.5.28}$$

C.6 Tabuchi (1998) (Tb) model

The Tb model introduces the internal structure of regions to the Km model. The main thrust of this model is that unlike the majority of regional models, the city boundary in each region is endogenously determined by the full-fledged monocentric city model of Alonso–Muth–Mills. This produces a rich structure of urban costs, because the tradeoff between commuting costs and land rents is explicit.

In this model, we have the previous three sectors (M, H, and A). The internal structure of each region is featureless, except that it is endowed with a single central business district (CBD) with negligible spatial extent. In each region, locations are indexed by the distance from the CBD, $x \geq 0$. At any point, the land endowment density is assumed to be unity. The total numbers of skilled and unskilled workers are given by H and L, respectively. The number of skilled workers in region i is denoted by h_i, whereas the spatial distribution (density) in that region is, allowing notational abuse, denoted by $h_i(x)$. Thus, we have

$$\int_0^{\bar{x}_i} h_i(x)\mathrm{d}x = h_i, \tag{C.6.1}$$

where $\bar{x}_i \geq 0$ is the city boundary in region i that is endogenously determined. Unskilled workers are employed by the A-sector and do not commute to the CBD, whereas skilled workers do. A skilled worker at distance x from the CBD incurs the generalized cost of commuting $T(x)$, which is measured by the numéraire.

Preference. The utility of a representative worker living in region i and located at x is given by

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{H}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + \gamma \ln C_i^{\mathrm{H}} + (1-\mu-\gamma)\ln C_i^{\mathrm{A}}, \tag{C.6.2}$$

where μ and γ with $\mu + \gamma < 1$ are the constant expenditure shares for M-sector goods and H-sector goods, respectively; C_i^{M} is the CES aggregate of M-sector goods defined by (C.1.2), C_i^{H} the consumption of housing space (H-sector goods), C_i^{A} the consumption of agricultural products (A-sector goods) in region i. M-sector goods are subject to iceberg transport costs, whereas those of the A-sector are not for both intra- and interregional transportation. H-sector goods are local and non-tradable. By

choosing A-sector goods as the numéraire, the budget constraint of a skilled worker at location x in region i is

$$C_i^A + r_i(x)C_i^H(x) + \sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi)q_{ji}(\xi)\mathrm{d}\xi + T_i(x) = y_i, \quad \text{(C.6.3)}$$

where $r_i(x)$ is the land rent prevailing at location x in region i, $T(x)$ the generalized cost of commuting from location x to the CBD, and y_i the income of the worker. We assume that $T(x)$ is differentiable and increasing in x with $T(0) = 0$. Note that $T(x)$ is independent of its population and is homogeneous among the regions. Given the price including the land rent profile $\{r_i(x)\}$, utility maximization yields

$$C_i^M(x) = \mu \frac{y_i(x)}{P_i}, \ C_i^H(x) = \gamma \frac{y_i(x)}{r_i(x)}, \ C_i^A(x) = (1 - \mu - \gamma)y_i(x), \ q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}}C_i^M, \quad \text{(C.6.4)}$$

where $y_i(x) = y_i - T(x)$ is the net income of a worker residing at x in region i. Following the tradition of urban economics, the model assumes absentee landowners who keep the rental revenue of housing, leading to $y_i = w_i$ for every skilled worker. Unskilled workers live outside the city and do not commute to the CBD. Thus, they face the agricultural land rent $r^A > 0$ and zero commuting cost as well as $y_i = 1$. For simplicity, we assume that r^A is the same across the regions. We also assume that the intracity transportation of M-sector goods is costless, meaning that unskilled and skilled workers face the same M-sector product price.

Internal structure of each region. As discussed, the difference compared with the Km model is that the internal structure of each region is now explicitly modeled by a monocentric city model. The standard first-order condition for the equilibrium spatial pattern is that

$$C_i^H(x)\frac{\mathrm{d}r_i(x)}{\mathrm{d}x} + \frac{\mathrm{d}T(x)}{\mathrm{d}x} = 0 \quad \text{(C.6.5)}$$

for $0 \le x \le \bar{x}$ with the boundary condition being $r_i(\bar{x}) = r^A$. In the following, we focus on a single region given the fixed values of w_i and h_i. For simplicity, we omit index i unless otherwise noted. By combining $C_i^H(x)$ in (C.6.4), we obtain the land rent profile $r(x)$ given w:

$$r(x) = \hat{r}\{1 - T(x)/w\}^{1/\gamma} \quad \text{(C.6.6)}$$

with $r(\bar{x}) = r^A$ at the city boundary $\bar{x}$ of the region. Thus, $\hat{r}$, the land rent at the CBD ($x = 0$) when the city boundary is at $\bar{x}$ and wage rate is w, is determined as

$$\hat{r}(\bar{x}, w) = \frac{r^A}{\{1 - T(\bar{x})/w\}^{1/\gamma}}. \quad \text{(C.6.7)}$$

We observe that $\hat{r} = r^A$ when $h_i = 0$ because $\bar{x} = 0$ and $T(0) = 0$. With notational abuse, the population

density function $h(x)$ in the region for the given $\bar{x}$ and w becomes

$$h(x) = \frac{a(x)}{C^{\mathrm{H}}(x)} = \frac{a(x)r(x)}{\gamma y(x)} = \frac{\hat{r}(\bar{x}, w)}{\gamma w}a(x)\{1 - T(x)/w\}^{1/\gamma - 1}, \tag{C.6.8}$$

where $a(x)$ is the land endowment at distance x. We here note that as $r(\bar{x}) = r^{\mathrm{A}}$,

$$h(\bar{x}) = \frac{a(\bar{x})r^{\mathrm{A}}}{\gamma(w - T(\bar{x}))}. \tag{C.6.9}$$

In Tabuchi (1998), it is assumed that $a(x) = 2\pi x$, meaning that the city is disk-shaped.

Comparative statistics for the internal structure of a region. Before studying the stability of the flat-earth equilibrium at the regional scale, we first investigate how changes in h_i and w_i affect the internal structure of a region. We note that the population density function $h(x)$ satisfies

$$\frac{\partial h(x)}{\partial \bar{x}} = \frac{T'(\bar{x})}{\gamma(w - T(\bar{x}))}h(x) > 0 \tag{C.6.10}$$

$$\frac{\partial h(x)}{\partial w} = \left(\frac{1 - \gamma}{\gamma(w - T(x))} - \frac{1}{\gamma(w - T(\bar{x}))} \right) h(x) < 0 \tag{C.6.11}$$

provided that $w - T(\bar{x}) > 0$, which must be the case because otherwise the utility of an agent at $\bar{x}$ becomes negative infinity. The latter inequality states that, as is standard in the literature, population density decreases as income increases. Define the function $H(\bar{x}, w)$ that returns the population in the interval $[0, \bar{x})$ by

$$H(\bar{x}, w) = \int_0^{\bar{x}} h(x)\mathrm{d}x. \tag{C.6.12}$$

Then, the location of the city boundary $\bar{x}$ for the given h and w is determined by the equation

$$h = H(\bar{x}, w), \tag{C.6.13}$$

where $\bar{x}$ becomes a function of h and w. For later use, we investigate the effects of h and w on $\bar{x}$. By applying the implicit function theorem to the equation $H(\bar{x}, w) - h = 0$, we have

$$\frac{\partial \bar{x}}{\partial h} = \frac{1}{h(\bar{x})} \quad \text{and} \quad \frac{\partial \bar{x}}{\partial w} = -\frac{1}{h(\bar{x})}\frac{\partial H}{\partial w}, \tag{C.6.14}$$

where we assume that w is determined in the region-scale trade balance, meaning that w and h are the independent variables. Note that $\partial H / \partial \bar{x} = h(\bar{x})$. From (C.6.11), we have

$$\frac{\partial H}{\partial w} = \int_0^{\bar{x}} \frac{\partial h(x)}{\partial w}\mathrm{d}x < 0, \tag{C.6.15}$$

which suggests that the population in the interval $[0, \bar{x})$ decreases when income increases. Then, from (C.6.14), we conclude that (i) the city boundary $\bar{x}$ is increasing in w_i and (ii) $\bar{x}$ is increasing in

h_i. Thus, we see that $\bar{x}$ is increasing in both h_i and w_i, which is standard. In addition, define the total (generalized) costs incurred by commuting in the region by

$$T_i = \int_0^{\bar{x}} T(x)h(x)\mathrm{d}x. \tag{C.6.16}$$

Then, we can show that

$$\frac{\partial T_i}{\partial h} = T(\bar{x}) + \frac{v}{h(\bar{x})}T_i > 0, \quad \frac{\partial T_i}{\partial w} = -\frac{\partial T_i}{\partial h}\frac{\partial H}{\partial w} > 0, \tag{C.6.17}$$

where v is the elasticity of land rent at the city boundary $\bar{x}$:

$$v \equiv -\frac{r'(\bar{x})}{r(\bar{x})} = \frac{T'(\bar{x})}{\gamma(w - T(\bar{x}))}. \tag{C.6.18}$$

Thus, the total commuting cost increases in both h_i and w_i ceteris paribus, which is also standard.

Short-run equilibrium. Consider the regional scale and recover the region indices. Given $\bar{x}_i$, total expenditure in region i net of commuting costs is given by $Y_i = w_i h_i - T_i + l_i$. The wage equation for the model is given by

$$w_i h_i = \mu \sum_{j \in \mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} h_k w_k^{1-\sigma} d_{kj}}(w_j h_j - T_j + l_j). \tag{C.6.19}$$

We impose the following constraint on w for normalization purposes:

$$\sum_{i \in \mathcal{K}}(w_i h_i - T_i) = \frac{\mu}{1-\mu}L, \tag{C.6.20}$$

where T_i depends on both h_i and w_i. Given the short-run wage, indirect utility for region i is obtained by evaluating it at the CBD ($x = 0$) since utility is equalized in each region:

$$v_i(h) = \bar{\kappa}\ln[\Delta_i] + \ln[y_i(\bar{x}_i)] \tag{C.6.21}$$

where $\Delta_i = \sum_{j \in \mathcal{K}} h_j w_j^{1-\sigma} d_{ji}$ and $y_i(\bar{x}_i) = w_i - T(\bar{x}_i)$.

Jacobian matrix at the flat-earth equilibrium. We compute as follows:

$$\nabla v(h) = \bar{\kappa}M^\top \mathrm{diag}[h]^{-1} - \mu M^\top \nabla w(h)\mathrm{diag}[w]^{-1} + \mathrm{diag}[y_i(\bar{x}_i)]^{-1}\nabla[y_i(\bar{x}_i)] \tag{C.6.22}$$

with M defined in line with the Km model and $\nabla[y_i(\bar{x}_i)] = \nabla w(h) - \nabla\mathrm{diag}[T(\bar{x}_i)]$, where we note that

$$\nabla \mathrm{diag}[T(\bar{x}_i)] = \mathrm{diag}[T'(\bar{x}_i)]\nabla[\bar{x}_i(h_i, w_i)] = \mathrm{diag}[T'(\bar{x}_i)]\big\{\mathrm{diag}[\partial\bar{x}_i/\partial h_i] + \mathrm{diag}[\partial\bar{x}_i/\partial w_i]\nabla w(h)\big\}.$$

Thus, by letting $\Psi_0 \equiv \mathrm{diag}[T'(\bar{x}_i)\partial\bar{x}_i/\partial h_i]$ and $\Psi_1 \equiv \mathrm{diag}[T'(\bar{x}_i)\partial\bar{x}_i/\partial w_i]$, we have

$$\nabla[y_i(\bar{x}_i)] = \nabla w(h) - (\Psi_0 + \Psi_1 \nabla w(h)) = -\Psi_0 + (I - \Psi_1)\nabla w(h). \tag{C.6.23}$$

As in the Km model, $\nabla_{\boldsymbol{w}} = [\partial/\partial w_i]$. For $\nabla \boldsymbol{w}(\boldsymbol{h})$, we have $\nabla \boldsymbol{w}(\boldsymbol{h}) = -(\nabla_{\boldsymbol{w}} \boldsymbol{W})^{-1}(\nabla \boldsymbol{W})$ with

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} = \text{diag}[\boldsymbol{h}] + \mu(\sigma - 1)(\text{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\,\text{diag}[\boldsymbol{Y}]\boldsymbol{M}^{\top})\,\text{diag}[\boldsymbol{w}]^{-1} - \mu\boldsymbol{M}\nabla_{\boldsymbol{w}}\boldsymbol{Y}, \quad \text{(C.6.24)}$$

$$\nabla \boldsymbol{W} = \text{diag}[\boldsymbol{w}] - \mu(\text{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}^{\top})\,\text{diag}[\boldsymbol{h}]^{-1} - \mu\boldsymbol{M}\nabla\boldsymbol{Y} \quad \text{(C.6.25)}$$

where $\boldsymbol{Y} = [Y_i] = [w_i h_i - T_i + l_i]$, $\nabla_{\boldsymbol{w}}\boldsymbol{Y} = \text{diag}[\boldsymbol{h}] - \nabla_{\boldsymbol{w}}\boldsymbol{T}$, and $\nabla\boldsymbol{Y} = \text{diag}[\boldsymbol{w}] - \nabla\boldsymbol{T}$.

Consider the flat-earth equilibrium in a symmetric racetrack economy with $l_i = l$. Let $\bar{w}$ and $\bar{T}$ be the uniform level of the nominal wage rate and total commuting cost in each region. Note that $\bar{T}$ is a function of $\bar{w}$ and $\bar{x}$. Given the commuting cost function $T(x)$ and location of the city boundary and wage $(\bar{x}, \bar{w})$, at the flat-earth equilibrium, we require

$$\bar{w}h - \bar{T}(\bar{x}, \bar{w}) = \frac{\mu}{1 - \mu}l \quad \text{(C.6.26)}$$

so that wages are normalized. Then, we can show that there exists a unique positive solution $(\bar{x}^*, \bar{w}^*)$ such that $\bar{w}^* - T(\bar{x}^*) > 0$ for the system of non-linear equations defined by (C.6.13) and (C.6.26) for the given h. By employing the solution $(\bar{x}^*, \bar{w}^*)$, total income in Y is given by $\bar{Y} = l/(1 - \mu)$. Define the ratios ϕ of the regional disposable income of skilled workers and $\hat{\phi}$ of regional total expenditure to the total nominal wage:

$$\phi \equiv \frac{\bar{w}h - \bar{T}}{\bar{w}h}, \quad \hat{\phi} \equiv \frac{\bar{Y}}{\bar{w}h}. \quad \text{(C.6.27)}$$

The latter implies that $\bar{Y}/\bar{w} = \hat{\phi}h$ and $\bar{Y}/h = \hat{\phi}\bar{w}$. Given $(\bar{x}^*, w^*)$, we define the $T(x)$-dependent positive constants ψ_0, ψ_1, ρ_0, and ρ_1 such that $\boldsymbol{\Psi}_0 = \psi_0\boldsymbol{I}$, $\boldsymbol{\Psi}_1 = \psi_1\boldsymbol{I}$, $\nabla\boldsymbol{Y} = \rho_0\bar{w}\boldsymbol{I}$, and $\nabla_{\boldsymbol{w}}\boldsymbol{Y} = \rho_1 h\boldsymbol{I}$. Then, we can calculate the Jacobian matrix of the payoff function at the flat-earth equilibrium as follows:

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1}\bar{\kappa}\bar{\boldsymbol{D}} - \bar{w}^{-1}\mu\bar{\boldsymbol{D}}\nabla\boldsymbol{w}(\bar{\boldsymbol{h}}) - \bar{y}^{-1}\psi_0\boldsymbol{I} + \bar{y}^{-1}(1 - \psi_1)\nabla\boldsymbol{w}(\bar{\boldsymbol{h}}), \quad \text{(C.6.28)}$$

$$= h^{-1}\bar{\kappa}\bar{\boldsymbol{D}} - \bar{y}^{-1}\psi_0\boldsymbol{I} + \{\bar{y}^{-1}(1 - \psi_1)\boldsymbol{I} - \bar{w}^{-1}\mu\bar{\boldsymbol{D}}\}\nabla\boldsymbol{w}(\bar{\boldsymbol{h}}), \quad \text{(C.6.29)}$$

where $\bar{y} \equiv y(\bar{x}^*) = \bar{w}^* - T(\bar{x}^*)$ is the net wage at $\bar{x}^*$ and $\nabla\boldsymbol{w}(\bar{\boldsymbol{h}}) = -(\nabla_{\boldsymbol{w}}\boldsymbol{W})^{-1}(\nabla\boldsymbol{W})$ with

$$\nabla \boldsymbol{W} = -\bar{w}\big[- (1 - \hat{\phi}\mu)\boldsymbol{I} + \rho_0\mu\bar{\boldsymbol{D}} - \hat{\phi}\mu\bar{\boldsymbol{D}}^2\big], , \quad \text{(C.6.30)}$$

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} = h\big[\{\hat{\phi}\mu(\sigma - 1) + 1\}\boldsymbol{I} - \rho_1\mu\bar{\boldsymbol{D}} - \hat{\phi}\mu(\sigma - 1)\bar{\boldsymbol{D}}^2\big]. \quad \text{(C.6.31)}$$

Illustration. Following Tabuchi (1998), we investigate the simplest case where the commuting cost function is linear with respect to distance: $T(x) = tx$. We also simplify the analysis by assuming that the internal structure of each region is one-dimensional and extends symmetrically around the CBD over the interval $[-\bar{x}, \bar{x}]$ á la Murata and Thisse (2005). Although this change strengthens the role of urban costs in each region, it does not affect the intrinsic properties of the model. For this case, By

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letting $a(x) = 1$, we obtain

$$\bar{x} = \frac{1}{t}(1 - \epsilon^\gamma)\bar{w}, \tag{C.6.32}$$

where the non-dimensional constant $\epsilon \in (0, 1)$ is defined by $\epsilon \equiv (1 + \hat{t}h)^{-1}$. The parameter $\hat{t} \equiv (t/2)/r^A$ is interpreted as a measure of the relative magnitude of commuting costs to land rents. As expected, $\bar{x}$ is decreasing in the generalized commuting cost per distance t. Then, solving (C.6.26) implies that

$$\bar{w} = \frac{1}{\phi} \cdot \frac{\mu}{1 - \mu} \cdot \frac{L}{H} \quad \text{and} \quad \phi = \frac{1}{1 + \gamma} \cdot \frac{1 - \epsilon^{1+\gamma}}{1 - \epsilon} \tag{C.6.33}$$

as well as $\bar{y} = \epsilon^\gamma \bar{w}$, $\bar{Y} = l/(1 - \mu)$, and $\hat{\phi} = \phi/\mu$. Then, we also have

$$\psi_0 = T'(\bar{x})\frac{\partial \bar{x}}{\partial h} = h^{-1}\bar{y}\gamma(1 - \epsilon), \quad \psi_1 = T'(\bar{x})\frac{\partial \bar{x}}{\partial w} = 1 - \epsilon^\gamma, \tag{C.6.34}$$

$$\rho_0 = \frac{1}{\bar{w}}\frac{\partial Y_i}{\partial h_i} = \frac{1}{\bar{w}}\left(w_i - \frac{\partial T_i}{\partial h_i}\right) = 1 - \gamma(1 - \epsilon)\phi, \quad \rho_1 = \frac{1}{h}\frac{\partial Y_i}{\partial w_i} = \frac{1}{h}\left(h_i - \frac{\partial T_i}{\partial w_i}\right) = \phi. \tag{C.6.35}$$

Summarizing computations up to here yields the analytical expression of $\nabla v(\bar{h})$ as follows:

$$\nabla v(\bar{h}) = h^{-1}\bar{\kappa}\bar{D} + (I - \mu\bar{D})\bar{w}^{-1}\nabla w(\bar{h}) - h^{-1}\hat{\gamma}I, \tag{C.6.36}$$

$$\nabla w(\bar{h}) = \bar{w}h^{-1}\left[\hat{c}_0 I + \hat{c}_1 \bar{D} + \hat{c}_2 \bar{D}^2\right]^{-1}\left[\bar{c}_0 I + \bar{c}_1 \bar{D} + \bar{c}_2 \bar{D}^2\right] \tag{C.6.37}$$

with the coefficients being

$$\begin{cases} \hat{c}_0 \equiv 1 + (\sigma - 1)\phi > 0, \\ \hat{c}_1 \equiv -\mu\phi < 0, \\ \hat{c}_2 \equiv -(\sigma - 1)\phi < 0, \end{cases} \quad \begin{cases} \bar{c}_0 \equiv -(1 - \phi) < 0, \\ \bar{c}_1 \equiv \mu(1 - \hat{\gamma}\phi) > 0, \\ \bar{c}_2 \equiv -\phi < 0 \end{cases} \tag{C.6.38}$$

where $\hat{\gamma} \equiv \gamma(1 - \epsilon)$. Note that ϕ and $\hat{\gamma}$ together summarize the net effects of the two types of urban costs; ϕ and $\hat{\gamma}$ represent those from commuting and non-tradable land, respectively. As a consequence, we have $\nabla v(\bar{h}) \simeq c_0 I + c_1 \bar{D} + c_2 \bar{D}^2$ with

$$c_0 = -\hat{\gamma}\left(\frac{1}{\sigma} + \frac{\sigma - 1}{\sigma}\phi\right) < 0, \tag{C.6.39}$$

$$c_1 = \mu\left(\frac{1}{\sigma - 1} + \frac{1}{\sigma}\right) > 0, \tag{C.6.40}$$

$$c_2 = -\left[\underbrace{\frac{\mu^2}{\sigma - 1}\left(\frac{\phi}{\sigma} + \frac{\sigma - 1}{\sigma}(1 - \hat{\gamma}\phi)\right)}_{=:\omega_0} + \underbrace{\frac{1}{\sigma}\phi - \hat{\gamma}\phi\frac{\sigma - 1}{\sigma}}_{=:\omega_1/\sigma}\right] \equiv -\left(\frac{\mu^2}{\sigma - 1}\omega_0 + \frac{1}{\sigma}\omega_1\right). \tag{C.6.41}$$

Remark C.5. Observe that if $\hat{t}$ and γ are both infinitesimally small so that there are virtually no urban costs, we have $\hat{\gamma} = \gamma(1 - \epsilon) \approx 0(1 - 1) = 0$ and $\phi \approx (1 + \gamma)^{-1} \approx 1$. Then, the coefficients c_0, c_1, and c_2

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reduce to those of the Km model, which is intuitive. We note that for general cases, the sign of c_2 is ambiguous. In particular, if γ is large relative to μ and in addition $1 - \epsilon$ is small ($\hat{t}$ or h is small), c_2 can be positive. This is because while housing is important relative to manufactured goods, commuting costs are quite low; this implies that a concentration of skilled workers is beneficial despite higher market competition on the side of firms.

C.7 Pflüger and Südekum (2008) (PS) model

The PS model builds on Pflüger (2004), with the only difference being that it introduces the housing sector (again denoted by H), which produces a local dispersion force.

Preference. The homogeneous preference of skilled workers is given by the following quasilinear form with respect to the A-sector good (numéraire):

$$U(C_i^M, C_i^H, C_i^A) = \mu \ln C_i^A + \gamma \ln C_i^H + C_i^A, \tag{C.7.1}$$

where C_i^M, C_i^H, and C_i^A are again the consumption of manufacturing aggregates, C_i^H housing goods, and C_i^A agricultural goods, respectively. Then, the indirect utility of a skilled worker in region i is obtained as

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] - \gamma(\ln[h_i + l_i] - \ln A_i) + w_i, \tag{C.7.2}$$

where $\Delta_i = \sum_{j \in \mathcal{K}} d_{ji} h_j$, and l_i and A_i denote the number of unskilled workers and amount of housing stock in region i, respectively. The nominal wage in region i is given by

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\Delta_j}(h_j + l_j) \tag{C.7.3}$$

as in the Pf model.

Jacobian matrix. At the flat-earth equilibrium with $l_i = l$ and $A_i = A$ for all i, we can show

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1} \left[-\gamma(1 + \epsilon)^{-1} \boldsymbol{I} + (\bar{\kappa} + \kappa)\bar{\boldsymbol{D}} - \kappa(1 + \epsilon)\bar{\boldsymbol{D}}^2 \right], \tag{C.7.4}$$

where $\epsilon \equiv L/H$ is the ratio of the total number of unskilled workers to that of skilled workers. We thus conclude that $c_0 = -\gamma(1 + \epsilon)^{-1} < 0$, $c_1 = \bar{\kappa} + \kappa > 0$, and $c_2 = -\kappa(1 + \epsilon) < 0$.

Numerical simulation. **Figure 11** and **Figure 12** assume Pflüger and Südekum (2008)'s model. The parameters are set to $\mu = 0.4$, $\sigma = 2.5$, $L = 4$, $H = 1$, $\gamma = 0.5$, and $A = 1$.

C.8 Murata and Thisse (2005) (MT) model

Similar to the Tb model, Murata and Thisse (2005) studied the interplay between commuting costs and interregional transport costs by employing a simplified yet reasonable specification. The internal structure of each region is assumed to be one-dimensional and featureless except that there is a given

CBD; the city expands symmetrically around the origin. There are only skilled and mobile workers, who choose their own residential region i and location $x \geq 0$ in that region, where the CBD is located at $x = 0$. The total number of skilled workers is fixed and assumed to be H.

The internal structure of a region. Land endowment equals unity everywhere in a region and workers are assumed to inelastically consume one unit of land. The opportunity cost of land is normalized to zero in every region. Then, the city spreads in the interval $\mathcal{X}_i \equiv [-\bar{x}_i, \bar{x}_i]$, where $\bar{x}_i \equiv h_i/2$ denotes the city boundary. Commuting costs take an iceberg form. Specifically, a worker located at x supplies

$$s(x) = 1 - 4\theta|x| \qquad\qquad x \in \mathcal{X}_i \tag{C.8.1}$$

unit of labor, where we require $\theta \in [0, 1/(2H))$ so that we have $s(x) \geq 0$ for all $x \in \mathcal{X}$ and for all region i at any configuration. Then, total effective labor supply in the CBD of region i is given by

$$S_i = \int_{\mathcal{X}_i} s(x)\mathrm{d}x = h_i(1 - \theta h_i). \tag{C.8.2}$$

Note that $S_i = h_i$ when commuting is costless: $\theta = 0$. Letting $r_i(x)$ be the land rent profile, at the equilibrium, this must satisfy

$$s(x)w_i - r_i(x) = \bar{w}_i, \qquad\qquad \forall x \in \mathcal{X}_i, \tag{C.8.3}$$

where $\bar{w}_i \equiv s(\bar{x}_i)w_i - r_i(\bar{x}_i) = s(\bar{x}_i)w_i = s(-\bar{x}_i)w_i = (1 - 2\theta h_i)w_i$ is the disposable wage level of a worker located at the boundary of the city. We thus have

$$r_i(x) = 2\theta(h_i - 2|x|)w_i, \qquad\qquad \forall x \in \mathcal{X}_i, \tag{C.8.4}$$

which means that the aggregate land rent in region i is

$$R_i \equiv \int_{\mathcal{X}_i} r_i(x)\mathrm{d}x = \theta w_i h_i^2. \tag{C.8.5}$$

Land is locally owned, and thus the income of a worker in region i and any location x is

$$y_i = s(x)w_i - r_i(x) + \frac{R_i}{h_i} = \bar{w}_i + \theta w_i h_i = (1 - \theta h_i)w_i. \tag{C.8.6}$$

Preference. The homogeneous preference of skilled workers in region i is given by

$$U(C_i^{\mathrm{M}}) = \ln C_i^{\mathrm{M}}, \tag{C.8.7}$$

where, as usual, C_i^{M} is the consumption of the CES aggregate defined by (C.1.2). The budget constraint of a mobile worker becomes

$$\sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi)q_{ji}(\xi)\mathrm{d}\xi = y_i, \tag{C.8.8}$$

where y_i denotes the income of the worker. It is immediately obvious that given y_i, utility maximization yields

$$C_i^{\mathrm{M}} = \frac{y_i}{P_i}, \quad q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}} C_i^{\mathrm{M}}, \tag{C.8.9}$$

where P_i is the price index in region i.

Firms. Manufacturing firms are assumed to be the same as in the Km model. Specifically, to produce x_i units of a good, a firm requires $\alpha + \beta x_i$ units of skilled labor. Thus, the cost function faced by a firm in region i is given by $C_i(x_i) = w_i(\alpha + \beta x_i)$. Profit maximization yields $p_{ij}(\xi)$ as in the Km model (C.1.10), which does not depend on ξ. Noting that the number of firms n_i in region i is given by $n_i = S_i/(\alpha + \beta x_i^*) = (\alpha \sigma)^{-1} S_i$, the price index in region i is given as

$$P_i = \frac{\beta \sigma}{\sigma - 1} \left(\frac{1}{\alpha \sigma} \sum_{j \in \mathcal{K}} S_j w_j^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)} \tag{C.8.10}$$

with $d_{ij} = \tau_{ij}^{1-\sigma}$ and $S_i = (1 - \theta h_i)h_i$.

Short-run equilibrium. Noting that aggregate income in region i is given by $Y_i = w_i S_i$, the wage equation for the MT model becomes

$$w_i S_i = \sum_{j \in \mathcal{K}} \frac{S_i w_i^{1-\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} S_k w_k^{1-\sigma} d_{kj}} w_j S_j. \tag{C.8.11}$$

To normalize w, we assume $\sum_{i \in \mathcal{K}} w_i S_i = W > 0$. Given the solution w to the equation, the indirect utility of workers in region i is obtained as

$$v_i(h) = \bar{\kappa} \ln[\Delta_i] + \ln[w_i] + \ln[1 - \theta h_i], \tag{C.8.12}$$

where $\bar{\kappa} = 1/(\sigma - 1)$ and $\Delta_i \equiv \sum_{k \in \mathcal{K}} h_i(1 - \theta h_i)w_k^{1-\sigma} d_{ki}$.

Jacobian matrix at the flat-earth equilibrium. We compute as follows:

$$\nabla v(h) = \bar{\kappa} M^{\top} \mathrm{diag}[S]^{-1} \mathrm{diag}[1 - 2\theta h_i] + (I - M) \mathrm{diag}[w]^{-1} \nabla w(h) - \theta \mathrm{diag}[1 - \theta h_i]^{-1} \tag{C.8.13}$$

where $\nabla w(h) = -(\nabla_w W)^{-1}(\nabla W)$ with

$$\nabla_w W = \mathrm{diag}[S] + (\sigma - 1)(\mathrm{diag}[MY] - M \mathrm{diag}[Y]M^{\top}) \mathrm{diag}[w]^{-1} - M \mathrm{diag}[S] \tag{C.8.14}$$

$$\nabla W = \left[\mathrm{diag}[w] - (\mathrm{diag}[MY] - MYM^{\top}) \mathrm{diag}[S]^{-1} - M \mathrm{diag}[w] \right] \mathrm{diag}[1 - 2\theta h_i] \tag{C.8.15}$$

with $Y = [Y_i] = [w_i(1 - \theta h_i)h_i]$ and $S = [S_i] = [(1 - \theta h_i)h_i]$. Note that $Y_i = w_i S_i$. Assume a symmetric racetrack economy. We have

$$\nabla_w W = (1 - \theta h)h \left[\sigma I + (\sigma - 1)\bar{D} \right] \left[I - \bar{D} \right], \tag{C.8.16}$$

$$\nabla W = -\bar{w}(1 - 2\theta h)\bar{D} \left[I - \bar{D} \right], \tag{C.8.17}$$

which in turn yields

$$\nabla v(\boldsymbol{h}) = \frac{1 - 2\theta h}{(1 - \theta h)h}\left(\left[\sigma\boldsymbol{I} + (\sigma - 1)\bar{\boldsymbol{D}}\right]^{-1}\left(\frac{1}{\sigma - 1} + \frac{1}{\sigma}\right)\bar{\boldsymbol{D}} - \frac{\theta h}{1 - 2\theta h}\boldsymbol{I}\right). \tag{C.8.18}$$

As a consequence, we obtain $\nabla v(\bar{\boldsymbol{h}}) \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}}$, where, with $\hat{\theta} \equiv (\theta h)/(1 - 2\theta h)$,

$$c_0 = -\hat{\theta}, \quad c_1 = (1 - \hat{\theta})\left(\frac{1}{\sigma - 1} + \frac{1}{\sigma}\right) - \hat{\theta}\frac{\sigma - 1}{\sigma}. \tag{C.8.19}$$

Remark C.6. We must require that $0 \leq \hat{\theta} < 1/\{2(K - 1)\} < 1$ to ensure that S_i is positive for all region i. In particular, when $H = 1$ and $K = 2$, meaning that $h = 1/2$ as in the original study, we have $\hat{\theta} = \theta/\{2(1 - \theta)\}$ and $\hat{\theta} \in (0, 1/2)$. Moreover, by letting $\gamma \equiv \hat{\theta}$ and $\mu \equiv 1 - \hat{\theta}$, the model is isomorphic to Helpman (1998)'s model with LL, albeit there is a restriction on γ.

C.9 Harris and Wilson (1978) (HW) model

The HW model is an archetypal economic geography model formulated in the field of geography well before mainstream economists started to emphasize the self-organization of the spatial allocation of economic activity. The model has fruitful applications in urban planning. A detailed analysis of the model can be found in Osawa et al. (2017). The model can also be interpreted as a spatial competition model with discrete locations but a continuum of firms.

Assumptions. We consider a city discretized into *K zones* and associated centroids. There is a continuum of retailing firms in each zone that operate a shop. The number of firms in zone i is denoted by $h_i \geq 0$; $\boldsymbol{h}$ denotes the spatial distribution of retailers. A fixed proportion of consumers resides in each zone. Consumers are assumed to inelastically buy retail goods from some shop located in the city. Total per capita consumer demand for a shopping activity in zone i is a constant O_i. Consumers' shopping behavior is captured by a set of origin-constrained gravity equations. For any given $\boldsymbol{h}$, consumer demand $S_{ij}(\boldsymbol{h})$ from zone i to j, measured as cash flow, is given by

$$S_{ij}(\boldsymbol{h}) = \frac{h_j^\alpha \exp[-\beta t_{ij}]}{\sum_{k \in \mathcal{K}} h_k^\alpha \exp[-\beta t_{ik}]}O_i, \tag{C.9.1}$$

where t_{ij} is the travel cost from zone i to j. The parameters $\alpha, \beta > 0$ are exogenous constants. The term h_i^α is the attractiveness of the retailers in the zone i, where α determines the economies of scale. We assume $\alpha > 1$ and hence there is increasing returns to scale. β dictates how fast demand decreases with the travel cost t_{ij} (respecting the original formulation, this section uses β instead of τ). Note that one may recast the demand function into the context of spatial competition by interpreting α^{-1} as the magnitude of product differentiation.

Payoff. The payoff (profit) of a retailer in zone i is defined as follows:

$$\Pi_i(\boldsymbol{h}) = \frac{\sum_{j \in \mathcal{K}} S_{ji}(\boldsymbol{h})}{h_i} - \kappa_i, \tag{C.9.2}$$

where κ_i is the fixed cost of entry. Assume that $O_i = 1$ and that $\kappa_i = \kappa$ for all i. Then, we have

$$\Pi(h) = M^\top - \kappa \mathbf{1} \ labeleq : Pi_vec \tag{C.9.3}$$

where $M \equiv \mathrm{diag}[D \, \mathrm{diag}[h]^\alpha \mathbf{1}]^{-1} D \, \mathrm{diag}[h]^{\alpha-1}$ with $d_{ij} \equiv \exp[-\beta t_{ij}]$.

Long-run equilibrium. The HW model is an open-city model. The total number of retailers at an equilibrium is thus determined from the following equilibrium condition: $h_i \Pi_i(h) = 0$, $h_i \geq 0$, $\Pi_i(h) \leq 0$. However, at any equilibrium, we have $\sum_{i \in \mathcal{K}} \kappa_i h_i = \sum_{i \in \mathcal{K}} O_i$; the set $\mathcal{D} \equiv \{h \in \mathbb{R}^K \mid \sum_{i \in \mathcal{K}} \kappa_i h_i = \sum_{i \in \mathcal{K}} O_i, \ h_i \geq 0\}$ is globally attracting.

Dynamics. Harris and Wilson (1978) assumed that the spatial pattern h gradually evolves in proportion to the profit $\Pi(h)$ and the state h. Specifically, we define $\dot{h} = F(h) \equiv \mathrm{diag}[h] \cdot \Pi(h) = [S_i(h) - \kappa_i h_i]$

Jacobian matrix at the flat-earth equilibrium. It is immediately clear that $J = \nabla F(\bar{h})$ is given by

$$J = \kappa \left\{ (\alpha - 1)I - \alpha \bar{D}^2 \right\}, \tag{C.9.4}$$

where I is the identity matrix and $\bar{D} \equiv D/d$ with $d \equiv \sum_{j \in \mathcal{K}} d_{0,j}$. We see that $J \simeq c_0 I + c_2 \bar{D}^2$ with

$$c_0 = 1 - \frac{1}{\alpha}, \ c_2 = -1. \tag{C.9.5}$$

It is clear that c_0 reflects the magnitude of the *local* increasing return. c_0 is positive as long as $\alpha > 1$; $\alpha < 1$ yields that the flat-earth equilibrium is always stable. $c_2 = -1$ represents, analogous to the FO model, firms' competition over demand from immobile consumers.

C.10 Beckmann (1976) (Bm) model

We formulate a discrete-space version of Beckmann (1976)'s spatial model of social interactions. Since the original formulation of Beckmann (1976) uses a linear communication cost, we introduce suitable modifications. Yet, as long as every consumer communicates with all other consumers, our modification does not alter the intrinsic properties of agglomeration and dispersion. In particular, whether possible equilibria are unimodal or multimodal does not change. We also avoid unnecessary complication and stick to the simplest possible specification.

Assumptions. Consider a city discretized into K areas. Each area i is endowed with fixed amount A_i of housing stocks. Housing stocks are owned by absentee landlords. The city is endowed with H homogeneous consumers that can choose his or her residential location and consume land and composite goods. The income of consumers is a fixed constant Y, which is sufficiently large.

Preference. In addition to land and composite goods, every consumer draws social utility because of his or her communication with others. Specifically, everyone in area i draws the following social utility:

$$S_i(h) = \log[\Delta_i], \tag{C.10.1}$$

where $\Delta_i \equiv \sum_{j \in \mathcal{K}} d_{ij} h_j$ with $d_{ij} \equiv \exp[-\tau \ell_{ij}]$. Note that Δ_i is an exponential accessibility function à la Fujita and Ogawa (1982). Given the spatial distribution of consumers h, the utility of residing area i takes the following quasilinear form:

$$U_i(z_i, s_i; h) = z_i + \gamma \log[s_i] + S_i(h), \tag{C.10.2}$$

where z_i and s_i are the consumption of the composite and housing goods, respectively, and γ is an exogenous constant. We set the composite good to the numéraire and the budget constraint of a worker in area i is

$$Y = z_i + r_i s_i, \tag{C.10.3}$$

where utility maximization yields $s_i = A_i / h_i$, $r_i = \alpha h_i / A_i$, and $z_i = Y - \gamma$. Then, by assuming $A_i = 1$ in every area and removing the constants, indirect utility in area i is given by

$$v_i(h) = \log[\Delta_i] - \gamma \log[h_i]. \tag{C.10.4}$$

Jacobian matrix at the flat-earth equilibrium. Assuming a racetrack economy, it is immediately clear that the Jacobian matrix at the flat-earth equilibrium is given by

$$\nabla v(\bar{h}) = h^{-1} \big[-\gamma I + \bar{D} \big]. \tag{C.10.5}$$

We thus see that $c_0 = -\gamma$ and $c_1 = 1$ for the model. Without any location-fixed factors, Mossay and Picard (2011) and Blanchet et al. (2016) are essentially the same model as the one presented here.

C.11   Takayama and Akamatsu (2011) (TA) model

Takayama and Akamatsu (2011) is a reduced-form partial equilibrium model that introduces a spatial competition effect à la Harris and Wilson (1978) into the Bm model. Specifically, in essence, they introduced firms that sell goods at a fixed price to spatially immobile consumers. The consumers in the Bm model are now workers; each worker inelastically provides a single unit of labor.

Immobile consumers. In each area, l_i immobile consumers with $\sum_i l_i = L$ demand a single unit of goods produced by firms; immobile consumers are assumed to engage in jobs in other industries. Given the spatial distribution $n = (n_i)_{i \in \mathcal{K}}$ of firms, demand from area j to i is given by the following origin-constrained gravity equation:

$$q_{ji} = \frac{\hat{d}_{ji}}{\sum_{k \in \mathcal{K}} \hat{d}_{jk} n_k} l_j \tag{C.11.1}$$

with $\hat{d}_{ij} \equiv \exp[-\hat{\tau} \ell_{ij}]$, whose microfoundation can be found at a CES preference or alternatively some taste heterogeneity.

Firms. A manufacturing firm produces a single unit of a manufactured good at a fixed price μ,

using a single unit of the labor of mobile consumers. Thus, we must have $n_i = h_i$. The profit function of the firm at i is given by

$$\Pi_i(\boldsymbol{h}) = \mu \sum_{j \in \mathcal{K}} \frac{\hat{d}_{ji}}{\sum_{k \in \mathcal{K}} \hat{d}_{jk} h_k} l_j - w_i. \tag{C.11.2}$$

For simplicity, we force zero profit for firms and abstract from commuting between different areas. Then, the wage of a mobile worker in area i equals

$$w_i(\boldsymbol{h}) = \mu \sum_{j \in \mathcal{K}} \frac{\hat{d}_{ji}}{\sum_{k \in \mathcal{K}} \hat{d}_{jk} h_k} l_j, \tag{C.11.3}$$

meaning that the indirect utility of the worker becomes

$$v_i(\boldsymbol{h}) = w_i(\boldsymbol{h}) + \log[\Delta_i] - \gamma \log[h_i]. \tag{C.11.4}$$

Jacobian matrix at the flat-earth equilibrium. Let $l_i = L/K$ for all i and assume that $d_{ij} = \hat{d}_{ij}$ for all i and j (i.e., $\tau = \hat{\tau}$). Then, we compute as follows:

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1} \big[-\gamma \boldsymbol{I} + \bar{\boldsymbol{D}} - \mu \epsilon \bar{\boldsymbol{D}}^2 \big], \tag{C.11.5}$$

where $\epsilon \equiv L/H$. Hence, we see that $c_0 = -\gamma$, $c_1 = 1$, and $c_2 = -\mu\epsilon$.

C.12 Allen and Arkolakis (2014) (AA) model

The AA model is formulated as a perfectly competitive Armington (1969)-based framework with positive (production) and negative (congestion) reduced-form *local* agglomeration externalities. We introduce a discrete-space version of the AA model, instead of the continuous-space version of the original study, to fit our context.

Assumptions. A fixed number H of mobile consumers choose residents. We denote the spatial pattern of consumers by $\boldsymbol{h}$. In each region i, a unique differentiated variety of a good is produced following Armington (1969). Production is assumed to be perfectly competitive and labor is the only factor of production. Each mobile consumer inelastically supplies a single unit of labor. As usual, we do not consider the commuting of workers between two regions. We denote the wage of workers by $\boldsymbol{w}$. The transportation of goods between regions takes an iceberg form; firms in i must export $\tau_{ij} > 0$ units of the good to meet a single unit of demand in region j.

In each region, the total factor productivity (TFP) and amenity are directly affected by the number of inhabitants, h_i. These externalities are *local* in the sense that they do not depend on the distance between regions. The number of consumers in each region does not affect its TFP or amenity; it is exclusively enjoyed by the agents located in each region. As the analysis in the present section demonstrates, such an assumption turns out to be insufficient for endogenously producing the polycentricity of spatial agglomeration patterns.

Preference. The utility function of a consumer in region i is defined as the following CES function:

$$u_i(\{q_{ji}\}) = a_i \cdot \left(\sum_{j \in \mathcal{K}} q_{ji}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \tag{C.12.1}$$

where q_{ji} is the quantity of the good variety produced in region $j \in \mathcal{K}$ and consumed in region i. The constant $\sigma > 1$ is the elasticity of substitution between varieties, and $a_i(h_i)$ is the local amenity. The local amenity deteriorates as the population h_i in i increases; this is defined by the following power function that produces a congestion effect:

$$a_i(h_i) = \bar{a}_i h_i^{-\beta}, \tag{C.12.2}$$

where $\bar{a}_i > 0$, $\beta \geq 0$ is the exogenously given constants. In particular, $\bar{a}_i$ represents the *unobserved amenity* in region i. When $\beta = 0$, there is no congestion effect and the local amenity is the exogenous constant $\bar{a}_i$.

The income of consumers comes only from the wage from production firms. We denote the price of the variety produced in j and consumed in i as p_{ji}. The wage in region i is denoted by $w_i \geq 0$. Then, the budget constraint of a consumer in i is given by the following equation:

$$w_i = \sum_{j \in \mathcal{K}} p_{ji} q_{ji}. \tag{C.12.3}$$

To normalize the wage, we impose a constraint $\sum_{i \in \mathcal{K}} w_i h_i = W$, which means that total income in the economy always equals the fixed constant W.

The utility maximization of consumers under a given price system p yields

$$q_{ji} = \frac{p_{ji}^{-\sigma}}{P_i^{1-\sigma}} w_i, \tag{C.12.4}$$

where P_i is the price index of the good in region i:

$$P_i \equiv \left(\sum_{k \in \mathcal{K}} p_{ki}^{1-\sigma} \right)^{1/(1-\sigma)}. \tag{C.12.5}$$

Production. Firms in region $i \in \mathcal{K}$ produce goods under perfect competition. As a result, the final price of the good produced in i and sold in j, which we denote by p_{ij}, equals

$$p_{ij} = \frac{w_i}{m_i} \tau_{ij}, \tag{C.12.6}$$

where m_i denotes the TFP in region i. To model a Marshallian agglomeration economy (Marshall, 1989) in a reduced form, the TFP in region i is assumed to be an increasing power function of its

population:

$$m_i(h_i) = \bar{m}_i h_i^{\alpha} \tag{C.12.7}$$

with $\bar{m}_i > 0$, $\alpha \geq 0$ being exogenous constants. If $\alpha = 0$, the TFP in region i is a given constant $\bar{m}_i$.

Short-run equilibrium. In the following, we set $\bar{m}_i = 1$, $\bar{a}_i = 1$ for all i to abstract from any first-nature advantages. In the short run, consumers are immobile across regions. We determine short-run indirect utility as a function of h under general equilibrium conditions, which consist of the PMCC and the ZPC of firms. First, by plugging (C.12.6) and (C.12.7) into (C.12.5), with $d_{ki} \equiv \tau_{ki}^{1-\sigma}$, we obtain

$$P_i = \left(\sum_{k \in \mathcal{K}} w_k^{1-\sigma} h_k^{\alpha(\sigma-1)} d_{ki} \right)^{1/(1-\sigma)}. \tag{C.12.8}$$

The ZPC of firms requires that total revenue in region i is exhausted. This yields the wage equation for the model:

$$w_i h_i = \sum_{j \in \mathcal{K}} \frac{w_i^{1-\sigma} h_i^{\alpha(\sigma-1)} d_{ij}}{\sum_{k \in \mathcal{K}} w_k^{1-\sigma} h_k^{\alpha(\sigma-1)} d_{kj}} w_j h_j. \tag{C.12.9}$$

Given the short-run equilibrium wage w, the indirect utility function is given by

$$v_i(h) = \frac{h_i^{-\beta} w_i}{P_i}. \tag{C.12.10}$$

Jacobian matrix at the flat-earth equilibrium. Direct computation shows that the Jacobian matrix of the payoff function $\nabla v(\bar{h})$ is given by

$$\nabla v(\bar{h}) = \left[\sigma I - \bar{D} - (\sigma - 1)\bar{D}^2 \right]^{-1} \left[-(\alpha + \beta - \gamma_0)I + (\alpha + \beta + \gamma_1)\bar{D} \right], \tag{C.12.11}$$

where $\gamma_0 \equiv \frac{1+\alpha}{\sigma}$ and $\gamma_1 \equiv \frac{1-\beta}{\sigma}$. Thus, we conclude that

$$\nabla v(\bar{h}) \simeq c_0 I + c_1 \bar{D}, \tag{C.12.12}$$

with $c_0 = -(\alpha + \beta - \gamma_0)$ and $c_1 = \alpha + \beta + \gamma_1$.

Numerical example. **Figure 9** assumes Allen and Arkolakis (2014)'s model. The parameters are set to $\alpha = 0.5$, $\beta = 0.3$, $\sigma = 6$, and $H = 10$.

Table 2: Exact mappings of economic geography models to the coefficients of $G(f) = c_0 + c_1 f + c_2 f^2$

Model class	Specific model	Local force	Global forces	
		c_0	c_1	c_2
Class (i)	Krugman (1991)	0	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1} + \frac{1}{\sigma}\right)$
	Puga (1999)	0	$\tilde{\mu}\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\left(\frac{\tilde{\mu}^2}{\sigma-1} + \frac{1}{\sigma} + \omega\right)$
	Forslid and Ottaviano (2003)	0	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma(\sigma-1)} + 1\right)$
	Pflüger (2004)	0	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\frac{\mu}{\sigma}\frac{L+H}{H}$
	Harris and Wilson (1978)	$1 - \frac{1}{\alpha}$	0	-1
Class (ii)	Helpman (1998)	$-\gamma$	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1} + \frac{1}{\sigma}\right) + \gamma$
	Redding and Sturm (2008)	$-\gamma$	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right) - \gamma\frac{\sigma-1}{\sigma}$	0
	Murata and Thisse (2005)	$-\hat{\theta}$	$(1-\hat{\theta})\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right) - \hat{\theta}\frac{\sigma-1}{\sigma}$	0
	Allen and Arkolakis (2014)	$-(\alpha + \beta) + \frac{1+\alpha}{\sigma}$	$(\alpha + \beta) + \frac{1-\beta}{\sigma}$	0
	Beckmann (1976)	$-\gamma$	1	0
Class (iii)	Tabuchi (1998)	$-\hat{\gamma}\left(\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}\phi\right)$	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1}\omega_0 + \frac{1}{\sigma}\omega_1\right)$
	Pflüger and Südekum (2008)	$-\gamma\frac{H}{L+H}$	$\mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$	$-\frac{\mu}{\sigma}\frac{L+H}{H}$
	Takayama and Akamatsu (2011)	$-\gamma$	1	$-\mu\frac{L}{H}$

Note: The positive (negative) coefficients indicate agglomeration (dispersion) forces. Observe that class (i) models incorporate a global dispersion force, class (ii) models include a local one, and class (iii) models comprise both forces. See the analyses above for the derivations and definitions of the parameters. The model of Mossay and Picard (2011) (and hence Blanchet et al. (2016)) is the equivalent of Beckmann (1976).

D Comparative statics: Role of local factors

The majority of structural exercises in the current stream of quantitative spatial economics employ local unobserved factors (i.e., heterogeneities in local amenities or the productivity of firms) to replicate the actual data, often under conditions where the uniqueness of the equilibrium is ensured (Redding and Rossi-Hansberg, 2017). For example, in the simplest form, structural residuals under fixed values of the main exogenous parameters of the model (e.g., the expenditure share of the manufactured goods μ or the elasticity of substitution σ) are given broad interpretations such as recovered "local amenities" and then used as exogenous parameters to conduct the counterfactual analyses. In this section, we explore the implications of such approaches by carrying out simple comparative static analyses.

D.1 Structure of equilibrium spatial patterns with location-fixed factors

The payoff function of an economic geography model can be written as $v_i(\boldsymbol{h}, \boldsymbol{A})$, where $\boldsymbol{A} \equiv (A_i)_{i \in \mathcal{K}}$ is the vector of the location-fixed factors. Two canonical examples show how such location-fixed factors are modeled in the literature.

The first and perhaps simplest example is a location-fixed factor in the payoff function:

$$v_i(\boldsymbol{h}, A_i) = \hat{v}_i(\boldsymbol{h}) + A_i, \tag{D.1.1}$$

where $\hat{v}_i(\boldsymbol{h})$ is the $\boldsymbol{A}$-independent component of $v_i(\boldsymbol{h}, A_i)$, which we term *local* heterogeneity. The specification (D.1.1) includes many models with location-fixed factors that *directly* affect the (indirect) utility of mobile workers. For instance, by taking the logarithm, the indirect utility function of Allen and Arkolakis (2014)'s model that incorporates location-fixed amenities reduces to (D.1.1). Such effects also arise from local non-tradable goods, with a representative example being Helpman (1998). As is evident from (C.4.11), when we let $A_i := (1 - \mu) \log[A_i]$, the model reduces to (D.1.1).

The second and more involved example is those location-fixed factors that affect interregional trade flows, which we term *global* heterogeneity. The regional model of Redding and Rossi-Hansberg (2017), §3, is an example. Owing to heterogeneities in the local productivity of firms A_i, The prices of manufactured goods differ across regions; then, the trade balance implies that the wage in region i depends on the whole pattern of $\boldsymbol{A}$. Thus, $v_i(\boldsymbol{h}, \boldsymbol{A})$ is (with slight notational abuse)

$$v_i(\boldsymbol{h}, \boldsymbol{A}) = v_i(\boldsymbol{h}, \boldsymbol{w}(\boldsymbol{h}, \boldsymbol{A})), \tag{D.1.2}$$

where $\boldsymbol{w}(\boldsymbol{h}, \boldsymbol{A}) = (w_i(\boldsymbol{h}, \boldsymbol{A}))$ denotes the wage vector. Krugman (1991)'s model is also an example, where one may interpret that A_i represents the number of immobile workers in region i or, alternatively, the region-specific productivity (as in Redding and Rossi-Hansberg (2017), §3).

We have seen that by assuming a racetrack economy and abstracting from the first-nature advantages as well as by letting $\boldsymbol{A} = \bar{\boldsymbol{A}} \equiv \bar{A}\boldsymbol{1}$, the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv h\boldsymbol{1}$ is always an equilibrium. The question asked in the present appendix is as follows: *What happens when we consider variation in*

the spatial pattern of the location-fixed factors? Does our classification obtained under no heterogeneities still matter?

Suppose that $\bar{h}$ is the unique stable equilibrium. Then, we may view that the equilibrium spatial pattern is a function of A so that $h = h(A)$. In the vicinity of $\bar{h}$, we have

$$h(A) = h(\bar{A} + \delta) \approx \bar{h} + J^A \delta, \tag{D.1.3}$$

where $\delta = (\delta_i) \equiv A - \bar{A} = (A_i - \bar{A})$ is the variation in A and $J^A \equiv [\partial h_i / \partial A_j]$ is the Jacobian matrix of the spatial pattern of mobile agents with respect to A evaluated at $\bar{A}$. We also define ϵ by

$$\epsilon \equiv \delta^\top (h - \bar{h}) = \delta^\top J^A \delta. \tag{D.1.4}$$

If $\epsilon = \sum_{i \in \mathcal{K}} \delta_i (h_i - \bar{h}) \geq 0$ for *any* imposed non-zero variation δ in the location-fixed factors, we have $\delta_i (h_i - \bar{h}) = (A_i - \bar{A})(h_i - \bar{h}) \geq 0$ for all $i \in \mathcal{K}$. This fact implies the following lemma.

Lemma D.1. Assume that $J^A = [\partial h_i / \partial A_i]$ is positive definite at $A = \bar{A}$ and consider a small variation $\delta = (\delta_i) \neq 0$ in A such that $A = \bar{A} + \delta$. Then, the sign of the variation in the location-fixed factor of region i, $\delta_i = A_i - \bar{A}$, and that of the marginal increase in its population, $h_i - h$, coincide.

The above lemma provides a sufficient condition *for any economic geography model* under which an increase of the location-fixed factor A_i implies population growth in region i and vice versa.

To employ Lemma D.1, we should evaluate J^A. Below, we show that this is represented by the Jacobian matrix of the payoff function. First, recall that an interior equilibrium with $h_i > 0$ for all i must be a solution to the following system of non-linear equations:

$$v(h, A) - \bar{v}(h, A)\mathbf{1} = 0, \tag{D.1.5}$$

where $\bar{v}(h, A) \equiv H^{-1} \sum_{i \in \mathcal{K}} v_i(h, A) h_i$ denotes the average payoff. The implicit function theorem regarding the equilibrium equation (D.1.5) implies that at $(\bar{h}, \bar{A})$, J^A is evaluated as follows:

$$J^A = [cE - (I - E)J]^{-1}[I - E]\hat{J}, \tag{D.1.6}$$

where $c \equiv h^{-1}\bar{v}$, $E \equiv K^{-1}\mathbf{1}\mathbf{1}^\top$ is a matrix whose elements are all $1/K$, $J \equiv [\partial v_i / \partial h_i]$, and $\hat{J} \equiv [\partial v_i / \partial A_i]$. All matrices are evaluated at the flat-earth pattern $(\bar{h}, \bar{A})$.

Since J^A is symmetric at the flat-earth equilibrium, it is positive definite if and only if its eigenvalues are all positive. However, because J^A is circulant, its eigenvalues are computable by adopting the same procedure as in our stability analysis (Lemma B.2). We conclude that the eigenvalues a_k of J^A are given by[63]

$$a_k = \begin{cases} 0, & k = 0, \\ -e_k^{-1}\hat{e}_k, & k = 1, 2, \ldots, K - 1, \end{cases} \tag{D.1.7}$$

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[63]We note that $I - E$ and E represent the projections onto the subspace of $\mathbb{R}^K$ defined by $\sum_{i \in \mathcal{K}} x_i = 0$ and its orthogonal subspace, respectively, and their eigenvalues are $(0, 1, 1, \ldots, 1)$ and $(1, 0, 0, \ldots, 0)$.

with e_k and $\hat{e}_k$ being the kth eigenvalues of J and $\hat{J}$, respectively, where we assume that $e_k \neq 0$. Moreover, the eigenvectors of J^A are again $\{\eta_k\}$ with $\eta_k = (\cos[\theta ki])$ with $k = 0, 1, \ldots, K-1$. Note that we have $a_0 = 0$. This is intuitive because it says that a uniform increase in A_i across the regions does not affect the spatial pattern—in other words, what matters is the *relative* variation in the location-fixed factors. Thus, without loss of generality, we rewrite $\delta = \sum_{k \in \mathcal{K}} C_k \eta_k$ and assume $C_0 = 0$, meaning that $\delta \cdot \mathbf{1} = 0$. We then have $h - \bar{h} = \sum_{k \in \mathcal{K}} C_k a_k \eta_k$ and

$$\epsilon = \delta^\top (h - \bar{h}) = \sum_{k \in \mathcal{K}} C_k^2 a_k. \tag{D.1.8}$$

If $a_k > 0$ for all $k \geq 1$, we have $\epsilon > 0$. Each a_k is an amplifying factor in the direction of η_k in the sense that if $\delta = \eta_k$, we obtain $h - \bar{h} = a_k \eta_k$.

That said, we have two questions regarding the properties of a_k. The first is obvious:

Question 1. *Is $a_k > 0$ for all $k \geq 1$?*

If true, from Lemma D.1, this implies that the relative advantage of a region implies a relative increase in its population and vice versa. As we see below, this is generally the case.

The second is important: *What happens on $\{a_k\}$ if we face a change (in particular, a decrease) in transport costs?* Put another way, does an increase in trade freeness r (see Section B.2) imply a strengthened role of first natures—or the converse? In concrete terms:

Question 2. *Is $\mathrm{d}a_k / \mathrm{d}r$ positive (or negative) for all $k \geq 1$?*

We see that because

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}r} = \sum_{i \in \mathcal{K}} C_k^2 \frac{\mathrm{d}a_k}{\mathrm{d}r}, \tag{D.1.9}$$

if $\mathrm{d}a_k / \mathrm{d}r$ happened to be positive for all $k \geq 1$, as r increases (τ decreases), the location-fixed factors matter more; the converse is also true.

D.2 Role of location-fixed factors: Model class matters

For simplicity, consider the simplest case, (*local* heterogeneity), as in (D.1.1). We note that for (D.1.1), we have $\hat{J} = I$ and thus $\hat{e}_k = 1$, which in turn implies that $a_k = -e_k^{-1}$. Recalling that if the flat-earth equilibrium is stable, we have $e_k < 0$ for all k, we see that $a_k > 0$. Thus, it must be that $\epsilon > 0$ for *any* relative variation δ in A. Thus, the answer to the first question is "yes": any relative first-nature (dis)advantage in terms of location-fixed amenities increases (decreases) the local population when the flat-earth equilibrium is stable—this is, of course, hardly a surprise.

We next turn our attention to the second question. As we see, asking the question reveals a major watershed between model classes (i) and (ii): *when the economy faces a decrease in transport costs, the effects of location-fixed advantages are typically in the opposite direction for classes (i) and (ii).*

For the class (i) models in the literature, there is a determinate implication regarding the effects of a decline in interregional transport costs on first-nature advantages. As long as the flat-earth pattern

is stable, we have[64]

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} > 0. \tag{D.2.1}$$

Thus, *the positive effects of the relative location-fixed advantages increase according to the decrease in inter-regional transport costs.* Under the stability of the flat-earth equilibrium, a decrease in interregional transport costs fosters more agglomeration in regions with relative advantages in amenity. In fact, this leads to the instability of the flat-earth equilibrium because at the first break point, we have $e_k = 0$ for some k and hence $a_k = \infty$ for that k. Thus, the model leads to regional divergence, even in the range of transport costs where the flat-earth equilibrium is stable.

For class (ii) models, *a decrease in transport costs has the opposite implication* compared with class (i) models. We illustrate this by using Helpman (1998)'s model. For the original model with PL, we have

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} < 0 \tag{D.2.2}$$

whenever the stability of the flat-earth equilibrium is ensured regardless of the level of r, by the condition $\sigma(1 - \mu) > 1$. Thus, regions once flourished by first-nature advantages due to larger endowments of housing space will decline if interregional transport costs decrease. Assuming different specifications of the local factors as in (D.1.2) does not alter the result. In fact, as we see, if we consider a variant model where A is interpreted as the heterogeneities in local productivity as in the regional model of Redding and Rossi-Hansberg (2017), §3 (see Section C.4), we have the same result: $a_k > 0$ and that $\mathrm{d}a_k/\mathrm{d}r < 0$; this result is also consistent with the numerical exercise conducted by the study. In short, in class (ii) models, the role of initial heterogeneity declines in line with decreasing transport costs.

The interpretation of the behavior of class (ii) models is straightforward. As the role of interregional transport costs declines, the local dispersion force dominates. Then, an agglomeration formed solely by its local advantages must face relative *second-nature* disadvantages because of local congestion compared with those formerly behind, leading to a relative decline in such a region.

In light of this, assumptions about landownership can affect the sign of $\mathrm{d}a_k/\mathrm{d}r$. In particular, LL, by redistributing local rental revenue, can relax the magnitude of the second-nature disadvantage in regions in which housing rent is high. If the expenditure share of the housing good is sufficiently high, via redistribution, this can overcome any relative second-nature disadvantage, meaning that

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[64] For all class (i) models in the literature, we have $e_k = G(f_k(r))/\phi(f_k(r))$ with a strictly positive and decreasing function $\phi(f)$ (see Appendix C). Noting that $\mathrm{d}f_k/\mathrm{d}r < 0$, this then implies that

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} = -\frac{\mathrm{d}e_k^{-1}}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\phi(f_k(r))}{G(f_k(r))}\right) = -\frac{\phi'(f_k)G(f_k) - \phi(f_k)G'(f_k)}{\{G(f_k)\}^2}\frac{\mathrm{d}f_k}{\mathrm{d}r} > 0,$$

where we note that $\phi'(f_k)G(f_k) - \phi(f_k)G'(f_k)$ is strictly positive since $\phi'(f_k) < 0$, $\phi(f_k) > 0$, and because the flat-earth equilibrium is stable $G(f_k) < 0$ and $G'(f_k) < 0$.

$\mathrm{d}a_k / \mathrm{d}r > 0$. If we assume LL in Helpman (1998) as in Redding and Sturm (2008), we obtain

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} \begin{cases} < 0, & \text{if} \quad \bar{\mu} < \mu < 1, \\ > 0, & \text{if} \quad 0 < \mu < \bar{\mu}, \end{cases} \tag{D.2.3}$$

with $\bar{\mu} \equiv \frac{2(\sigma-1)^2}{2\sigma^2 - 2\sigma + 1} < \frac{\sigma-1}{\sigma}$, which confirms the above speculation. This result illustrates the basic role of a local dispersion force and typically less featured assumptions on landownership.

Thus, whether the second-nature causation of an economic geography model boosts first-nature advantages in line with decreasing transport costs or not depends on the model class to which it belongs.

Below, in addition to the simplest case (D.1.1), we provide examples of *global* heterogeneity where the payoff is given by (D.1.2). For this case, we have $\nabla_A v = \nabla_w v \nabla_A w$. Because a_k ($k \geq 1$) is the kth eigenvalue of $(\nabla_h v)^{-1}(\nabla_A v)$, we first evaluate the two matrices and then their product. Given any wage equation $W(h, w, A) = 0$ that incorporates local factors A, we have the following computation:

$$\nabla_h v = \{\phi(\bar{D})\}^{-1} G_{\mathrm{H}}(\bar{D}), \ \nabla_A v = \{\phi(\bar{D})\}^{-1} G_{\mathrm{A}}(\bar{D}), \tag{D.2.4}$$

where we define the matrix polynomials ϕ, G_{H}, and G_{A} of $\bar{D}$ by

$$\phi(\bar{D}) \equiv (\nabla_w W)^{-1}, \ G_{\mathrm{H}}(\bar{D}) \equiv \nabla_h v \nabla_w W - \nabla_w v \nabla_h W, \ G_{\mathrm{A}}(\bar{D}) \equiv \nabla_A v \nabla_w W - \nabla_w v \nabla_A W. \tag{D.2.5}$$

By employing these formula, we see $e_k = G_{\mathrm{H}}(f_k)/\phi(f_k)$ and $\hat{e}_k = G_{\mathrm{A}}(f_k)/\phi(f_k)$, meaning that we have $a_0 = 0$, and, for $k \geq 1$,

$$a_k = -\frac{G_{\mathrm{A}}(f_k)}{G_{\mathrm{H}}(f_k)}. \tag{D.2.6}$$

This in turn implies

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} = -\frac{G'_{\mathrm{A}}(f_k)G_{\mathrm{H}}(f_k) - G_{\mathrm{A}}(f_k)G'_{\mathrm{H}}(f_k)}{\{G_{\mathrm{H}}(f_k)\}^2} \frac{\mathrm{d}f_k}{\mathrm{d}r}. \tag{D.2.7}$$

However, since we have $\mathrm{d}f_k / \mathrm{d}r < 0$, we conclude

$$\mathrm{sgn}\,\frac{\mathrm{d}a_k}{\mathrm{d}r} = \mathrm{sgn}\left[G'_{\mathrm{A}}(f_k)G_{\mathrm{H}}(f_k) - G_{\mathrm{A}}(f_k)G'_{\mathrm{H}}(f_k) \right]. \tag{D.2.8}$$

Basically, location-fixed factors that affect trade flows can be modeled by employing either of the two forms in the following examples. These two examples demonstrate that the above implication, namely *model class matters even when uniqueness is the case*, holds true for the cases when the level of the location-fixed factors in a region affects the nominal wages in other regions.

Example D.1 (Heterogeneous local productivity (Redding and Rossi-Hansberg, 2017, §3)). The productivity of firms differs across regions and thus affects the regional *share* in trade flows. The wage

equation for the model is defined by (C.4.19):

$$W_i(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{A}) = w_i h_i - \sum_{j \in \mathcal{K}} \frac{h_i A_i w_i^{1-\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} h_k A_k w_k^{1-\sigma} d_{kj}} w_j h_j = 0. \tag{D.2.9}$$

Without heterogeneities in the per capita housing space, the indirect utility function is

$$v_i = \frac{\mu}{\sigma - 1} \ln[\Delta_i] + \mu \ln[w_i] - (1 - \mu) \ln[h_i] \tag{D.2.10}$$

with $\Delta_i \equiv \sum_{k \in \mathcal{K}} h_j A_i w_j^{1-\sigma} d_{ji}$. By employing these formulae, we compute as follows:

$$\nabla_{\boldsymbol{h}} v = \frac{1}{h} \left(\frac{\mu}{\sigma - 1} \bar{\boldsymbol{D}} - (1 - \mu) \boldsymbol{I} \right), \ \nabla_{\boldsymbol{w}} v = \frac{1}{\bar{w}} \mu (\boldsymbol{I} - \bar{\boldsymbol{D}}), \ \nabla_{\boldsymbol{A}} v = \frac{1}{A} \frac{\mu}{\sigma - 1} \bar{\boldsymbol{D}}, \tag{D.2.11}$$

$$\nabla_{\boldsymbol{h}} \boldsymbol{W} = -\bar{w} \bar{\boldsymbol{D}} (\boldsymbol{I} - \bar{\boldsymbol{D}}), \ \nabla_{\boldsymbol{w}} \boldsymbol{W} = h \{ \sigma \boldsymbol{I} - (\sigma - 1) \bar{\boldsymbol{D}} \} (\boldsymbol{I} + \bar{\boldsymbol{D}}), \ \nabla_{\boldsymbol{A}} \boldsymbol{W} = -\frac{1}{A} \bar{w} h (\boldsymbol{I} - \bar{\boldsymbol{D}}) (\boldsymbol{I} + \bar{\boldsymbol{D}}). \tag{D.2.12}$$

These formulae imply that without heterogeneities in the per capita housing space, we have

$$G_{\mathrm{H}}(f) = \frac{1}{\sigma} (1 - f) \left[-(1 - \mu) + \left(\frac{\mu \sigma}{\sigma - 1} - \frac{\sigma - 1}{\sigma} \right) f \right] \tag{D.2.13}$$

$$G_{\mathrm{A}}(f) = -\frac{h}{A} \frac{\mu}{\sigma - 1} (1 - f) \left[(\sigma - 1) + \sigma f \right] < 0. \tag{D.2.14}$$

By employing these formulae, we can show that whenever the equilibrium is unique ($\sigma(1 - \mu) > 1$), we have $G_{\mathrm{H}}(f) < 0$ and thus $a_k \geq 0$ for all k. It also follows that $\mathrm{d}a_k / \mathrm{d}r < 0$ for all $k \geq 1$. We also note that $G_{\mathrm{H}}(f) < 0$ implies the stability of $\bar{h}$. Further, if there are no exogenous heterogeneities in $\boldsymbol{A}$, the model is isomorphic to Redding and Sturm (2008) and Allen and Arkolakis (2014) regarding the second-nature mechanism.

Example D.2 (Heterogeneous local market size (Krugman, 1991)). Consider Krugman (1991)'s model. Assuming there are first-nature heterogeneities in the local endowments of immobile agents, we can model the heterogeneities in *market size*. For the model, the wage equation is

$$W_i(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{A}) = w_i h_i - \mu \sum_{j \in \mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k \in \mathcal{K}} h_k w_k^{1-\sigma} d_{kj}} (w_i h_i + A_i) = 0, \tag{D.2.15}$$

where A_i is the number of immobile workers in region i. We compute as follows:

$$\nabla_{\boldsymbol{h}} v = \frac{1}{h} \frac{\mu}{\sigma - 1} \bar{\boldsymbol{D}}, \ \nabla_{\boldsymbol{w}} v = \frac{1}{\bar{w}} (\boldsymbol{I} - \mu \bar{\boldsymbol{D}}), \ \nabla_{\boldsymbol{A}} v = \boldsymbol{0}, \tag{D.2.16}$$

$$\nabla_{\boldsymbol{h}} \boldsymbol{W} = -\bar{w} \bar{\boldsymbol{D}} (\mu \boldsymbol{I} - \bar{\boldsymbol{D}}), \ \nabla_{\boldsymbol{w}} \boldsymbol{W} = h \{ \sigma \boldsymbol{I} - \mu \bar{\boldsymbol{D}} - (\sigma - 1) \bar{\boldsymbol{D}}^2 \}, \ \nabla_{\boldsymbol{A}} \boldsymbol{W} = -\mu \bar{\boldsymbol{D}}. \tag{D.2.17}$$

Then, we have

$$G_{\mathrm{H}}(f) = \frac{1}{\sigma} \left[\left(\frac{\mu}{\sigma - 1} + \frac{\mu}{\sigma} \right) f - \left(\frac{\mu^2}{\sigma - 1} + \frac{1}{\sigma} \right) f^2 \right], \tag{D.2.18}$$

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$$G_A(f) = \frac{\mu}{\bar{w}} f(1 - \mu f) > 0. \tag{D.2.19}$$

By employing these formulae, we can show that $a_k \geq 0$ for all k and that $da_k/dr > 0$ for all $k \geq 1$ whenever the flat-earth equilibrium is stable (i.e., $G_H(f) < 0$).

Remark D.1. Some models, e.g., Redding and Turner (2015), §20.3, employ both local and global heterogeneities such that

$$v_i(h, A, B) = v_i(h, w(h, A)) + B_i, \tag{D.2.20}$$

where $A = (A_i)$ and $B = (B_i)$ are exogenous constants that reflect global and local heterogeneities, respectively. Since A and B are not related to each other, the Jacobian matrix with respect to these two heterogeneities is given by a block-diagonal form and the effects of each heterogeneity can be studied separately.

D.3   Numerical examples

This section provides numerical examples to complement the above formal analysis, which focused on infinitesimally small variations in A. Below, by focusing on the most canonical form of the location-fixed factors as in (D.1.1), we add an extra positive constant term A_0 to the indirect utility of region 0, meaning that the region has an exogenous advantage. Our numerical results suggest that the drawn formal conclusions correctly predict the tendency in agglomeration patterns even when a strong location-fixed effect is imposed.

Figure 16 and **Figure 17** report the results of our numerical experiments under three representative settings, namely a class (ii) model under the uniqueness of the equilibrium and class (i) and (ii) models under a multiplicity of equilibria. In line with the numerical examples discussed in Section 5 (**Figure 8**, and **Figure 9**), Krugman (1991) and Allen and Arkolakis (2014) are employed for the examples for classes (i) and (ii), respectively. We note that the latter is isomorphic to Helpman (1998) with LL (i.e., Redding and Sturm, 2008; Redding and Rossi-Hansberg, 2017).

The figures show the population share of region 0 at stable equilibria, $\lambda_0 \equiv h_i/H$, against τ for the four settings of A_0 in $\{0, 0.001, 0.005, 0.01\}$. $A_0 = 0$ is the baseline case with no location-fixed advantage. Under our parameter setting, A_0 accounts for $0.5 \sim 100\%$ of the indirect utility of region 0 and hence has significant effects on the equilibrium patterns.

Figure 16 reports the evolutionary paths of λ_0 for the model proposed by Allen and Arkolakis (2014) [class (ii)] under the *uniqueness of the equilibrium*. The parameters are the same as in **Figure 9** except that we let $\beta = 0.6$. This implies $\alpha + \beta \leq 0$ and hence the equilibrium is unique regardless of the level of transport costs (see Section 5.2). Compared with the baseline case $A_0 = 0$, λ_0 is larger for the other cases ($A_0 = 0.001, 0.005, 0.01$); this corresponds to the condition $a_k > 0$. In addition, λ_0 is increasing in A_0, which is intuitive. Furthermore, λ_0 decreases in line with τ, which is consistent with $da_k/dr < 0$.

Figure 17 reports the evolutionary paths of λ_0 for the models proposed by Krugman (1991) [class (i)] and Allen and Arkolakis (2014) [class (ii)] under a *multiplicity of equilibria*. The basic model
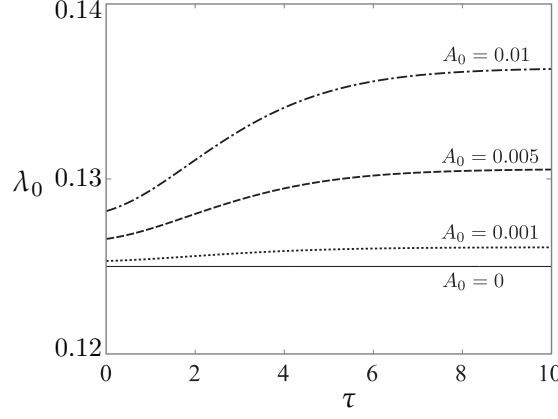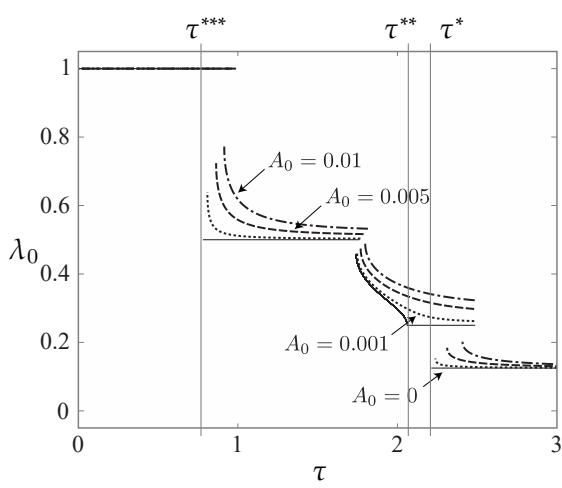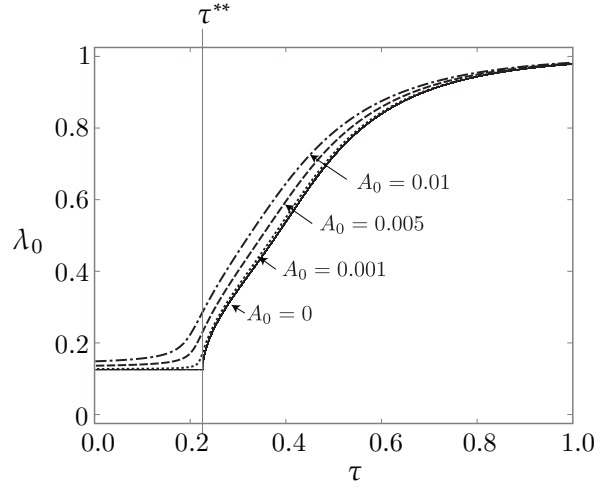
Figure 16: Population share of region 0 under the uniqueness of the equilibrium [Allen and Arkolakis (2014)'s model]

Panel A: Krugman (1991)'s model   Panel B: Allen and Arkolakis (2014)'s model

Figure 17: Population share of region 0 under a multiplicity of equilibria

parameters other than A_0 are the same as in **Figure 8** and **Figure 9**. We confirm that the figures are also consistent with our predictions: that (a) $a_k > 0$ and that (b) $\mathrm{d}a_k/\mathrm{d}r > 0$ for the class (i) models and $\mathrm{d}a_k/\mathrm{d}r < 0$ for the class (ii) models, provided that $\bar{h}$ is stable. For all $A_0 = 0.001, 0.005, 0.01$, λ_0 is greater than that for $A_0 = 0$, which confirms $a_k > 0$. Moreover, by focusing on the ranges $\tau \in (\tau^*, \infty)$ (for Panel A) and $\tau \in (0, \tau^{**})$ (for Panel B), the curves confirm (b). Although our predictions do not cover $\tau \in (0, \tau^*)$ for Panel A, a similar relation robustly holds true: as long as the global structure of the spatial pattern is unchanged (i.e., bifurcation is not encountered), a monotonic decrease in τ implies a greater role of location-fixed advantages in region 0.