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Commodity Spot, Forward, and Futures Prices with a Firm's Optimal Strategy

NAKAJIMA Katsushi
Ritsumeikan Asia Pacific University

Research Institute of Economy, Trade & Industry, IAA
The Research Institute of Economy, Trade and Industry
http://www.rieti.go.jp/en/

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Abstract

This paper studies commodity spot, forward, and futures prices under a continuous-time setting. The model is an enhanced version of Nakajima (2015) which was modeled through discrete time. Our model considers a firm, which uses an input commodity to produce an output commodity, stores the commodity, and trades forwards or futures commodities to hedge. Through the Hamilton-Jacobi-Bellman equation and Feynman-Kac formula, we derive relations between spot, forward, and futures prices. The convenience yield can be interpreted as shadow price of storage, short selling constraints, and limits of risk. We compare our result with the existing models and conduct a numerical analysis. The optimal production plan and trading strategy for spot commodities and forwards are also derived. The model can be easily modified to consider cash settlement or hedging using output commodity forward contracts.

Keywords: Commodity prices, Convenience yield, Forward prices, Futures price, Production, Storage

JEL classification: G12, G13, Q02

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1 Introduction

As China and other emerging countries evolve, the needs of commodities such as energy and metals are stronger than ever. On the other hand, the financialization of commodities is another aspect of recent trends in this market. Thus, commodities are an interesting area to be studied from financial and economical points of view. In this paper, we propose a model which includes financial and economical aspects of commodity spot, forward, and futures prices.

One of the most important key factors of commodity spot, forward, and futures is the convenience yield. This was recognized by Kaldor (1939). Kaldor realized that there were differences among commodity spot, forward, and expected prices which were constructed by interest rates, carrying cost, risk premium, and the convenience yield.[1] Working (1949) argued that the convenience yield increases as the amount of storage become scarce through examples from wheat markets which is now called as the theory of storage. Brennan (1958) analyzed the theory of storage empirically. He defined the net total cost of storage to be the cost of physical storage plus the risk-aversion factor minus the convenience yield and estimated the marginal risk-aversion factor minus the marginal convenience yield using agricultural products.

In 1990, Gibson and Schwartz (1990) introduced mean reverting convenience yields with geometric Brownian spot prices and analyzed futures prices on crude oil. The model enriched the earlier paper to include the stochastic behavior of commodity spot prices and convenience yields. Schwartz (1997) enhanced their model to include mean-reverting interest rates. This was generalized by Miltersen and Schwartz (1998) to incorporate futures convenience yields and forward interest rates through the Heath-Jarrow-Morton (1992) framework. Another generalization was done by Casassus and Collin-Dufresne (2005). However, in these papers, the convenience yield is exogenous and defined to be the difference of commodity spot, forward, and futures prices. In our paper, the convenience yield is not assumed explicitly but is derived as a by-product of storage constraints on a firm.

On the other hand, there were some research studies from economical aspects. Deaton and Laroque (1992) analyzed the commodity price with storage by modeling producer-consumers and risk-neutral inventory holders under equilibrium. However, they did not include futures or forward markets and thus convenience yield was not analyzed. Routledge, Seppi, and Spatt (2000) extended Deaton and Laroque's model and studied the spot and the forward prices including the convenience yield. On the other hand, Gorton, Hayashi, and Seppi (2013) revealed the relation between the convenience yield and inventory through empirical analysis with risk-averse hedger. Casassus et al. (2013) used production rates and utility functions in their model and showed that convenience yield can be expressed as the marginal productivity rate. However, all of these models do not build from the firm's profit maximization and thus the

[1] In Kaldor's (1939) paper, he used forward instead of futures. Since interest rate is deterministic in his paper, there will be no difference between forward and futures price as Cox, et al. (1981) state.

relation between commodity spot prices, forward, and futures prices were not analyzed concretely.

However, Nakajima (2015) proposed a model which includes the firm's profit maximization and consumer-speculator's utility maximization under a discrete-time setting and derived the equation for the commodity spot price, forward price, and futures prices. He interpreted that the convenience yield could be decomposed into cost and yield parts which were restrictions on spot, forward, or futures storage. Thus, in his model the convenience yield are not an exogenous factor but endogenously determined.

In this paper, we enhance Nakajima's model (2015) into a continuous-time setting. That is, we model a representative firm of an industry which takes one-input commodity, produce one-output commodity, trade forward and store input commodity. There are trading constraints and storage constraints. Since the stochastic control have been developed throughout these years, we apply these tools. We use the Hamilton-Jacobi-Bellman equation and the Feynman-Kac formula in order to derive the optimal condition and the spot-forward price relation. This allows us to compare with the other financial stochastic model such as the Gibson-Schwartz model and thus we can reinterpret convenience yield as shadow prices on constraints used by the firm.

In Section 2, we set up a one-input and one-output model with forward trading. Here we introduce other models which consider hedging by forward under cash settlement, futures, and hedging by forward on the output commodity. In Section 3, we provide equations for spot commodity prices, future spot commodity prices, forward prices, and futures prices and discuss their implications. We show correspondences with other existing models such as the Gibson-Schwartz model and the Schwartz model. Also, we analyze our model numerically through the Gibson-Schwartz model with parameters estimated by Schwartz. We derive the optimal amount of buying and using the commodity and the optimal trading strategy for forward. Furthermore, we show that the existence of a speculator-consumer agent implies that we have a different way of pricing commodity forward. Section 4 concludes.

2 The One-Input and One-Output Model

2.1 A Firm

We consider a representative firm which uses commodity 2, e.g. coal or crude oil, to produce commodity 1, e.g. electricity or heating oil. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ be a filtered probability space. P is the risk-neutral probability and the corresponding natural probability is P_N which will be considered when we are discussing speculators' behavior.[2] Let $p : D_p \to \mathbb{R}$ be a production function for commodity 1 which uses commodity 2.[3] The prices of spot commodities at time

[2]We assume that there is no approximate arbitrage and thus there exists a risk-neutral probability. See Duffie (2001), Chapter 6, Section K, Proposition, p.121.

[3]In this paper, $\mathbb{R}_+$ is $\{x|x \geq 0\}$ and $\mathbb{R}_+^N$ is$\{(x_n)_{n=1,\dots,N}|x_n \geq 0\}$.

t are $S_n(t), n = 1, 2$. We assume that the storage cost depends on a storage price process $S_3(t)$ which is positive and $\mathcal{F}_t$-adapted. These prices satisfy the following stochastic differential equations.

$$\mathrm{d}S_n(t) = S_n(t)\left\{\mu_{S_n}(t)\mathrm{d}t + \sigma_{S_n}(t)\cdot \mathrm{d}B(t)\right\}, 0 \le t \le T, n = 1, 2$$

where $B(t)$ is a d-dimensional standard Brownian motion. We assume that $\{\mathcal{F}_t\}_{0\le t\le T}$ is a natural filtration generated by $B(t)$ and augmented by P-null sets in $\mathcal{F}$. Commodity 2 can be stored and there are forward contracts for commodity 2. The prices of forward commodities at time t which mature at T_m are $F_2(t, T_m), 0 \le t \le T_m, T_m \in \#F_2(t, T)$ where we denote $\#F_2(t, T)$ to be a finite number of commodity forward that matures between $[t, T]$. The forward price $F_2(t, T_m)$ follows the stochastic differential equation:

$$\mathrm{d}F_2(t, T_m) = F_2(t, T_m)\left\{\mu_{F_2}(t, T_m)\mathrm{d}t + \sigma_{F_2}(t, T_m)\cdot \mathrm{d}B(t)\right\}, 0 \le t \le T_m$$

There are many finite maturities $(T_1, \cdots, T_M)$ for forward and we assume that there is a forward contract which matures at T. The forward commodity is physically delivered with spot commodity at maturity. Therefore, $F_2(t, t) = S_2(t)$ for any $t \in \#F_2(0, T)$. Let $R : D_R \to \mathbb{R}$ be a cost function of storage of physical commodity 2 and storage price process $S_3(t)$ which is positive and $\mathcal{F}_t$-adapted. We define the domain of p and R in the next section. The Heath-Jarrow-Morton type forward interest rate is $f(t, s)$ which are modeled by the following stochastic differential equations[4]

$$\mathrm{d}f(t, T) = \mu_f(t, T)\mathrm{d}t + \sigma_f(t, T)\mathrm{d}B(t)$$

and the spot interest rate is $r(t)$ which is $f(t, t) = r(t)$. Therefore the price of the bank account at time t is $P_0(t) = \exp(\int_0^t r(u)\mathrm{d}u)$. We also assume that $\mu_{S_2}(\cdot), \mu_f(\cdot, T), \sigma_{S_2}(\cdot), \sigma_{F_2}(\cdot, T), \sigma(\cdot, T)$ are continuous on $[0, T]$. We denote $\mathrm{E}_t(\cdot)$ as the conditional expectation given $\mathcal{F}_t$.

We use the following notations. $q_{S_2,b}(t)$ is the amount of spot commodity 2 bought at time t, $q_{S_2,u}(t)$ is the amount of spot commodity 2 used at time t, $q_{F_2,b}(t, T_m), T_m \in \#F_2(0, T)$ is the amount of future commodity 2 which matures at time T_m bought at time t. Note that the amount of purchased future commodity 2 $q_{F_2,b}(t, T_m), T_m \in \#F_2(0, t)$ which are matured before time t are 0, because there are no forward traded after it matures. We use the notation $q(t) = (q_{S_2,b}(t), q_{S_2,u}(t), (q_{F_2,b}(t, T_m))_{t\le T_m\le T})$ for these amounts and $q = (q(t))_{0\le t\le T}$.

The amount of stored commodity is

$$\begin{aligned} x_{S_2}(t) &= x_{S_2,0} + \int_0^t q_{S_2,b}(s) - q_{S_2,u}(s)\mathrm{d}s \\ &\quad + \sum_{T_m\in\#F_2(0,t)} 1_{t\ge T_m} x_{F_2}(T_m, T_m), 0 \le t \le T \end{aligned}$$

[4]Here we assume conditions C.1, C.2, and C.3 from Heath, Jarrow, and Morton (1992).

and the amount of storage of forward commodity which matures at time T_m are

$$x_{F_2}(t,T_m) = x_{F_2,0,T_m} + \int_0^t q_{F_2,b}(s,T_m)\mathrm{d}s, 0 \le t \le T, T_m \in \#F_2(t,T).$$

In this model, we adopt physical delivery for trading forward. The amount of storage of forward commodity which reached maturity $x_{F_2}(t,t)$ are spot commodities and the firm can use it for production. Although we introduce other models for cash settlement or hedging by forward on output commodities, the model with physical delivery will be the one we mainly focus on.

We now formulate a stochastic control problem for a firm. Let us define

$$x(t) = (x_{S_2}(t), (x_{F_2}(t,T_m))_{T_m \in \#F_2(0,T)}, r(t), S_1(t), S_2(t), S_3(t), (F_2(t,T_m))_{T_m \in \#F_2(0,T)})$$

which the stochastic controlled system is

$$\begin{aligned} \mathrm{d}x(t) &= \mu_x(t,x(t),q(t))\mathrm{d}t + \sigma_x(t,x(t),q(t))\mathrm{d}B(t) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} & \mu_x(t,x(t),q(t)) \\ = \; & (q_{S_2,b}(t) - q_{S_2,u}(t) + \sum_{T_m \in \#F_2(t,T)} 1_{t=T_m} x_{F_2}(t,T_m), (q_{F_2,b}(t,T_m))_{T_m \in \#F_2(t,T)}, \\ & \mu_f(t,t), (\mu_{S_n}(t) S_n(t))_{n=1,2,3}, (\mu_{F_2}(t,T_m) F_2(t,T_m))_{T_m \in \#F_2(t,T)})^\top \\ & \sigma_x(t,x(t)) \\ = \; & \left(0, 0, \sigma_f(t,t), (\sigma_{S_n}(t) S_n(t))_{n=1,2,3}, (\sigma_{F_2}(t,T_m) F_2(t,T_m))_{T_m \in \#F_2(t,T)}\right)^\top . \end{aligned}$$

The first two terms are the amount of storage $(x_{S_2}(t), \{x_{F_2}(t,T_m))$, followed by interest rate $r(t)$, commodity spot prices $(S_n(t))$, and forward prices $(F_2(t,T_m))$. Note that the firm can control the amount of storage but it can not control interest rates and prices.

The firm's objective is to maximize its profit.[5]

$$\begin{aligned} \sup_{q \in \mathcal{Q}} \quad & \mathrm{E}\Bigg[\int_0^T \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t)))\mathrm{d}t \\ & - \int_0^T \sum_{T_m \in \#F_2(t,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t,T_m) F_2(t,T_m)\mathrm{d}t \\ & + \mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} x_{S_2}(T) S_2(T) \Bigg]. \end{aligned} \tag{2}$$

[5]Here we assume that there is a risk-neutral probability and the firm's expected discounted value of profits is defined under this probability.

where

$$
\begin{aligned}
\mathcal{Q} \;=\; & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process } , x_{S_2}(t) \geq 0, 0 \leq t \leq T, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, 0 \leq t \leq T, \\
& 0 \leq x_{F_2}(t, T_m) \leq K, 0 \leq t \leq T, T_m \in \#F_2(t, T), \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, 0 \leq t \leq T, \\
& L_{F_2,b} \leq q_{F_2,b}(t, T_m) \leq K_{F_2,b}, 0 \leq t \leq T, T_m \in \#F_2(t, T)\}.
\end{aligned}
$$

The first two terms are the firm's profit from its main business. The third term is the storage cost of spot commodity 2. Therefore, these three terms represent sales minus cost, including the storage cost. The fourth term is the cost of purchasing forward contracts on commodity 2. The last term is the income from disposing storage at time T. This is a continuous-time version of Nakajima model (2015).

Note that the firm does not short sell spot commodities or forward contracts. The firm has upper limits for the forward contract in order to limit the forward price risk. Forward can be stored without storage costs unless it is matured. Define q^* to be the optimal control and x^* to be the corresponding optimal state process. Moreover, the model is based on a risk-neutral agent which is adopted by Deaton and Laroque (1992) and Routledge et al. (2000). The difference with these models are that in this model we consider the production planning. The economic intuition of this model is that the firm is not just deciding the production planning, but also the trading strategy to maximize its profits while controlling its storage amount.

Although a typical example of commodities are coal and electricity (dark spread) for this model, there are other examples in the real world which are: Natural gas and electricity (spark spread), crude oil and heating oil or other petroleum products (crack spread), natural gas and natural gas liquids (frac spread), and soybean and soybean meal (crush spread).

2.2 Variants of the Model

In this section, we introduce other variants of the above model which are forward under cash settlement, futures, and hedging by forward on output commodity. However, the results are somewhat the same comparing to the first model.

2.3 Forward under Cash Settlement

If the forward are settled by cash then the firm's problem will be

$$
\begin{aligned}
\sup_{q\in\mathcal{Q}} \quad & \mathrm{E}\Bigg[\int_0^T \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t)))\mathrm{d}t \\
& -\int_0^T \sum_{T_m\in\#F_2(t,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t,T_m)F_2(t,T_m)\mathrm{d}t \\
& + \sum_{T_m\in\#F_2(0,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} x_{F_2}(T_m,T_m)S_2(T_m) \\
& + \mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} x_{S_2}(T)S_2(T)\Bigg].
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
\mathcal{Q} \quad = \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t)\geq 0, 0\leq t\leq T, \\
& 0\leq q_{S_2,u}(t)\leq K_{S_2,u}, 0\leq t\leq T, \\
& 0\leq x_{F_2}(t,T_m)\leq K, 0\leq t\leq T, T_m\in\#F_2(t,T), \\
& L_{S_2,b}\leq q_{S_2,b}(t)\leq K_{S_2,b}, 0\leq t\leq T, \\
& L_{F_2,b}\leq q_{F_2,b}(t,T_m)\leq K_{F_2,b}, 0\leq t\leq T, T_m\in\#F_2(t,T)\}
\end{aligned}
$$

and the amount of stored commodity is

$$
x_{S_2}(t) = x_{S_2,0} + \int_0^t q_{S_2,b}(s) - q_{S_2,u}(s)\mathrm{d}s, 0\leq t\leq T.
$$

The difference between physical delivery and cash settlement is that for cash settlement it settles with cash at maturity so we do not need to consider the forward position after it matures as a spot commodity. Therefore, the storage cost function includes only the amount of spot commodities.

The results between the model based on physical delivery and that of cash settlement are somewhat same.

2.4 Using Futures for Hedging through Cash Settlement

Let us see if the firm use futures to hedge its profit. We assume that the futures are continuously resettled and settled by cash and the futures price is a martingale under risk-neutral probability.[6] We denote $G_2(t,T)$ as its price at

[6] For continuous resettlement on futures and its prices, see Duffie (2001), Chapter 8, Section C and D.

time t which matures at time T. The firm's profit maximization problem is

$$\sup_{q\in\mathcal{Q}} \quad \mathrm{E}\Bigg[\int_0^T \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))\mathrm{d}t$$
$$+\int_0^T \sum_{T_m\in\#G_2(t,T)} \int_t^{T_m} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_m)dG_2(s,T_m)\mathrm{d}t$$
$$+\mathrm{e}^{-\int_0^T r(u)\mathrm{d}u}x_{S_2}(T)S_2(T)\Bigg]. \tag{4}$$

where

$$\begin{aligned}\mathcal{Q} \quad = \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t) \geq 0, 0 \leq t \leq T,\\ & 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, 0 \leq t \leq T,\\ & 0 \leq x_{G_2}(t,T_m) \leq K, 0 \leq t \leq T, T_m \in \#G_2(t,T),\\ & L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, 0 \leq t \leq T,\\ & L_{G_2,b} \leq q_{G_2,b}(t,T_m) \leq K_{G_2,b}, 0 \leq t \leq T, T_m \in \#G_2(t,T)\},\end{aligned}$$

$$x_{S_2}(t) = x_{S_2,0} + \int_0^t q_{S_2,b}(s) - q_{S_2,u}(s)\mathrm{d}s, 0 \leq t \leq T.$$

2.5 Hedging by Forward on an Output Commodity

Instead of controlling for the amount of inputs, the firm may want to control the amount of outputs. In this case, $q_{S_1,b}(t)$ is the amount of spot commodity 1 bought at time t, $q_{S_1,u}(t)$ is the amount of spot commodity 1 used at time t, $q_{F_1,b}(t,T_m), T_m \in \#F_1(0,T)$ is the amount of future commodity 1 which matures at time T_m bought at time t. Again, the amount of purchased forward commodity 1 $q_{F_1,b}(t,T_m), T_m \in \mathcal{F}_1(0,t)$ which are matured before time t are 0, because there are no forward traded after it matures. The amount of future commodity 1 can be used as spot commodities after the maturity. We again use the notation $q(t) = (q_{S_1,b}(t), q_{S_1,u}(t), (q_{F_1,b}(t,T_m))_{T_m\in\#F_1(t,T)})$ for these amounts and $q = (q(t))_{0\leq t\leq T}$. The amount of stored commodity 1 is

$$\begin{aligned}x_{S_1}(t) \quad = \quad & x_{S_1,0} + \int_0^t q_{S_1,b}(s) - q_{S_1,u}(s)\mathrm{d}s\\ & + \sum_{T_m\in\#F_1(0,t)} 1_{t\geq T_m} x_{F_1}(T_m,T_m), 0 \leq t \leq T\end{aligned}$$

where the amount of storage of forward commodity which matures at time u are

$$x_{F_1}(t,u) = x_{F_1,0,u} + \int_0^t q_{F_1,b}(s,u)\mathrm{d}s, 0 \leq t \leq T, u \in \#F_1(t,T).$$

Again, we denote $\#F_1(t,T)$ to be a finite number of commodity forward 1 that matures between $[t,T]$. Now, the firm's objective is to maximize the following profit function.

$$\begin{aligned}\sup_{q\in\mathcal{Q}} \quad & \mathrm{E}\Bigg[\int_0^T \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(q_{S_1,u}(t)S_1(t) - p_0(q_{S_1,b}(t))S_2(t) - R(x_{S_1}(t), S_3(t)))\mathrm{d}t \\ & -\int_0^T \sum_{T_m\in\#F_1(t,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_1,b}(t,T_m)F_1(t,T_m)\mathrm{d}t \\ & +\mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} x_{S_2}(T)S_1(T)\Bigg] \end{aligned} \tag{5}$$

where

$$\begin{aligned}\mathcal{Q} \quad = \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_1}(t) \geq 0, 0 \leq t \leq T, \\ & 0 \leq q_{S_1,u}(t) \leq K_{S_1,u}, 0 \leq t \leq T, \\ & 0 \leq x_{F_1}(t,T_m) \leq K, 0 \leq t \leq T, T_m \in \#F_1(t,T), \\ & L_{S_1,b} \leq q_{S_1,b}(t) \leq K_{S_1,b}, 0 \leq t \leq T, \\ & L_{F_1,b} \leq q_{F_1,b}(t,T_m) \leq K_{F_1,b}, 0 \leq t \leq T, T_m \in \#F_1(t,T)\}.\end{aligned}$$

Note that in this case, the amount of input is substituted by $p_0(q_{S_1,b}(t))$ which is a function of the amount of output. This function p_0 can be acknowledged as a conversion formula from output commodity to input commodity. The firm controls the amount of output commodity $q_{S_1,b}(t)$ which can take negative values and therefore it can buy or sell amounts of input commodity $p_0(q_{S_1,b}(t))$. When $p_0(q_{S_1,b}(t))$ takes a negative value it implies selling of the input commodity. It also controls for the cost of storage for the output commodity and hedging amount of forward on output commodity. These two variants will produce similar results as the main model produces.

2.6 Speculators' Utility Maximization Problem

In this section, we introduce the speculator-consumer agent. We assume P_N to be the natural probability. Suppose there are J agents. The agent j is defined by the utility function u_j. Agent j consumes commodity 1 and trades the money market account, zero-coupon bond, and forward commodity contract, but does not trade spot commodities. The price of money market account and zero-coupon bond will be denoted as $P_0(t)$ and $P(t,T)$, respectively. The zero-coupon bond which the agent trades will only be those that have the same maturities with the forward commodity contract. They own some share of the firm and this share is fixed. Therefore a part of the firm's profit $\theta_{\pi,j}\pi(t)$ at time t will be agent j's income.

Furthermore, we assume that the forward commodity price follows

$$\begin{aligned}\mathrm{d}F_2(t,T_m) \quad = \quad & F_2(t,T_m)\left\{\mu_{F_2}(t,T_m)\mathrm{d}t + \sigma_{F_2}(t,T_m)\cdot \mathrm{d}B_{P_N}(t)\right\}, \\ & 0 \leq t \leq T_m\end{aligned}$$

under natural probability P_N.

Let $c_{1,j}(t)$ be the amount of consumption of commodity 1 at time t. Let C be the space of nonnegative adapted processes in $\boldsymbol{R}$ for consumption at time $0 \leq t \leq T$, C_T be the space of nonnegative random variable in $\boldsymbol{R}$ for consumption at time T, and Θ be a space of $\{\mathcal{F}(t)\}$-progressively measurable, $\boldsymbol{R}^{2\#F_2(0,T)+1}$-valued process for trading strategies. Let

$$\begin{aligned}
S(t) &= (S_1(t), S_2(t), S_3(t)), \\
\theta_{P,j}(t) &= (\theta_{P_0,j}(t), (\theta_{P,j}(t,T_m))_{T_m \in \#F_2(t,T)}) \\
\theta_{F_2,j}(t) &= ((\theta_{F_2,j}(t,T_m))_{T_m \in \#F_2(t,T)}) \\
\theta_j(t) &= (\theta_{P,j}(t), \theta_{F_2,j}(t))
\end{aligned}$$

Agent j maximizes the following expected utility.

$$\sup_{(c_{1,j},\theta_j)\in\mathcal{A}_j} \mathrm{E}_{P_N}\left[\int_0^T u_j(t, c_{1,j}(t))\mathrm{d}t + U(W(T))\right] \tag{6}$$

where

$$\begin{aligned}
\mathcal{A}_j(t_0,T) &= \Bigg\{(c_{1,j}(\cdot), C_1, \theta(\cdot)) \in C \times C_T \times \Theta : W_j(t) = W_j(0) \\
&\quad + \int_0^t \theta_{P_0,j}(t)dP_0(t) + \int_0^t \sum_{T_m \in \#F_2(t,T)} \theta_{P,j}(t,T_m)dP(t,T_m) \\
&\quad + \int_0^t \sum_{T_m \in \#F_2(t,T)} \theta_{F_2,j}(t,T_m)dF_2(t,T_m) - \int_0^t c_{1,j}(t)S_1(t), \\
&\quad c_{1,j}(t) \geq 0, \theta_{F_2,j}(t,t) = 0, 0 \leq t \leq T\Bigg\}, \\
W_j(0) &= W_{j,0}, \\
W_j(t) &= \theta_{P_0,j}(t)P_0(t) + \sum_{T_m \in \#F_2(t,T)} \theta_{P,j}(t,T_m)P(t,T_m) \\
&\quad + \sum_{T_m \in \#F_2(t,T)} \theta_{F_2,j}(t,T_m)F_2(t,T_m), 0 \leq t \leq T.
\end{aligned}$$

and E_{P_N} is the expectation operator under P_N. Notice that the position of forward vanishes for each period which means the speculator clears out at maturity.

3 Implications under the One-Input and One-Output Model

3.1 Spot Price, Forward Prices, Futures Price, and Convenience Yield

Now we provide the relation between spot prices, forward prices, futures prices, and convenience yields under the model.

We assume the following conditions on storage function R and production function p.

Assumption 1. $R(q, S_3)$ is a convex function of q and differentiable with respect to q. There exists a constant K which satisfies $|R(x,s)| \leq K(1+|(x,s)|^k)$. and a function $h_R \in L^1(\Omega, P)$ such that $|\partial_q R| \leq h_R$[7] where $\partial_q R$ denote the partial derivative with respect to the amount of storage. Furthermore, R is defined on $D_R = (-\epsilon, \infty) \times \mathbb{R}_+$ where $\epsilon > 0$.

This condition was assumed in Nakajima (2015). The domain D_R of R is defined in order to calculate the partial derivative at 0. The firm optimizes its profit for only nonnegative q since it can not take negative amounts for storage. Therefore, this last assumption is only to calculate the partial derivative at the boundary.

Assumption 2. p is nondecreasing, concave, and differentiable.

Now we define the value function and its assumption. Let us decompose the period into subperiods which are delimited by the maturities of forward.

$$[T_0, T_1], [T_1, T_2] \cdots, [T_{M-2}, T_{M-1}], [T_{M-1}, T_M]$$

where $T_0 = 0$ and $T_M = T$. Define

$$\begin{aligned}
& J_\pi(t_0, x; q(\cdot)) \\
= \; & \mathrm{E}\Bigg[\int_{t_0}^{T} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))\mathrm{d}t \\
& - \int_{t_0}^{T} \sum_{T_m \in \# F_2(t,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t, T_m) F_2(t, T_m)\mathrm{d}t \\
& + \mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} x_{S_2}(T) S_2(T) \Bigg].
\end{aligned}$$

The value function of the optimization problem (2) is

$$V_\pi(t_0, x) = \sup_{q(\cdot) \in \mathcal{Q}(t_0, T)} J(t_0, x; q(\cdot)). \tag{7}$$

$$V_\pi(T, x) = \mathrm{e}^{-\int_{t_0}^{T} r(u)\mathrm{d}u} (x_{S_2} + x_{F_{2,T}}) S_2(T), x \in \mathbb{R}^{5+2\# F_2(0,T)} \tag{8}$$

[7] $L^1(\Omega, P)$ is a space of integrable function on Ω with respect to the measure P.

where

$$
\begin{aligned}
\mathcal{Q}(t_0,T) &= \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t)\geq 0, t_0\leq t\leq T,\\
&\quad 0\leq q_{S_2,u}(t)\leq K_{S_2,u}, t_0\leq t\leq T,\\
&\quad 0\leq x_{F_2}(t,T_m)\leq K, t_0\leq t\leq T, T_m\in \#F_2(t,T),\\
&\quad L_{S_2,b}\leq q_{S_2,b}(t)\leq K_{S_2,b}, t_0\leq t\leq T,\\
&\quad L_{F_2,b}\leq q_{F_2,b}(t,T_m)\leq K_{F_2,b}, t_0\leq t\leq T, T_m\in\#F_2(t,T)\}.
\end{aligned}
$$

and $x=(x_{S_2},(x_{F_2,T_m})_{T_m\in\#F_2(0,T)},r,S_1,S_2,S_3,(F_2)_{T_m\in\#F_2(0,T)})$.

We also need the following assumption in order to derive the relation between commodity spot and forward prices.

Assumption 3. $V_\pi(t,x)\in C^{1,3}([T_{m-1},T_m]\times\mathbb{R}^{5+2\#F_2(T_{m-1},T_m)})$ and $\partial_{tx}V_\pi$ is a continuous function on $[T_m-1,T_m]\times\mathbb{R}^{5+2\#F_2(T_{m-1},T_m)}$ for each time interval $[T_{m-1},T_m]$.

The relation between commodity spot and forward prices is derived in the following proposition.

Proposition 3.1. Let Assumptions 1, 2, and 3 hold. Suppose that there exists an optimal solution for the problem (2). Then the spot and forward price satisfy the following equations.

$$
\begin{aligned}
S_2(t) &= E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}S_2(T)\middle|\mathcal{F}_t\right]+\lambda_{S_2}(t)\\
&\quad -E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] &(9)\\
F_2(t,T_m) &= P(t,T_m)^{-1}\Bigg(E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}S_2(T)|\mathcal{F}_t\right]+\lambda_{F_2,0}(t,T_m)\\
&\quad -E\left[\int_{T_m}^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]\Bigg) &(10)\\
S_2(t) &= E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]F_2(t,T_m)+\lambda_{F_2}(t,T_m)\\
&\quad -E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] &(11)\\
S_2(t) &= p'(q^*_{S_2,u}(t))S_1(t)+(\lambda_{S_2,q_u,l}(t,x)-\lambda_{S_2,q_u,u}(t,x)) &(12)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{S_2}(t) &= \lambda_{S_2,x}(t)+\lambda_{S_2,q_b,l}(t)-\lambda_{S_2,q_b,u}(t)\\
\lambda_{F_2,0}(t,T_m) &= -\lambda_{F_2,x_l}(t,T_m)+\lambda_{F_2,x_u}(t,T_m)-\lambda_{F_2,q_b,l}(t,T_m)\\
&\quad +\lambda_{F_2,q_b,u}(t,T_m)\\
\lambda_{F_2}(t,T_m) &= \lambda_{S_2}(t)+\lambda_{F_2,0}(t,T_m)\\
P(t,T_m) &= E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]
\end{aligned}
$$

and $q^*_{S_2,b}, q^*_{S_2,u}, q^*_{F_2,b}$ be the optimal solution and x^* be the corresponding optimal state process.

Proof. See the Appendix. □

$\lambda_{F_2}(t,T_m)$ is the residual between the commodity spot price and the discounted forward price minus the discounted storage cost.[8] Therefore, it is natural to interpret this $\lambda_{F_2}(t,T_m)$ as the convenience yield or in other words the benefit of holding spot commodity. This convenience yield can be decomposed by marginal storage cost, shadow prices of storage, short selling constraints, and limits of risk. It can also be decomposed to the cost part and the yield part. The cost consists of marginal storage cost and shadow prices associated with the upper limit of purchasing spot commodities, the lower limit of purchasing forward, the nonnegativity of forward storage. The yield is composed of shadow prices associated with the nonnegativity of storage commodity, the lower limit of purchasing spot commodity, the upper limit of forward storage, the upper limit of purchasing forward.

We can also derive the dynamics of $\lambda_{F_2}(t,T_m)$.

Corollary 3.1. Let Assumptions 1, 2, and 3 hold. Suppose that there exists an optimal solution for the problem (2). Then the dynamics of $\lambda_{F_2}(t,T_m)$ are

$$\begin{aligned}
& \mathrm{d}\lambda_{F_2}(t,T_m) \\
= \; & S_2(t)\left\{\mu_{S_2}(t)\mathrm{d}t + \sigma_{S_2}(t)\cdot \mathrm{d}B(t)\right\} \\
& -P(t,T_m)F_2(t,T_m)\{(\mu_P(t,T_m)+\mu_{F_2}(t,T_m)\mathrm{d}t - \sigma_P(t,T_m)^\top \sigma_{F_2}(t,T_m))\mathrm{d}t \\
& +(\sigma_P(t,T_m)-\sigma_{F_2}(t,T_m))\cdot \mathrm{d}B(t)\} \\
& +\Bigg\{ -r(t)\left(E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}} R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]\right) \\
& +\partial_{x_{S_2}} R(x^*_{S_2}(t),S_3(t))\Bigg\}\mathrm{d}t \\
& +\Bigg\{\partial_x E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}} R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] \\
& \cdot\sigma(t,x^*(t),u^*(t))\Bigg\}\mathrm{d}B(t).
\end{aligned}$$

[8] These findings were also indicated in Nakajima (2015). See Nakajima (2015) for more details.

where

$$\begin{aligned}
\sigma_P(t, T_m) &= -\sum_{T_m \in \# F_2(t,T)} \sigma_f(t, T_m) \\
\mu_P(t, T_m) &= r(t) - \sum_{T_m \in \# F_2(t,T)} \mu_f(t, T_m) + 1/2\sigma_P(t, T_m)^\top \sigma_P(t, T_m) \\
&\quad + \sum_{T_m \in \# F_2(t,T_m)} \sigma_P(t, T_m)\gamma(t).
\end{aligned}$$

Proof. See the Appendix. □

This gives us the dynamics of the Lagrange multiplier $\lambda_{F_2}(t, T_m)$ which have a correspondence with the dynamics of traditional convenience yield which are assumed in Gibson-Schwartz model or Schwartz model.

We can also derive the relation between commodity spot and futures prices in a similar manner.

Proposition 3.2. Let Assumptions 1, 2, and 3 hold. Suppose that there exists an optimal solution for the problem (4). Then the spot and forward price satisfy the following equations.

$$\begin{aligned}
S_2(t) &= E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
&\quad - E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad (13) \\
S_2(t) &= P(t, T_m)G_2(t, T_m) + \lambda_{G_2}(t, T_m) \\
&\quad - E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad (14) \\
S_2(t) &= p'(q^*_{S_2,u}(t))S_1(t) + (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x)) \qquad (15)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{S_2}(t) &= \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
\lambda_{G_2,0}(t, T_m) &= \mathrm{Cov}\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}, S_2(T_m) \middle| \mathcal{F}_t\right] \\
\lambda_{G_2}(t, T_m) &= \lambda_{S_2}(t) + \lambda_{G_2,0}(t, T_m)
\end{aligned}$$

and $q^*_{S_2,b}, q^*_{S_2,u}, q^*_{F_2,b}$ be the optimal solution and x^* be the corresponding optimal state process.

Proof. See the Appendix. □

3.2 Comparison with Other Commodity Pricing Models

Let us compare the results with the existing models. We will show a correspondence between the convenience yield from the existing models and the optimal

Lagrange multipliers. However, since the optimal Lagrange multipliers in our model are endogenous variables and the convenience yield in the existing models such as Schwartz (1997) model are exogenous variables, there can be no equivalence among these models.

3.2.1 The Gibson and Schwartz (1990) model

If the dynamics of the commodity spot price $S_2(t)$ and the convenience yield $\delta_2(t)$ are

$$\begin{aligned} \mathrm{d}S_2(t) &= (r-\delta_2(t))S_2(t)\mathrm{d}t + \sigma_{S_2}S_2(t)\mathrm{d}B_1(t) && (16)\\ \mathrm{d}\delta_2(t) &= \kappa_{\delta_2}(\alpha_{\delta_2}-\delta_2(t)-\theta_{\delta_2})\mathrm{d}t + \sigma_{\delta_2}\mathrm{d}B_2(t) && (17)\end{aligned}$$

and the interest rate r is deterministic, then the Gibson-Schwartz (1990) model asserts that

$$S_2(t) = F_2(t,T)\mathrm{e}^{-r(T-t)+\delta_2(t)\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})-A(T-t)}$$

where $A(T-t)$ is determined by the parameters including volatilities.[9] Comparing with equation (11), we have

$$\begin{aligned} &\mathrm{e}^{-r(T-t)}F_2(t,T) - E\left[\int_t^T \mathrm{e}^{-r(s-t)}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] + \lambda(t,T)\\ =\;& F_2(t,T)\mathrm{e}^{-r(T-t)+\delta_2(t)\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})-A(T-t)}\end{aligned}$$

Thus, we have a correspondence between the Lagrange multipliers $\lambda_{F_2}(t,T)$ and the convenience yield $\delta_2(t)$ which is[10]

$$\begin{aligned} &\delta_2(t)\\ =\;& \kappa_{\delta_2}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})^{-1}\Bigg\{A(T-t)+\ln\Bigg(1-F_2(t,T)^{-1}\\ &\cdot\Bigg\{E\left[\int_t^T \mathrm{e}^{-r(s-T)}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] - e^{r(T-t)}\lambda(t,T)\Bigg\}\Bigg)\Bigg\}. \end{aligned} \tag{18}$$

Therefore, our model is compatible with the Gibson-Schwartz model.

[9] See Gibson and Schwartz (1990) and Schwartz (1997) for details on $A(T-t)$. Here we slightly modified the notation of $A(T-t)$ and removed $-r(T-t)$ outside.

[10] Note that κ_{δ_2} and $A(T-t)$ are also part of the Gibson-Schwartz model. Therefore, to be precise, we have a correspondence between the adjusted convenience yield and the optimal Lagrange multipliers.

$$\begin{aligned} &\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})\delta_2(t)-A(T-t)\\ =\;& \ln\left(1-F_2(t,T)^{-1}\left\{E\left[\int_t^T \mathrm{e}^{-r(s-T)}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]-e^{r(T-t)}\lambda(t,T)\right\}\right).\end{aligned}$$

3.2.2 The Schwartz (1997) model

One of the benchmark models is the Schwartz (1997) model. If the dynamics of the commodity spot price $S_2(t)$ and the convenience yield $\delta_2(t)$ are

$$\begin{aligned}
\mathrm{d}S_2(t) &= (r-\delta_2(t))S_2(t)\mathrm{d}t + \sigma_{S_2}S_2(t)\mathrm{d}B_1(t)\\
\mathrm{d}\delta_2(t) &= \kappa_{\delta_2}(\alpha_{\delta_2}-\delta_2(t))\mathrm{d}t + \sigma_{\delta_2}\mathrm{d}B_2(t)\\
\mathrm{d}r(t) &= \kappa_r(\alpha_r - r(t))\mathrm{d}t + \sigma_r\mathrm{d}B_2(t)
\end{aligned}$$

then the Schwartz (1990) model asserts that

$$\begin{aligned}
S_2(t) &= G_2(t,T)P(t,T)\mathrm{e}^{\delta_2(t)\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})-A(T-t)}\\
P(t,T) &= \exp\{-r(t)\kappa_r^{-1}(1-\mathrm{e}^{\kappa_r(T-t)})+\kappa_r^{-2}(\kappa_r\alpha_r+\sigma_{S_2,r})\\
&\quad \times((1-\mathrm{e}^{-\kappa_r(T-t)})-\kappa_r(T-t))-(4\kappa_r^3)^{-1}\sigma_r^3(4(1-\mathrm{e}^{-\kappa_r(T-t)})\\
&\quad -(1-\mathrm{e}^{-2\kappa_r(T-t)})-2\kappa_r(T-t))\}
\end{aligned}$$

where $A(T-t)$ is determined by the parameters including volatilities.[11] Comparing with equation (11), we have

$$\begin{aligned}
&P(t,T)G_2(t,T) - E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]+\lambda_{G_2}(t,T)\\
=\ & G_2(t,T)P(t,T)\mathrm{e}^{\delta_2(t)\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})-A(T-t)}
\end{aligned}$$

Thus, we have a correspondence between the Lagrange multipliers $\lambda_{G_2}(t,T)$ and the convenience yield $\delta_2(t)$ which is[12]

$$\begin{aligned}
\delta_2(t) &= \kappa_{\delta_2}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})^{-1}\Bigg(A(T-t)+\ln\Bigg(1-(G_2(t,T)P(t,T))^{-1}\\
&\quad \cdot E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]-\lambda_{G_2}(t,T)\Bigg)\Bigg)
\end{aligned}$$

Therefore, our model is compatible with the Schwartz model.

[11] See Schwartz (1997) for details on $A(T-t)$. Here we slightly modified the notation of $A(T-t)$ and removed the interest rate term into the zero-coupon bond price $P(t,T)$ for maturity T outside.

[12] Note that κ_{δ_2} and $A(T-t)$ are also part of the Gibson-Schwartz model. Therefore, to be precise, we have a correspondence between the adjusted convenience yield and the optimal Lagrange multipliers.

$$\begin{aligned}
&\kappa_{\delta_2}^{-1}(1-\mathrm{e}^{-\kappa_{\delta_2}(T-t)})\delta_2(t)-A(T-t)\\
=\ & \ln\left(1-(G_2(t,T)P(t,T))^{-1}E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]-\lambda_{G_2}(t,T)\right)
\end{aligned}$$

3.2.3 The Casassus and Collin-Dufresne (2005) model

Now, let us examine Casassus and Collin-Dufresne (2005) model. Suppose that the commodity spot price $S_2(t)$, the convenience yield $\delta_2(t)$, and the interest rate $r(t)$ are assumed as follows.

$$\begin{aligned}\ln S_2(t) &= \phi_0 + \phi_Y^\top Y(t)\\ r(t) &= \psi_0 + \psi_1 Y_1(t)\\ dY(t) &= -\kappa_Y Y(t)dt + dB_Y(t).\end{aligned}$$

and they derived[13]

$$\begin{aligned}\ln G_2(t,T) &= A(T-t) + B(T-t)^\top Y(t)\\ E_t[S_2(T) - S_2(t)] &= \int_t^T (r(s) - \delta_2(s))S_2(s)ds\\ \delta_2(t) &= r(t) - \frac{1}{2}\phi_Y^\top \phi_Y + \phi_Y^\top \kappa_Y Y(t).\end{aligned}$$

Let us define

$$\begin{aligned}X(t) &= (\ln S_2(t), \ln G_2(t,T), r(t))^\top\\ c(T-t) &= (\phi_0, A(T-t), \psi_0)^\top\\ M(T-t) &= (\phi_Y^\top, B(T-t)^\top, (\psi_1, 0, 0)^\top)\end{aligned}$$

If $M(T-t)$ is invertible, then

$$Y(t) = M(T-t)^{-1}(X(t) - c(T-t))$$

Therefore, if we substitute

$$\begin{aligned}&\ln S_2(t)\\ =\;& \ln\Bigg(E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]G_2(t,T)\\ &-E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}} R(x_{S_2}^*(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] + \lambda_{G_2}(t,T)\Bigg),\end{aligned}$$

we can calculate the convenience yield $\delta_2(t)$ through $X(t)$ and $Y(t)$. Thus, we have a correspondence between the convenience yield and the optimal Lagrange multiplier.

3.2.4 Semi-discretized Miltersen-Schwartz (1998) model

Another example of this model is a semi-discretized Miltersen-Schwartz (1998) model. The term structure for $\delta_2(t,T)$ and $f(t,T)$ is continuous for time t but

[13] For details see Casassus and Collin-Dufresne (2005).

discretized for maturity T. Suppose that the dynamics of the commodity spot price $S_2(t)$, the futures convenience yield $\delta_2(t,T_m)$, and the forward interest rate $f(t,T_m)$ are

$$\begin{aligned} \mathrm{d}S_2(t) &= \mu_{S_2}(t)S_2(t)\mathrm{d}t + \sigma_{S_2}(t)S_2(t)\mathrm{d}B_1(t) \\ \mathrm{d}\delta_2(t,T_m) &= \mu_{\delta_2}(t,T_m)\mathrm{d}t + \sigma_{\delta_2}(t,T_m)\mathrm{d}B_2(t) \\ \mathrm{d}f(t,T_m) &= \mu_f(t,T_m)\mathrm{d}t + \sigma_f(t,T_m)\mathrm{d}B_3(t). \end{aligned}$$

The relation between the commodity spot and futures price under Miltersen-Schwartz model (1998) is

$$S_2(t) = G_2(t,T)E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]\mathrm{e}^{\sum_{T_m\in\#F(t,T)}\delta_2(t,T_m)(T_m-T_{m-1})}$$

Comparing with equation (11), we have

$$\begin{aligned} & G_2(t,T)E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]\mathrm{e}^{\sum_{T_m\in\#F(t,T)}\delta_2(t,T_m)(T_m-T_{m-1})} \\ = {} & E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]G_2(t,T) - E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] \\ & +\lambda_{G_2}(t,T) \end{aligned}$$

This equation implies

$$\begin{aligned} & \sum_{T_m\in\#F(t,T)}\delta_2(t,T_m)(T_m-T_{m-1}) \\ = {} & \ln\left(1-\left(G(t,T)E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right]\right)^{-1}\right. \\ & \left.\cdot\left(E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}}R(x^*_{S_2}(s),S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]+\lambda_{G_2}(t,T)\right)\right) \end{aligned}$$

By calculating the difference equation, we have the convenience yield $\delta_2(t,T_m)$. Therefore, there is a correspondence between the discretized Miltersen-Schwartz futures convenience yield (1998) and the optimal Lagrange multipliers.

3.3 Numerical Analysis

Now we will see how we can use the existing model and interpret the Lagrange multiplier $\lambda_{F_2}(t,T)$ in this model. Let us use the Gibson-Schwartz model (16) with the parameters estimated by Schwartz (1997) for crude oil.

$\sigma_{S_2} = 0.393, \sigma_{\delta_2} = 0.527, \rho(S_2,\delta_2) = 0.766, \kappa_{\delta_2} = 1.876, \alpha_{\delta_2} = 0.106, \theta_{\delta_2} = 0.198.$

Suppose that the interest rate is 5%, crude oil futures price which matures in one year is 35 dollars.[14] Furthermore, it is assumed that the storage cost function

[14] The interest rate is non-stochastic under the Gibson-Schwartz model, so futures and forward prices must be equal.

is linear and its marginal storage cost is \$0.4 per barrel and the current $\delta(t)$ is 0.106 which is same as the long-term mean α_{δ_2}. Using equation (18), we can calculate the Lagrange multiplier $\lambda_{F_2}(t,T)$ and see its behavior.

Figure 1 shows the correspondence between the convenience yield $\delta_2(t)$ and the Lagrange multiplier $\lambda_{F_2}(t,T)$. The units of the convenience yield are rates, but the units of the Lagrange multiplier are dollars which are intuitive to traders and financial manager. If $\delta_2(t)$ is 10% which is near the long-term mean $alpha_{\delta_2}$, then $\lambda_{F_2}(t,T)$ is around 3 dollars. This means that the benefit of holding a spot commodity comparing to holding futures is 10% or 3 dollars per barrel. We can see from the figure that as $\delta_2(t)$ increases, $\lambda_{F_2}(t,T)$ also increases.

Figure 1: The traditional convenience yield $\delta_2(t)$ and the Lagrange multiplier $\lambda_{F_2}(t,T)$. The Lagrange multipliers are calculated for each current convenience yield $\delta_2(t)$ using equation (18) under the Gibson-Schwartz (1997) model with parameters estimated by Schwartz (1997) for crude oil.

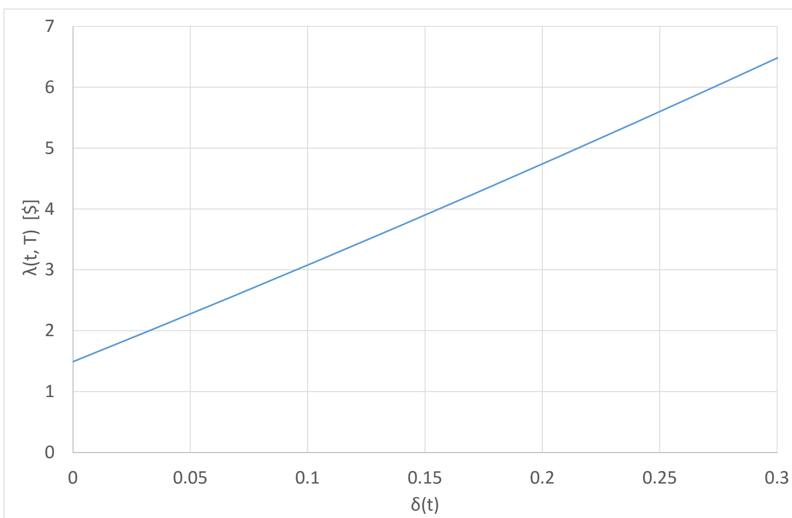
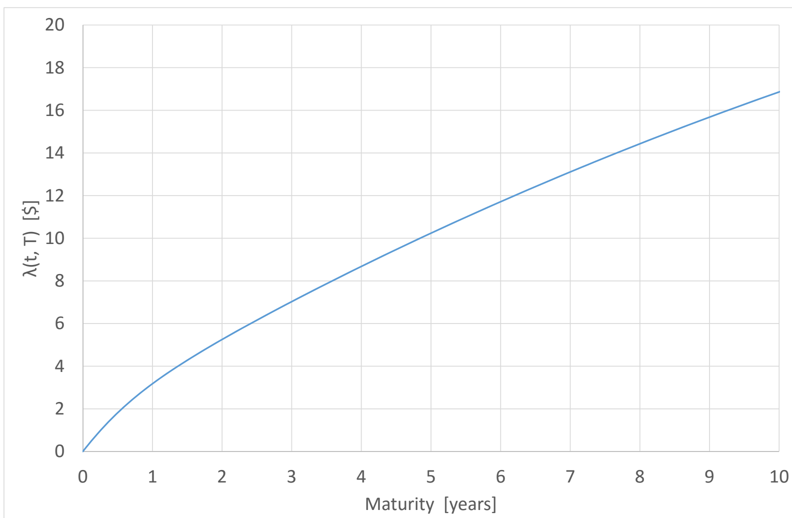


Figure 2 indicates the effect of time to maturity T on the Lagrange multiplier. If the time to maturity is only one year the Lagrange multiplier is 3 dollars, but if the time to maturity is 10 years the Lagrange multiplier is around 17 dollars. Therefore, the benefit of storage is large when the maturity is long. On the other hand, as time to maturity increases, the Lagrange multiplier also increases but the increasing rate falls.

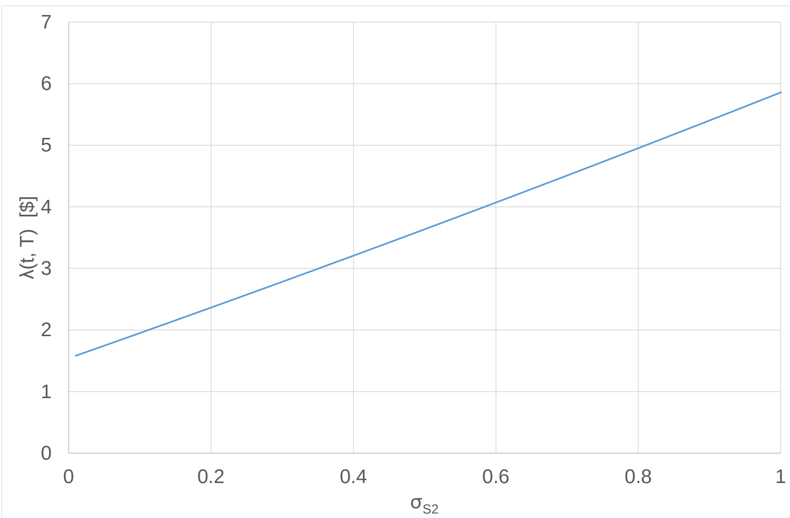
The relation between the Lagrange multiplier and the volatility σ_{S_2} is il-

Figure 2: Time to maturity T and the Lagrange multiplier $\lambda_{F_2}(t,T)$. The Lagrange multipliers are calculated for each maturity using equation (18) under the Gibson-Schwartz (1997) model with parameters estimated by Schwartz (1997) for crude oil.



lustrated in Figure 3. The figure shows that as σ_{S_2} increases, $\lambda_{F_2}(t,T)$ also increases. This implies that if the spot commodity price becomes more volatile, then the benefit of holding a spot commodity becomes large. Since the spot price can not be negative, the downside risk is limited and thus the benefit of holding spot commodities rises as the volatility of spot commodity price increases.

Figure 3: Volatility of spot commodity price σ_{S_2} and the Lagrange multiplier $\lambda_{F_2}(t,T)$. The Lagrange multipliers are calculated for each volatility of spot commodity price σ_{S_2} using equation (18) under the Gibson-Schwartz (1997) model with parameters estimated by Schwartz (1997) for crude oil.



3.4 Optimal Production Plan and Trading Strategy

In order to derive the optimal production plan and trading strategy, we need the following assumptions.

Assumption 4. *R* is strictly convex function of x. *R* is essentially smooth on x.[15]

Assumption 5. p is strictly concave and essentially smooth.

[15] A convex function f is essentially smooth for $C = int(domf)$ if C is not empty, f is differentiable throughout C, and $\lim_{n\to\infty} ||\nabla f(x_n)|| = +\infty$ whenever $x_1, x_2, \cdots$, is a sequence in C converging to a boundary point x of C.

We can derive the optimal amount of spot commodities and forward with these assumptions.

Proposition 3.3. Let Assumptions 1–5 hold. Let $S_1(t)$ be positive. Suppose the problem (2) has an optimal solution for any x_0. Then the optimal solution is

$$
\begin{aligned}
& q^*_{S_2,u}(t) = I_p\left(\frac{S_2(t) - e^{rt}\lambda_{S_2,u}(t)}{S_1(t)}\right) \\
& (x^*_{S_2}(t), (x^*_{F_2}(t,T_m))_{T_m \in \#F_2(t,T)}) = I_{R,t}(x_{0,t,T})
\end{aligned}
$$

where I_p is the inverse of p',

$$
\begin{aligned}
I_{R,t}(x_{0,t,T}) &= \phi_t^{-1}(x_{0,t,T}) \\
\phi_t(x_{t,T}) &= E\left[\int_t^T e^{-\int_0^u r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
x_{t,T} &= (x_{S_2}(t), (x_{F_2}(t,T_m))_{T_m \in \#F_2(t,T)}) \\
x_{0,t,T} &= \begin{pmatrix} x_{0,t,T,S_2} \\ (x_{0,t,T,F_2,T_m})_{T_m \in \#F_2(t,T_m)} \end{pmatrix} \\
x_{0,t,T,S_2} &= -S_2(t) + E\left[e^{-\int_t^T r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] + \lambda_{S_2,b}(t), \\
x_{0,t,T,F_2,T_m} &= -e^{-\int_t^{T_m} r(u)\mathrm{d}u} F_2(t,T_m) + E\left[e^{-\int_t^T r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] \\
&\quad +\lambda_{F_2,b_l}(t,T_m) - \lambda_{F_2,b_u}(t,T_m)
\end{aligned}
$$

and the optimal trading strategy is

$$
\begin{aligned}
dx_{S_2}(t) &= (q_{S_2,b}(t) - q_{S_2,u}(t))\mathrm{d}t + \sum_{T_m \in \#F_2(0,t)} 1_{t=T_m} x_{F_2}(t,T_m), \\
&\quad 0 \le t \le T \\
dx_{F_2}(t,T_m) &= q_{F_2,b}(t,T_m)\mathrm{d}t, 0 \le t \le T, T_m \in \#F_2(t,T).
\end{aligned}
$$

Proof. See the Appendix. □

The firm buys $q^*_{S_2,b}(t)$ and use $q^*_{S_2,u}(t)$ commodity 2, and trades forward $q^*_{F_2,b}(t,s)$. $q^*_{F_2,b}(t,s)$ is the hedging strategy for the firm. Although the optimal amount used $q^*_{S_2,u}(t)$ is determined by the two commodity prices, the optimal amount of buying $q^*_{S_2,b}(t)$ do not depend on $S_1(t)$ explicitly.

Examples of the production function and the storage cost function are shown in Nakajima (2015).

3.5 The Speculator's Valuation of Forward Prices

We now turn to the result for the speculator. Let us assume the following conditions.

Assumption 6. u_j is strictly concave and differentiable. There exists a function $h_{u_j} \in L^1(\Omega, P)$ where $|\partial u_j(t,\cdot)/\partial c_1| \leq h_{u_j}$, t=0,...,T, s=t+1,...,T. Furthermore, u_j is essentially smooth.

Define

$$J_u(t_0,(W_j);(c_{1,j}(\cdot),\theta_j(\cdot))) \quad = \quad E\left[\int_{t_0}^{T} u_j(t,c_{1,j}(t))\mathrm{d}t + U_j(W_j(T))\right].$$

The value function of the optimization problem (6) is

$$V_{u_j}(t_0,(W_j)) = \sup_{(c_1(\cdot),\theta(\cdot))\in\mathcal{A}_j(t_0,T)} J_{u_j}(t_0,W_j;(c_{1,j}(\cdot),\theta_j(\cdot))). \tag{19}$$

$$V_{u_j}(T,(W_j)) = U_j(W_j) \tag{20}$$

where

$$\begin{aligned}
& \mathcal{A}_j(t_0,T) \\
= & \Big\{(c_{j,1}(\cdot),\theta_j(\cdot)) \in C\times\Theta : W_j(t) = W_j(0) + \\
& + \int_{t_0}^{T}\theta_j(t)dX(t) - \int_{t_0}^{T} c_{1,j}(t)S_1(t)\mathrm{d}t, \\
& c_{1,j}(t)\geq 0, \theta_{F_2,j}(t,t) = 0, t_0\leq t\leq T\Big\} \\
& \int_{t_0}^{T}\theta_j(t)dX(t) \\
= & \int_{t_0}^{T}\theta_{P_0,j}(t)dP_0(t) + \sum_{T_m\in\#F_2(t_0,T)}\int_{t_0}^{T_m}\theta_{P,j}(t,T_m)dP(t,T_m) \\
& + \sum_{T_m\in\#F_2(t_0,T)}\int_{t_0}^{T_m}\theta_{F_2,j}(t,T_m)dF_2(t,T_m)
\end{aligned}$$

We assume the following condition.

Assumption 7. $V_{u_j}(t,W) \in C^{1,3}([T_{M-1},T_M]\times\mathbb{R})$ for each time interval $[T_{M-1},T_M]$ and $\partial_{tW}V_{u_j}$ is a continuous function.

The following result is a modification of intertemporal asset pricing theory.[16]

Proposition 3.4. Let Assumption 6 and 7 hold. Suppose there exists a consumer who faces optimization problem (6) and there exists an optimal solution and assume that the optimal consumption $c_1^*(t)$ is positive. Furthermore, assume that all the wealth at time T is consumed, i.e. $W^*(T) = C_1^*(T)S_1(T)$ and

[16]See for example Duffie (2001), Chapter 10, Section G.

define $U_{T,j}(C) = U_j(W)$ where $W = C \cdot S(T)$. Then

$$F_2(t,T) = E_{P_N}\left[\frac{\partial_c U_{T,j}(C_1^*(T))/S_1(T)}{\partial_c u_j(t,c_1^*)/S_1(t)} S_2(T)\middle|\mathcal{F}_t\right] \tag{21}$$

Proof. See the Appendix. □

$S_1(t)$ is used as a numeraire price. It is the usual intertemporal price relation. The convenience yield does not explicitly appear in the above result. However, if we compare (12) and (21), we may interpret that convenience yields are included in the marginal utility.

Corollary 3.2. Let Assumptions 1–6 hold. Suppose that there exists an optimal solution for problems (2) and (6). Assume that the optimal consumption $c_1^*(t)$ and the wealth process $W^*(t)$ is positive. Then

$$\begin{aligned} & E_{P_N}\left[\frac{\partial_c U_{T,j}(C_1^*)/S_1(T)}{\partial_c u_j(t,c_1^*)/S_1(t)} S_2(T)\middle|\mathcal{F}_t\right] - E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} S_2(T)|\mathcal{F}_t\right] P(t,T)^{-1} \\ = & -E\left[\int_T^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x_{S_2}^*(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] P(t,T)^{-1} - \lambda_{F_2,0}(t,T) \end{aligned}$$

This corollary states that the differences in the valuation of forward prices between a risk-neutral entity (a firm) and a risk averse entity (a speculator) consist of the future marginal storage cost plus the convenience yield on forward. Another interpretation is that a part of convenience yield is implicitly included in the intertemporal adjustment term $\frac{\partial U_{T,j}(C_1^*(T))/\partial c_1}{\partial u_j(t,c_1^*(t))/\partial c_1} \frac{S_2(T)}{S_1(T)/S_1(t)}$.

4 Conclusion

In this paper, we modeled a firm which uses an input commodity to produce an output commodity and also trades forward or futures on the input commodity in a continuous-time framework. The firm can also store input commodities by paying storage costs. This extends Nakajima model (2015) which was modeled using discrete-time. We compared the result with the Gibson-Schwartz model, the Schwartz model, the Miltersen-Schwartz model, and others. Although our model can be compared to existing models such as the Gibson-Schwartz model or the Schwartz model, our model does not assume any dynamics of the convenience yield explicitly. We analyzed our model numerically under the Gibson-Schwartz model.

The model implied that the optimal Lagrange multiplier can be deemed as the convenience yield. We also derived the dynamic of the optimal Lagrange multiplier.

Furthermore, we derived the optimal production and trading strategy for spot commodities and forward. We introduced two models which consider forward under cash settlement, futures, and hedging using futures on an output commodity. Our model can be generalized to include multiple-input and multiple-output models and we still have the same result.

If we introduce the speculator, we can see that convenience yields are implicitly included in the intertemporal adjustment term. In other words, the valuation of commodity forward and futures can be done in two aspects, which include a risk-averse agent (speculator-consumer) and a risk-neutral agent (firm).

For future analysis, our model can be incorporated into a general equilibrium analysis by introducing a multiple-input and multiple-output model. We can analyze the storage effect through demand and supply analysis. Furthermore, it is interesting to relax the complete market assumption and investigate how it affects the spot, the forward, and the futures price relation with convenience yields.

A Proof of Proposition 3.1

Let us define

$$
\begin{aligned}
& J_\pi(t_0, x; q(\cdot)) \\
= \; & \mathrm{E}\Bigg[\int_{t_0}^{T} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t)))\mathrm{d}t \\
& - \int_{t_0}^{T} \sum_{T_m \in \# F_2(t,T)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t, T_m) F_2(t, T_m)\mathrm{d}t \\
& + \mathrm{e}^{-\int_0^{T} r(u)\mathrm{d}u} x_{S_2}(T) S_2(T) \Bigg].
\end{aligned}
$$

The dynamic programming principle[17] states

$$
\begin{aligned}
& V_\pi(t_0, x) \\
= \; & \sup_{q(\cdot) \in \mathcal{Q}(t_0, T)} E\Bigg[\int_{t_0}^{t_1} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} f(t, x(t; t_0, x, q(\cdot)), q(t))\mathrm{d}t \\
& - \int_{t_0}^{t_1} \sum_{T_m \in \# F_2(t,t_1)} \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t, T_m) F_2(t, T_m)\mathrm{d}t \\
& + V_\pi(t_1, x(t_1; t_0, x, q(\cdot))) \Bigg], 0 \le t_0 \le t_1 \le T. \qquad \text{(A.1)}
\end{aligned}
$$

where

$$
f(t, x(t), q(t)) \quad = \quad (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t)))
$$

We start from the last period $[T_{M-1}, T_M]$ which the corresponding optimal control problem is

$$
\begin{aligned}
\sup_{q \in \mathcal{Q}(T_{M-1}, T_M)} \quad & \mathrm{E}\Bigg[\int_{T_{M-1}}^{T_M} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) \\
& - R(x_{S_2}(t), S_3(t)))\mathrm{d}t - \mathrm{e}^{-\int_0^{T_M} r(u)\mathrm{d}u} q_{F_2,b}(t, T_M) F_2(t, T_M)\mathrm{d}t \\
& + \mathrm{e}^{-\int_0^{T_M} r(u)\mathrm{d}u} x_{S_2}(T_M) S_2(T_M) \Bigg]. \qquad \text{(A.2)}
\end{aligned}
$$

[17]For the dynamic programming principle, see Nisio (2015), Chapter 2, Section 2, Proposition 2.4, and Fleming and Soner (2006), Chapter IV, Section IV.7, Corollary 7.2 and Remark 7.1.

where

$$
\begin{aligned}
& \mathcal{Q}(T_{M-1}, T_M) \\
= \; & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t) \geq 0, T_{M-1} \leq t \leq T_M, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, T_{M-1} \leq t \leq T_M, \\
& 0 \leq x_{F_2}(t, T_M) \leq K, T_{M-1} \leq t \leq T_M, \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, T_{M-1} \leq t \leq T_M, \\
& L_{F_2,b} \leq q_{F_2,b}(t, T_M) \leq K_{F_2,b}, T_{M-1} \leq t \leq T_M\},
\end{aligned}
$$

The Hamilton-Jacobi-Bellman equation[18] for (A.2) is

$$
\partial_t V_\pi(t, x) + \sup_{q \in Q(T_{M-1}, T_M)} G_{T_M}(t, x, q, \partial_x V_\pi(t, x), \partial_{xx} V_\pi(t, x), V_\pi(t, x)) = 0 \tag{A.3}
$$

$$
V_\pi(T_M, x) = \mathrm{e}^{-\int_0^{T_M} r(u)\mathrm{d}u}(x_{S_2} + x_{F_2, T_M}) S_2(T_M), x \in \mathbb{R} \tag{A.4}
$$

where ∂_x and ∂_{xx} are the partial derivatives and

$$
\begin{aligned}
& G_{T_M}(t, x, q, V_1, V_2, V) \\
= \; & \frac{1}{2}\mathrm{tr}(V_2 \sigma_x(t, x) \sigma_x(t, x)^\top) + V_1 \cdot \mu_x(t, x, q) \\
& +(p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t))) \\
& -E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \middle| \mathcal{F}_t\right] q_{F_2,b}(t, T_M) F_2(t, T_M) - r(t) V, \\
& Q(T_{M-1}, T_M) \\
= \; & \{q : (q_{S_2,b}(t) - q_{S_2,u}(t)) 1_{x_{S_2}(t)=0} \geq 0, T_{M-1} \leq t \leq T_M, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, T_{M-1} \leq t \leq T_M, \\
& q_{F_2,b}(t, T_M) 1_{K \geq x_{F_2}(t, T_M)} \leq 0, q_{F_2,b}(t, T_M) 1_{x_{F_2}(t, T_M) \geq 0} \geq 0, T_{M-1} \leq t \leq T_M \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, T_{M-1} \leq t \leq T_M, \\
& L_{F_2,b} \leq q_{F_2,b}(t, T_M) \leq K_{F_2,b}, T_{M-1} \leq t \leq T_M\}.
\end{aligned}
$$

Here $1_{x_{S_2}(t)=0}, 1_{K \geq x_{F_2}(t, T_M)}$, and $1_{x_{F_2}(t, T_M) \geq 0}$ are indicator functions. Note that $0 \leq x_{S_2}(t)$ and $0 \leq x_{F_2}(t, T) \leq K, 0 \leq t \leq T$ are equivalent to $(q_{S_2,b}(t) - q_{S_2,u}(t)) 1_{x_{S_2}(t)=0} \geq 0$ and $q_{F_2,b}(t, T) 1_{K \geq x_{F_2}(t, T)} \leq 0, q_{F_2,b}(t, T) 1_{x_{F_2}(t, T) \geq 0} \geq 0, 0 \leq t \leq T$ which was also used by Presman, Sethi, and Zhang (1995).

We solve the following optimization problem

$$
\sup_{q \in Q(T_{M-1}, T_M)} G_{T_M}(t, x, q, \partial_x V_\pi(t, x), \partial_{xx} V_\pi(t, x), V_\pi(t, x))
$$

[18]For the Hamilton-Jacobi-Bellman equation, see Nisio (2015), Chapter 2, Section 2, p.56 and Fleming and Soner (2006), Chapter IV, Section IV.4, Theorem 4.1, and Chapter IV, Section IV.3, Remark 3.3.

which the first order condition is

$$\begin{aligned} 0 &= \partial_{x_{S_2}} V_\pi(t,x) - S_2(t) + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) && \text{(A.5)}\\ 0 &= -\partial_{x_{S_2}} V_\pi(t,x) + p'(q^*_{S_2,u}(t))S_1(t) + \lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x) && \text{(A.6)}\\ 0 &= \partial_{x_{F_2,T_M}} V_\pi(t,x) - E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] F_2(t,T_M) + \lambda_{F_2,x_l}(t,T_M) \\ &\quad -\lambda_{F_2,x_u}(t,T_M) + \lambda_{F_2,q_b,l}(t,T_M) - \lambda_{F_2,q_b,u}(t,T_M) && \text{(A.7)} \end{aligned}$$

From the Hamilton-Jacobi-Bellman equation (A.3), we have

$$\begin{aligned} & G_{T_M}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)) \\ = \; & 0 \\ \geq \; & G_{T_M}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x)) + \partial_t V_\pi(t,x). \end{aligned}$$

where $(x^*(t),q^*(t))$ is the optimal solution.[19] Since $V_\pi \in C^{1,3}([T_{M-1},T_M] \times \mathbb{R}^{5+2\#F_2(T_{M-1},T_M)})$ and $\partial_{tx}V_\pi$ being continuous, we have

$$\begin{aligned} & 0 \\ = \; & \partial_x G_{T_M}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)). \end{aligned}$$

By the definition of G_{T_M},

$$\begin{aligned} 0 &= \partial_{tx}V_\pi(t,x^*(t)) + \partial_{xx}V_\pi(t,x^*(t))\mu_x(t,x^*(t),q^*(t)) \\ &\quad +\partial_x\mu_x(t,x^*(t),q^*(t))\partial_x V_\pi(t,x^*(t)) \\ &\quad +\frac{1}{2}\mathrm{tr}\left(\sigma_x(t,x^*(t),q^*(t))^\top \partial_{xxx}V(t,x^*(t))\sigma_x(t,x^*(t),q^*(t))\right) \\ &\quad +\sum_{j=1}^{d}\left(\partial_x\sigma_x^j(t,x^*(t),q^*(t))\right)^\top\left(\partial_{xx}V_\pi(t,x^*(t))\sigma_x(t,x^*(t),q^*(t))\right)^j \\ &\quad +V_\pi(t,x^*(t))\partial_x r(t) + r(t)\partial_x V_\pi(t,x^*(t)) + \partial_x f_{T_M}(t,x^*(t),q^*(t)), \end{aligned}$$

where

$$\begin{aligned} & f_{T_M}(t,x(t),q(t)) \\ = \; & (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t))) \\ & -E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] q_{F_2,b}(t,T_M)F_2(t,T_M), \\ & \mathrm{tr}(\sigma_x^\top \partial_{xxx}V_\pi\sigma_x) \\ = \; & \left(\mathrm{tr}(\sigma_x^\top\left(\partial_{xx}(\partial_x V_\pi)^1\right)\sigma_x),\cdots,\mathrm{tr}(\sigma_x^\top\left(\partial_{xx}\left(\partial_x V_\pi\right)^n\right)\sigma_x)\right)^\top \end{aligned}$$

and

$$\partial_x V_\pi = \left(\left(\partial_x V_\pi\right)^1,\cdots,\left(\partial_x V_\pi\right)^n\right)^\top$$

[19] The argument here is a modification of some part of a proof from Yong and Zhou (1999), Chapter 5, Section 4.1, pp.252-253.

For $\partial_{x_{S_2}} V_\pi$ and $\partial_{x_{F_2,T_M}} V_\pi$, we have

$$\begin{aligned}
&\partial_{x_{S_2} t} V_\pi(t, x^*(t)) + \partial_{x_{S_2} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
&+\frac{1}{2}\mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{S_2} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right) \\
&-r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t)) = 0 \qquad \text{(A.8)} \\
&\partial_{x_{F_2,T_M} t} V_\pi(t, x^*(t)) + \partial_{x_{F_2,T_M} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
&+\frac{1}{2}\mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{F_2,T_M} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right) \\
&-r(t)\partial_{x_{F_2,T_M}} V_\pi(t, x^*(t)) = 0 \qquad \text{(A.9)}
\end{aligned}$$

Applying Feynman-Kac formula[20] for (A.8) and (A.9), we have

$$\begin{aligned}
&\partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_M, x^*(T_M)) \middle| \mathcal{F}_t\right] \\
&-E\left[\int_t^{T_M} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
= \;& E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] \\
&-E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
&\partial_{x_{F_2,T_M}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \partial_{x_{F_2,T_M}} V_\pi(T_M, x^*(T_M)) \middle| \mathcal{F}_t\right] \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_M, x^*(T_M)) \middle| \mathcal{F}_t\right] \\
= \;& E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right].
\end{aligned}$$

Substituting this into equation (A.5) and (A.7), we have

$$\begin{aligned}
S_2(t) \;=\; & E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
&-E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{(A.10)} \\
F_2(t,T) \;=\; & \left[E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) | \mathcal{F}_t\right] + \lambda_{F_2,x_l}(t,T_M) - \lambda_{F_2,x_u}(t,T_M)\right. \\
&\left.+\lambda_{F_2,q_b,l}(t,T_M) - \lambda_{F_2,q_b,u}(t,T_M)\right]\left(E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} | \mathcal{F}_t\right]\right)^{-1} \qquad \text{(A.11)}
\end{aligned}$$

[20]For Feynman-Kac formula, see Pham (2009), Theorem 1.3.17.

which also derives

$$\begin{aligned}
S_2(t) \quad = \quad & E\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] F_2(t,T) \\
& -E\left[\int_t^T \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) \\
& -\lambda_{S_2,q_b,u}(t) - \lambda_{F_2,x_l}(t,T_M) + \lambda_{F_2,x_u}(t,T_M) \\
& -\lambda_{F_2,q_b,l}(t,T_M) + \lambda_{F_2,q_b,u}(t,T_M) \qquad \text{(A.12)}
\end{aligned}$$

Furthermore, from equation (A.5) and (A.6) we have

$$S_2(t) = p'(q^*_{S_2,u}(t))S_1(t) + (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x)) \qquad \text{(A.13)}$$

By induction and the dynamic programming principle (A.1), we consider the following optimization problem.

$$\begin{aligned}
\sup_{q\in\mathcal{Q}(T_{m-1},T_m)} \quad & \mathrm{E}\Bigg[\int_{T_{m-1}}^{T_m} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) \\
& -R(x_{S_2}(t), S_3(t))) - \mathrm{e}^{-\int_0^{T_m} r(u)\mathrm{d}u} q_{F_2,b}(t,T_m)F_2(t,T_m)\mathrm{d}t \\
& +V_\pi(T_m, x(T_m;T_{m-1}, x(T_{m-1}), q(\cdot)))\Bigg]. \qquad \text{(A.14)}
\end{aligned}$$

where

$$\begin{aligned}
& \mathcal{Q}(T_{m-1}, T_m) \\
= \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t) \geq 0, T_{m-1} \leq t \leq T_m, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, T_{m-1} \leq t \leq T_m, \\
& 0 \leq x_{F_2}(t,T_m) \leq K, T_{m-1} \leq t \leq T_m\}, \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, T_{m-1} \leq t \leq T_m, \\
& L_{F_2,b} \leq q_{F_2,b}(t,T_m) \leq K_{F_2,b}, T_{m-1} \leq t \leq T_m\}.
\end{aligned}$$

and the Hamilton-Jacobi-Bellman equation for (A.14) is

$$\partial_t V_\pi(t,x) + \sup_{q\in Q(T_{m-1},T_M)} G_{T_m}(t,x,q,\partial_x V_\pi(t,x), \partial_{xx}V_\pi(t,x), V_\pi(t,x)) = 0 \qquad \text{(A.15)}$$

where

$$
\begin{aligned}
& G_{T_m}(t,x,q,V_1,V_2,V) \\
= \quad & \frac{1}{2}\mathrm{tr}(V_2\sigma_x(t,x,q)\sigma_x(t,x,q)^\top) + V_1\cdot\mu_x(t,x,q) \\
& +(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t))) \\
& -E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] q_{F_2,b}(t,T_m)F_2(t,T_m) - r(t)V, \\
& Q(T_{m-1},T_m) \\
= \quad & \{q : (q_{S_2,b}(t) - q_{S_2,u}(t))1_{x_{S_2}(t)=0} \geq 0, T_{m-1}\leq t\leq T_m, \\
& 0\leq q_{S_2,u}(t)\leq K_{S_2,u}, T_{m-1}\leq t\leq T_m, \\
& q_{F_2,b}(t,T_m)1_{K\geq x_{F_2}(t,T_m)}\leq 0, q_{F_2,b}(t,T_m)1_{x_{F_2}(t,T_m)\geq 0}\geq 0, T_{m-1}\leq t\leq T_m \\
& L_{S_2,b}\leq q_{S_2,b}(t)\leq K_{S_2,b}, T_{m-1}\leq t\leq T_m, \\
& L_{F_2,b}\leq q_{F_2,b}(t,T_m)\leq K_{F_2,b}, T_{m-1}\leq t\leq T_m\}.
\end{aligned}
$$

and $\mathcal{S}^d$ denote the set of symmetric $d\times d$ matrices.

We have the following optimization problem

$$
\sup_{q\in Q(T_{m-1},T_m)} G_{T_m}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x(t)))
$$

which the first order condition is

$$
\partial_{x_{S_2}}V_\pi(t,x) - S_2(t) + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) = 0 \tag{A.16}
$$

$$
-\partial_{x_{S_2}}V_\pi(t,x) + p'(q^*_{S_2,u}(t))S_1(t) + \lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x) = 0 \tag{A.17}
$$

$$
\begin{aligned}
& \partial_{x_{F_2,T_m}}V_\pi(t,x) - E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] F_2(t,T_m) + \lambda_{F_2,x_l}(t,T_m) - \lambda_{F_2,x_u}(t,T_m) \\
& +\lambda_{F_2,q_b,l}(t,T_m) - \lambda_{F_2,q_b,u}(t,T_m) = 0
\end{aligned} \tag{A.18}
$$

From the Hamilton-Jacobi-Bellman equation (A.15), we have

$$
\begin{aligned}
& G_{T_m}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)) \\
= \quad & 0 \\
\geq \quad & G_{T_m}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x)) + \partial_t V_\pi(t,x).
\end{aligned}
$$

where $(x^*(t),q^*(t))$ is the optimal solution. Since $V_\pi\in C^{1,3}([T_{M-1},T_M]\times\mathbb{R}^{5+2\#F_2(T_{m-1},T_m)})$ and $\partial_{tx}V_\pi$ being continuous, we have

$$
\begin{aligned}
& 0 \\
= \quad & \partial_x G_{T_m}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)).
\end{aligned}
$$

By the definition of G_{T_m},

$$\begin{aligned}
&\partial_{tx} V_\pi(t, x^*(t)) + \partial_{xx} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
&+\partial_x \mu_x(t, x^*(t), q^*(t)) \partial_x V_\pi(t, x^*(t)) \\
&+\frac{1}{2} \mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{xxx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right) \\
&+\sum_{j=1}^{d} \left(\partial_x \sigma_x^j(t, x^*(t), q^*(t))\right)^\top \left(\partial_{xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right)^j \\
&-V_\pi(t, x^*(t)) \partial_x r(t) - r(t) \partial_x V_\pi(t, x^*(t)) + \partial_x f_{T_{M-1}}(t, x^*(t), q^*(t)) = 0,
\end{aligned} \tag{A.19}$$

where

$$\begin{aligned}
& f_{T_m}(t, x(t), q(t)) \\
= \;& (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t))) \\
& -E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \middle| \mathcal{F}_t\right] q_{F_2,b}(t, T_m) F_2(t, T_m).
\end{aligned}$$

For $\partial_{x_{S_2}} V_\pi$ and $\partial_{x_{F_2,T_m}} V_\pi$, we have

$$\begin{aligned}
&\partial_{x_{S_2} t} V_\pi(t, x^*(t)) + \partial_{x_{S_2} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
&+\frac{1}{2} \mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{S_2} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right) \\
&-r(t) \partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t)) = 0 \qquad (A.20) \\
&\partial_{x_{F_2,T_m} t} V_\pi(t, x^*(t)) + \partial_{x_{F_2,T_m} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
&+\frac{1}{2} \mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{F_2,T_m} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t))\right) \\
&-r(t) \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) = 0 \qquad (A.21)
\end{aligned}$$

Since

$$x_{S_2}(t) = x_{S_2,0} + \int_0^t q_{S_2,b}(s) - q_{S_2,u}(s) \mathrm{d}s + \sum_{T_m \in \# F_2(0,t)} x_{F_2}(T_m, T_m),$$

it is easy to see that

$$J_\pi(T_m, x_{T_m} + h_{S_2}; q(\cdot)) = J_\pi(T_m, x_{T_m} + h_{F_2,T_m}; q(\cdot))$$

where h_{S_2} have h in the first element and other elements are zero and h_{F_2,T_m} have h in the $1 + m$ th element and other elements are zero. Therefore, $V_\pi(T_m, x_{T_m} + h_{S_2}) = V_\pi(T_m, x_{T_m} + h_{F_2,T_m})$ and thus $\partial_{x_{S_2}} V_\pi(T_m, x_{T_m}) = \partial_{x_{F_2,T_m}} V_\pi(T_m, x_{T_m})$.

Applying Feynman-Kac formula for (A.20) and (A.21), we have

$$
\begin{aligned}
& \partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_m, x^*(T_m)) \middle| \mathcal{F}_t\right] \\
& - E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
& \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \partial_{x_{F_2,T_m}} V_\pi(T_m, x^*(T_m)) \middle| \mathcal{F}_t\right] \\
= \;& E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_m, x^*(T_m)) \middle| \mathcal{F}_t\right]
\end{aligned}
$$

Furthermore, by induction

$$
\begin{aligned}
& \partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] \\
& - E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{(A.22)} \\
& \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) \\
= \;& E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] \\
& - E\left[\int_{T_m}^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{(A.23)}
\end{aligned}
$$

Substituting this into equation (A.16) and (A.18), we have

$$
\begin{aligned}
S_2(t) \;=\;& E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
& - E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{(A.24)} \\
F_2(t, T_m) \;=\;& \Bigg(E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) | \mathcal{F}_t\right] \\
& - E\left[\int_{T_m}^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
& + \lambda_{F_2,x_l}(t, T_m) - \lambda_{F_2,x_u}(t, T_m) + \lambda_{F_2,q_b,l}(t, T_m) - \lambda_{F_2,q_b,u}(t, T_m) \Bigg) \\
& \cdot \left(E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} | \mathcal{F}_t\right] \right)^{-1} \qquad \text{(A.25)}
\end{aligned}
$$

which also derives

$$
\begin{aligned}
S_2(t) \quad = \quad & E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}\middle|\mathcal{F}_t\right] F_2(t,T_m) \\
& -E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] + \lambda_{S_2,x}(t) \\
& +\lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) - \lambda_{F_2,x_l}(t,T_m) + \lambda_{F_2,x_u}(t,T_m) \\
& -\lambda_{F_2,q_b,l}(t,T_m) + \lambda_{F_2,q_b,u}(t,T_m) \qquad \text{(A.26)}
\end{aligned}
$$

Again, from equation (A.16) and (A.17) we have

$$
S_2(t) \quad = \quad p'(q^*_{S_2,u}(t))S_1(t) + (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x)) \qquad \text{(A.27)}
$$

B Proof of Proposition 3.2

Let us define

$$
\begin{aligned}
& J_\pi(t_0,x;q(\cdot)) \\
= \quad & \mathrm{E}\left[\int_{t_0}^{T} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))\mathrm{d}t\right. \\
& + \int_{t_0}^{T} \sum_{T_m\in\#G_2(t,T)} \int_t^{T_m} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_m) dG_2(s,T_m)\mathrm{d}t \\
& \left. +\mathrm{e}^{-\int_0^T r(u)\mathrm{d}u} x_{S_2}(T)S_2(T)\right].
\end{aligned}
$$

The value function of the optimization problem (4) is

$$
\begin{aligned}
V_\pi(t_0,x) \quad &= \quad \sup_{q(\cdot)\in\mathcal{Q}(t_0,T)} J(t_0,x;q(\cdot)). \\
V_\pi(T,x) \quad &= \quad \mathrm{e}^{-\int_{t_0}^T r(u)\mathrm{d}u}(x_{S_2})S_2(T), x\in\mathbb{R}^{5+2\#G_2(0,T)} \qquad \text{(B.28)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Q}(t_0,T) \quad = \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t)\geq 0, t_0\leq t\leq T, \\
& 0\leq q_{S_2,u}(t)\leq K_{S_2,u}, t_0\leq t\leq T, \\
& 0\leq x_{G_2}(t,T_m)\leq K, t_0\leq t\leq T, T_m\in\#G_2(t,T), \\
& L_{S_2,b}\leq q_{S_2,b}(t)\leq K_{S_2,b}, t_0\leq t\leq T, \\
& L_{G_2,b}\leq q_{G_2,b}(t,T_m)\leq K_{G_2,b}, t_0\leq t\leq T, T_m\in\#G_2(t,T)\}
\end{aligned}
$$

and $x = (x_{S_2}, (x_{G_2,T_m})_{T_m\in\#G_2(0,T)}, r, S_1, S_2, S_3, (G_2)_{T_m\in\#G_2(0,T)})$.

The dynamic programming principle states

$$\begin{aligned}
& V_\pi(t_0,x) \\
= \quad & \sup_{q(\cdot)\in\mathcal{Q}(t_0,T)} E\Bigg[\int_{t_0}^{t_1} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u} f(t,x(t;t_0,x,q(\cdot)),q(t))\mathrm{d}t \\
& + \int_{t_0}^{t_1} \sum_{T_m\in\#G_2(t,t_1)} \int_t^{T_m} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_m) dG_2(s,T_m)\mathrm{d}t \\
& + V_\pi(t_1,x(t_1;t_0,x,q(\cdot)))\Bigg], 0\le t_0\le t_1\le T. \qquad \text{(B.29)}
\end{aligned}$$

where

$$\begin{aligned}
& f(t,x(t),q(t)) \\
= \quad & (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t)))
\end{aligned}$$

We start from the last period $[T_{M-1},T_M]$ which the corresponding optimal control problem is

$$\begin{aligned}
\sup_{q\in\mathcal{Q}(T_{M-1},T_M)} \quad & \mathrm{E}\Bigg[\int_{T_{M-1}}^{T_M} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) \\
& - R(x_{S_2}(t),S_3(t)))\mathrm{d}t \\
& + \int_{T_{M-1}}^{T_M}\int_t^{T_M} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_M)dG_2(s,T_M)\mathrm{d}t \\
& + \mathrm{e}^{-\int_0^{T_M} r(u)\mathrm{d}u} x_{S_2}(T_M)S_2(T_M)\Bigg]. \qquad \text{(B.30)}
\end{aligned}$$

where

$$\begin{aligned}
& \mathcal{Q}(T_{M-1},T_M) \\
= \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t)\ge 0, T_{M-1}\le t\le T_M, \\
& 0\le q_{S_2,u}(t)\le K_{S_2,u}, T_{M-1}\le t\le T_M, \\
& 0\le x_{G_2}(t,T_M)\le K, T_{M-1}\le t\le T_M, \\
& L_{S_2,b}\le q_{S_2,b}(t)\le K_{S_2,b}, T_{M-1}\le t\le T_M, \\
& L_{G_2,b}\le q_{G_2,b}(t,T_M)\le K_{G_2,b}, T_{M-1}\le t\le T_M\},
\end{aligned}$$

The Hamilton-Jacobi-Bellman equation for (B.30) is

$$\partial_t V_\pi(t,x) + \sup_{q\in Q(T_{M-1},T_M)} G_{T_M}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x)) = 0 \qquad \text{(B.31)}$$

$$V_\pi(T_M,x) = \mathrm{e}^{-\int_0^{T_M} r(u)\mathrm{d}u} x_{S_2} S_2(T_M), x\in\mathbb{R} \qquad \text{(B.32)}$$

where ∂_x and ∂_{xx} are the partial derivatives and

$$\begin{aligned}
& G_{T_M}(t,x,q,V_1,V_2,V) \\
= \; & \frac{1}{2}\mathrm{tr}(V_2\sigma_x(t,x)\sigma_x(t,x)^\top) + V_1\cdot\mu_x(t,x,q) \\
& +(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t))) \\
& +E\left[\int_t^{T_M} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_M)dG_2(s,T_M)\middle|\mathcal{F}_t\right] \\
& -r(t)V, \\
& Q(T_{M-1},T_M) \\
= \; & \{q : (q_{S_2,b}(t) - q_{S_2,u}(t))1_{x_{S_2}(t)=0} \geq 0, T_{M-1}\leq t\leq T_M, \\
& 0\leq q_{S_2,u}(t)\leq K_{S_2,u}, T_{M-1}\leq t\leq T_M, \\
& q_{G_2,b}(t,T_M)1_{K\geq x_{G_2}(t,T_M)}\leq 0, q_{G_2,b}(t,T_M)1_{x_{G_2}(t,T_M)\geq 0}\geq 0, T_{M-1}\leq t\leq T_M \\
& L_{S_2,b}\leq q_{S_2,b}(t)\leq K_{S_2,b}, T_{M-1}\leq t\leq T_M, \\
& L_{G_2,b}\leq q_{G_2,b}(t,T_M)\leq K_{G_2,b}, T_{M-1}\leq t\leq T_M\}.
\end{aligned}$$

We solve the following optimization problem

$$\sup_{q\in Q(T_{M-1},T_M)} G_{T_M}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x))$$

which the first order condition is

$$0 = \partial_{x_{S_2}}V_\pi(t,x) - S_2(t) + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \tag{B.33}$$

$$0 = -\partial_{x_{S_2}}V_\pi(t,x) + p'(q^*_{S_2,u}(t))S_1(t) + \lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x) \tag{B.34}$$

$$\begin{aligned}
0 = \; & \partial_{x_{G_2,T_M}}V_\pi(t,x) + E\left[\int_t^{T_M}\mathrm{e}^{-\int_0^s r(u)\mathrm{d}u}dG_2(s,T_M)\middle|\mathcal{F}_t\right] \\
& +\lambda_{G_2,x_l}(t,T_M) - \lambda_{G_2,x_u}(t,T_M) + \lambda_{G_2,q_b,l}(t,T_M) - \lambda_{G_2,q_b,u}(t,T_M)
\end{aligned}\tag{B.35}$$

From the Hamilton-Jacobi-Bellman equation (B.31), we have

$$\begin{aligned}
& G_{T_M}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)) \\
= \; & 0 \\
\geq \; & G_{T_M}(t,x,q,\partial_x V_\pi(t,x),\partial_{xx}V_\pi(t,x),V_\pi(t,x)) + \partial_t V_\pi(t,x).
\end{aligned}$$

where $(x^*(t),q^*(t))$ is the optimal solution.[21] Since $V_\pi\in C^{1,3}([T_{M-1},T_M]\times\mathbb{R}^{5+2\#G_2(T_{M-1},T_M)}$ and $\partial_{tx}V_\pi$ being continuous, we have

$$\begin{aligned}
& 0 \\
= \; & \partial_x G_{T_M}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)),\partial_{xx}V_\pi(t,x^*(t)),V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)).
\end{aligned}$$

[21] The argument here is a modification of some part of a proof from Yong and Zhou (1999), Chapter 5, Section 4.1, pp.252-253.

By the definition of G_{T_M},

$$\begin{aligned}
0 \quad = \quad & \partial_{tx} V_\pi(t, x^*(t)) + \partial_{xx} V_\pi(t, x^*(t))\mu_x(t, x^*(t), q^*(t)) \\
& + \partial_x \mu_x(t, x^*(t), q^*(t))\partial_x V_\pi(t, x^*(t)) \\
& + \frac{1}{2}\mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{xxx} V(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))\right) \\
& + \sum_{j=1}^{d}\left(\partial_x \sigma_x^j(t, x^*(t), q^*(t))\right)^\top \left(\partial_{xx} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))\right)^j \\
& + V_\pi(t, x^*(t))\partial_x r(t) + r(t)\partial_x V_\pi(t, x^*(t)) + \partial_x f_{T_M}(t, x^*(t), q^*(t)),
\end{aligned}$$

where

$$\begin{aligned}
& f_{T_M}(t, x(t), q(t)) \\
= \quad & (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t))) \\
& + E\left[\int_t^{T_M} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t, T_M) dG_2(s, T_M)\middle|\mathcal{F}_t\right], \\
& \mathrm{tr}(\sigma_x^\top \partial_{xxx} V_\pi \sigma_x) \\
= \quad & \left(\mathrm{tr}(\sigma_x^\top \left(\partial_{xx}(\partial_x V_\pi)^1\right)\sigma_x), \cdots, \mathrm{tr}(\sigma_x^\top \left(\partial_{xx}\left(\partial_x V_\pi\right)^n\right)\sigma_x)\right)^\top
\end{aligned}$$

and

$$\partial_x V_\pi = \left(\left(\partial_x V_\pi\right)^1, \cdots, \left(\partial_x V_\pi\right)^n\right)^\top$$

For $\partial_{x_{S_2}} V_\pi$ and $\partial_{x_{G_2,T_M}} V_\pi$, we have

$$\begin{aligned}
& \partial_{x_{S_2} t} V_\pi(t, x^*(t)) + \partial_{x_{S_2} x} V_\pi(t, x^*(t))\mu_x(t, x^*(t), q^*(t)) \\
& + \frac{1}{2}\mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{S_2} xx} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))\right) \\
& - r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t)) = 0 \qquad \text{(B.36)} \\
& \partial_{x_{G_2,T_M} t} V_\pi(t, x^*(t)) + \partial_{x_{G_2,T_M} x} V_\pi(t, x^*(t))\mu_x(t, x^*(t), q^*(t)) \\
& + \frac{1}{2}\mathrm{tr}\left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{G_2,T_M} xx} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))\right) \\
& - r(t)\partial_{x_{G_2,T_M}} V_\pi(t, x^*(t)) = 0 \qquad \text{(B.37)}
\end{aligned}$$

Applying Feynman-Kac formula[22] for (B.36) and (B.37), we have

$$\begin{aligned}
& \partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= \ & E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_M, x^*(T_M)) \Big| \mathcal{F}_t\right] \\
& - E\left[\int_t^{T_M} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \Bigg| \mathcal{F}_t\right] \\
= \ & E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \Bigg| \mathcal{F}_t\right] \\
& - E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \Bigg| \mathcal{F}_t\right] \qquad \text{(B.38)}
\end{aligned}$$

$$\begin{aligned}
& \partial_{x_{G_2, T_M}} V_\pi(t, x^*(t)) \\
= \ & E\left[\mathrm{e}^{-\int_t^{T_M} r(u)\mathrm{d}u} \partial_{x_{G_2, T_M}} V_\pi(T_M, x^*(T_M)) \Big| \mathcal{F}_t\right] \\
= \ & 0. \qquad \text{(B.39)}
\end{aligned}$$

Substituting equation (B.38) into equation (B.33), we have

$$\begin{aligned}
S_2(t) = \ & E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \Big| \mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
& - E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \Bigg| \mathcal{F}_t\right] \qquad \text{(B.40)}
\end{aligned}$$

On the other hand, since futures are martingale under risk-neutral probability,

$$E\left[\int_t^{T_m} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} dG_2(s, T_m) \Bigg| \mathcal{F}_t\right] = 0$$

$$0 = \lambda_{G_2,x_l}(t, T_M) - \lambda_{G_2,x_u}(t, T_M) + \lambda_{G_2,q_b,l}(t, T_M) - \lambda_{G_2,q_b,u}(t, T_M)$$

from equation (B.35).

Therefore

$$\begin{aligned}
& G_2(t, T) \\
= \ & \Bigg(S_2(t) - \mathrm{Cov}\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u}, S_2(T) \Big| \mathcal{F}_t\right] - \lambda_{S_2,x}(t) - \lambda_{S_2,q_b,l}(t) + \lambda_{S_2,q_b,u}(t) \\
& + E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \Bigg| \mathcal{F}_t\right]\Bigg) \\
& \cdot \left(E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} \Big| \mathcal{F}_t\right]\right)^{-1} \qquad \text{(B.41)}
\end{aligned}$$

[22] For Feynman-Kac formula, see Pham (2009), Theorem 1.3.17.

Furthermore, from equation (B.33) and (B.34) we have

$$S_2(t) = p'(q^*_{S_2,u}(t))S_1(t) + (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x)) \quad \text{(B.42)}$$

By induction and the dynamic programming principle (B.29), we consider the following optimization problem.

$$\begin{aligned}
\sup_{q\in\mathcal{Q}(T_{m-1},T_m)} \quad & \mathrm{E}\Bigg[\int_{T_{m-1}}^{T_m} \mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) \\
& -R(x_{S_2}(t), S_3(t)))\mathrm{d}t \\
& + \int_{T_{m-1}}^{T_m} \sum_{T_k\in\#G_2(t,T)} \int_t^{T_k} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_k)dG_2(s,T_k)\mathrm{d}t \\
& +V_\pi(T_m, x(T_m; T_{m-1}, x(T_{m-1}), q(\cdot)))\Bigg]. \qquad \text{(B.43)}
\end{aligned}$$

where

$$\begin{aligned}
& \mathcal{Q}(T_{m-1},T_m) \\
= \quad & \{q : q \text{ is } \mathcal{F}_t\text{-adapted process }, x_{S_2}(t) \geq 0, T_{m-1} \leq t \leq T_m, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, T_{m-1} \leq t \leq T_m, \\
& 0 \leq x_{G_2}(t,T_m) \leq K, T_{m-1} \leq t \leq T_m, \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, T_{m-1} \leq t \leq T_m, \\
& L_{G_2,b} \leq q_{G_2,b}(t,T_m) \leq K_{G_2,b}, T_{m-1} \leq t \leq T_m\},
\end{aligned}$$

and the Hamilton-Jacobi-Bellman equation for (B.43) is

$$\partial_t V_\pi(t,x) + \sup_{q\in Q(T_{m-1},T_m)} G_{T_m}(t,x,q,\partial_x V_\pi(t,x), \partial_{xx}V_\pi(t,x), V_\pi(t,x)) = 0 \quad \text{(B.44)}$$

where

$$
\begin{aligned}
& G_{T_m}(t,x,q,V_1,V_2,V) \\
= \; & \frac{1}{2}\mathrm{tr}(V_2\sigma_x(t,x)\sigma_x(t,x)^\top) + V_1 \cdot \mu_x(t,x,q) \\
& +(p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t),S_3(t))) \\
& + \sum_{T_k \in \# G_2(t,T)} E\left[\int_t^{T_k} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} q_{G_2,b}(t,T_k)dG_2(s,T_k)\middle|\mathcal{F}_t\right] \\
& -r(t)V, \\
& Q(T_{m-1},T_m) \\
= \; & \{q : (q_{S_2,b}(t) - q_{S_2,u}(t))1_{x_{S_2}(t)=0} \geq 0, T_{m-1} \leq t \leq T_m, \\
& 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, T_{m-1} \leq t \leq T_m, \\
& q_{G_2,b}(t,T_m)1_{K \geq x_{G_2}(t,T_m)} \leq 0, q_{G_2,b}(t,T_m)1_{x_{G_2}(t,T_m)\geq 0} \geq 0, T_{m-1} \leq t \leq T_m \\
& L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, T_{m-1} \leq t \leq T_m, \\
& L_{G_2,b} \leq q_{G_2,b}(t,T_m) \leq K_{G_2,b}, T_{m-1} \leq t \leq T_m\}.
\end{aligned}
$$

We have the following optimization problem

$$\sup_{q \in Q(T_{m-1},T_m)} G_{T_m}(t,x,q,\partial_x V_\pi(t,x), \partial_{xx}V_\pi(t,x), V_\pi(t,x(t)))$$

which the first order condition is

$$
\begin{aligned}
0 &= \partial_{x_{S_2}} V_\pi(t,x) - S_2(t) + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) && \text{(B.45)} \\
0 &= -\partial_{x_{S_2}} V_\pi(t,x) + p'(q^*_{S_2,u}(t))S_1(t) + \lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x) && \text{(B.46)} \\
0 &= \partial_{x_{G_2,T_M}} V_\pi(t,x) + E\left[\int_t^{T_k} \mathrm{e}^{-\int_0^s r(u)\mathrm{d}u} dG_2(s,T_k)\middle|\mathcal{F}_t\right] \\
& \quad +\lambda_{G_2,x_l}(t,T_m) - \lambda_{G_2,x_u}(t,T_m) + \lambda_{G_2,q_b,l}(t,T_m) - \lambda_{G_2,q_b,u}(t,T_m) && \text{(B.47)}
\end{aligned}
$$

From the Hamilton-Jacobi-Bellman equation (B.44), we have

$$
\begin{aligned}
& G_{T_m}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)), \partial_{xx}V_\pi(t,x^*(t)), V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)) \\
= \; & 0 \\
\geq \; & G_{T_m}(t,x,q,\partial_x V_\pi(t,x), \partial_{xx}V_\pi(t,x), V_\pi(t,x)) + \partial_t V_\pi(t,x).
\end{aligned}
$$

where $(x^*(t), q^*(t))$ is the optimal solution. Since $V_\pi \in C^{1,3}([T_{m-1},T_m] \times \mathbb{R}^{5+2\# G_2(T_{m-1},T_m)}$ and $\partial_{tx}V_\pi$ being continuous, we have

$$
\begin{aligned}
& 0 \\
= \; & \partial_x G_{T_m}(t,x^*(t),q^*(t),\partial_x V_\pi(t,x^*(t)), \partial_{xx}V_\pi(t,x^*(t)), V_\pi(t,x^*(t))) + \partial_t V_\pi(t,x^*(t)).
\end{aligned}
$$

By the definition of G_{T_m},

$$
\begin{aligned}
0 \quad = \quad & \partial_{tx} V_\pi(t, x^*(t)) + \partial_{xx} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
& + \partial_x \mu_x(t, x^*(t), q^*(t)) \partial_x V_\pi(t, x^*(t)) \\
& + \frac{1}{2} \mathrm{tr} \left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{xxx} V(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)) \right) \\
& + \sum_{j=1}^{d} \left(\partial_x \sigma_x^j(t, x^*(t), q^*(t)) \right)^\top \left(\partial_{xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)) \right)^j \\
& + V_\pi(t, x^*(t)) \partial_x r(t) + r(t) \partial_x V_\pi(t, x^*(t)) + \partial_x f_{T_m}(t, x^*(t), q^*(t)),
\end{aligned}
\tag{B.48}
$$

where

$$
\begin{aligned}
& f_{T_m}(t, x(t), q(t)) \\
= \quad & (p(q_{S_2,u}(t)) S_1(t) - q_{S_2,b}(t) S_2(t) - R(x_{S_2}(t), S_3(t))) \\
& + \sum_{T_k \in \# G_2(t,T)} E\left[\int_t^{T_k} \mathrm{e}^{-\int_0^s r(u) \mathrm{d}u} q_{G_2,b}(t, T_k) dG_2(s, T_k) \middle| \mathcal{F}_t \right],
\end{aligned}
$$

For $\partial_{x_{S_2}} V_\pi$ and $\partial_{x_{G_2,T_m}} V_\pi$, we have

$$
\begin{aligned}
& \partial_{x_{S_2} t} V_\pi(t, x^*(t)) + \partial_{x_{S_2} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
& + \frac{1}{2} \mathrm{tr} \left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{S_2} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)) \right) \\
& - r(t) \partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t)) = 0
\end{aligned}
\tag{B.49}
$$

$$
\begin{aligned}
& \partial_{x_{G_2,T_m} t} V_\pi(t, x^*(t)) + \partial_{x_{G_2,T_m} x} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) \\
& + \frac{1}{2} \mathrm{tr} \left(\sigma_x(t, x^*(t), q^*(t))^\top \partial_{x_{G_2,T_m} xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)) \right) \\
& - r(t) \partial_{x_{G_2,T_m}} V_\pi(t, x^*(t)) = 0
\end{aligned}
\tag{B.50}
$$

Applying Feynman-Kac formula for (B.49) and (B.50), we have

$$\begin{aligned}
& \partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= & E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \partial_{x_{S_2}} V_\pi(T_m, x^*(T_m)) \middle| \mathcal{F}_t\right] \\
& -E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
= & E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] \\
& -E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
& \partial_{x_{G_2,T_m}} V_\pi(t, x^*(t)) \\
= & E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \partial_{x_{G_2,T_m}} V_\pi(T_m, x^*(T_m)) \middle| \mathcal{F}_t\right] \\
= & 0.
\end{aligned}$$

Substituting this into equation (B.45) and (B.47), we have

$$\begin{aligned}
S_2(t) = & E\left[\mathrm{e}^{-\int_t^{T} r(u)\mathrm{d}u} S_2(T) \middle| \mathcal{F}_t\right] + \lambda_{S_2,x}(t) + \lambda_{S_2,q_b,l}(t) - \lambda_{S_2,q_b,u}(t) \\
& -E\left[\int_t^{T} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \qquad \text{(B.51)}
\end{aligned}$$

On the other hand, since futures are martingale under risk-neutral probability,

$$0 = \lambda_{G_2,x_l}(t, T_m) - \lambda_{G_2,x_u}(t, T_m) + \lambda_{G_2,q_b,l}(t, T_m) - \lambda_{G_2,q_b,u}(t, T_m).$$

Therefore

$$\begin{aligned}
& G_2(t, T_m) \\
= & \Bigg(S_2(t) - \mathrm{Cov}\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u}, S_2(T_m) \middle| \mathcal{F}_t\right] - \lambda_{S_2,x}(t) - \lambda_{S_2,q_b,l}(t) + \lambda_{S_2,q_b,u}(t) \\
& +E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right]\Bigg) \\
& \cdot\left(E\left[\mathrm{e}^{-\int_t^{T_m} r(u)\mathrm{d}u} \middle| \mathcal{F}_t\right]\right)^{-1} \qquad \text{(B.52)}
\end{aligned}$$

Furthermore, from equation (B.45) and (B.46) we have

$$S_2(t) = p'(q^*_{S_2,u}(t)) S_1(t) + (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x)) \qquad \text{(B.53)}$$

C Proof of Corollary 3.1

From (A.26) we have

$$\begin{aligned}
& \lambda_{F_2}(t, T_m) \\
= \; & S_2(t) - P(t, T_m)F_2(t, T_m) \\
& + E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t \right].
\end{aligned}$$

Differentiating this equation leads to

$$\begin{aligned}
& \mathrm{d}\lambda_{F_2}(t, T_m) \\
= \; & \mathrm{d}S_2(t) - F_2(t, T_m)\mathrm{d}P(t, T_m) - P(t, T_m)\mathrm{d}F_2(t, T_m) - \mathrm{d}P(t, T_m)\mathrm{d}F_2(t, T_m) \\
& + \mathrm{d}E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t \right] \qquad \text{(C.1)}
\end{aligned}$$

From Heath, Jarrow, and Morton (1992), the stochastic differential equation for $P(t, T_m)$ is

$$\begin{aligned}
\mathrm{d}P(t, T_m) &= P(t, T_m)\{(\mu_P(t, T_m))\mathrm{d}t + \sigma_P(t, T_m) \cdot dB(t)\}. \\
\sigma_P(t, T_m) &= - \sum_{T_m \in \# F_2(t,T)} \sigma_f(t, T_m) \\
\mu_P(t, T_m) &= r(t) - b(t, T_m) + \sum_{T_m \in \# F_2(t,T_m)} \sigma_P(t, T_m)\gamma(t) \\
&= r(t) - \sum_{T_m \in \# F_2(t,T)} \mu_f(t, T_m) + 1/2\sigma_P(t, T_m)^\top \sigma_P(t, T_m) \\
&\quad + \sum_{T_m \in \# F_2(t,T_m)} \sigma_P(t, T_m)\gamma(t)
\end{aligned}$$

under risk-neutral probability. Here $\gamma(t)$ is the market price of risk. Furthermore, from equation (A.22) and (A.23),

$$\begin{aligned}
& E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t \right] \\
= \; & -\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t))
\end{aligned}$$

which implies

$$\begin{aligned}
& \mathrm{d}E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t \right] \\
= \; & -\mathrm{d}\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \mathrm{d}\partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)).
\end{aligned}$$

Applying Ito's formula to $\partial_{x_{S_2}} V_\pi$ and $\partial_{x_{F_2,T_m}} V_\pi$, we have

$$\begin{aligned}
& \mathrm{d}\partial_{x_{S_2}} V_\pi(t, x^*(t)) \\
= & -\{-r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} f_{T_m}(t, x^*(t), q^*(t))\}\mathrm{d}t \\
& +\partial_{x_{S_2} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\mathrm{d}B(t) \\
& \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) \\
= & -\{-r(t)\partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) + \partial_{x_{F_2,T_m}} f_{T_m}(t, x^*(t), q^*(t))\}\mathrm{d}t \\
& +\partial_{x_{F_2,T_m} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\mathrm{d}B(t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \mathrm{d}E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
= & -\mathrm{d}\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \mathrm{d}\partial_{x_{F_2,T_m}} V_\pi(t, x^*(t)) \\
= & \{-r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} f_{T_m}(t, x^*(t), q^*(t))\}\mathrm{d}t \\
& -\partial_{x_{S_2} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\mathrm{d}B(t) \\
& -\{-r(t)\partial_x V_\pi(t, x^*(t)) + \partial_x f_{T_m}(t, x^*(t), q^*(t))\}\mathrm{d}t \\
& +\partial_{x_{F_2,T_m} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\mathrm{d}B(t) \qquad \text{(C.2)}
\end{aligned}$$

Substituting (C.2) and (C.2) into (C.1), we have

$$\begin{aligned}
& \mathrm{d}\lambda_{F_2}(t, T_m) \\
= & S_2(t)\{\mu_{S_2}(t)\mathrm{d}t + \sigma_{S_2}(t)\cdot \mathrm{d}B(t)\} \\
& -P(t,T_m)F_2(t,T_m)\{(\mu_P(t,T_m) + \mu_{F_2}(t,T_m) - \sigma_P(t,T_m)^\top \sigma_{F_2}(t,T_m))\mathrm{d}t \\
& +(\sigma_P(t,T_m) - \sigma_{F_2}(t,T_m))\cdot \mathrm{d}B(t)\} \\
& +\{-r(t)(\partial_{x_{S_2}} V_\pi(t, x^*(t)) - \partial_{x_{F_2,T_m}} V_\pi(t, x^*(t))) + \partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t))\}\mathrm{d}t \\
& +\{-\partial_{x_{S_2} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t)) \\
& +\partial_{x_{F_2,T_m} x} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\}\mathrm{d}B(t) \\
= & S_2(t)\{\mu_{S_2}(t)\mathrm{d}t + \sigma_{S_2}(t)\cdot \mathrm{d}B(t)\} \\
& -P(t,T_m)F_2(t,T_m)\{(\mu_P(t,T_m) + \mu_{F_2}(t,T_m) - \sigma_P(t,T_m)^\top \sigma_{F_2}(t,T_m))\mathrm{d}t \\
& +(\sigma_P(t,T_m) - \sigma_{F_2}(t,T_m))\cdot \mathrm{d}B(t)\} \\
& +\Bigg\{ - r(t)\left(E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right]\right) \\
& +\partial_{x_{S_2}} R(x^*_{S_2}(t), S_3(t))\Bigg\}\mathrm{d}t \\
& +\Bigg\{\partial_x E\left[\int_t^{T_m} \mathrm{e}^{-\int_t^s r(u)\mathrm{d}u} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))\mathrm{d}s \middle| \mathcal{F}_t\right] \\
& \cdot\sigma_x(t, x^*(t), u^*(t))\Bigg\}\mathrm{d}B(t).
\end{aligned}$$

D Proof of Proposition 3.3

Here we derive the optimal production plan and trading strategy.

Lemma D.1. For any t,

$$E_t\left[\int_t^{T_m} e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\mathrm{d}s\right]$$

is a strictly convex function.

Proof. We denote $x_1, x_2 \in \mathbb{R}^{5+2\#F(t,T)}$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{Q}$.[23] Furthermore, for any t and $\theta \in (0,1)$, let $x_\theta = \theta x_1 + (1-\theta)x_2$ and $q_\theta(\cdot) = \theta q_1 + (1-\theta)q_2(\cdot)$. Note that $\mathcal{Q}$ is convex. $x_\theta(s), t \le s \le T$ should satisfy the differential equations (1) with $x_\theta(t) = x_\theta$. Since $\mu_x(t, x(t), q(t))$ and $\sigma_x(t, x(t), q(t))$ is linear, we have

$$x_\theta(s) = \theta x_1(s) + (1-\theta)x_2(s)$$

Since R is strictly convex then

$$R(x_\theta, S_3(s)) > \theta R(x_1, S_3(s))) + (1-\theta)R(x_2, S_3(s))$$

Since the strict inequality is true almost everywhere, we have

$$\begin{aligned} & E\left[\int_t^{T_m} e^{-\int_0^s r(u)\mathrm{d}u} R(x_\theta(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] \\ > \ & \theta E_t\left[\int_t^{T_m} e^{-\int_0^s r(u)\mathrm{d}u} R(x_1(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right] \\ & +(1-\theta)E\left[\int_t^{T_m} e^{-\int_0^s r(u)\mathrm{d}u} R(x_2(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]. \end{aligned}$$

□

The next lemma shows the existence of an inverse of gradient of ϕ_t at $x_{t,T} = (x_{S_2}(t), (x_{F_2}(t, T_m))_{T_m \in \#F_2(t,T)})$.[24]

$$\phi_t(x_{t,T}) = E\left[\int_t^{T} e^{-\int_0^u r(u)\mathrm{d}u} R(x(s), S_3(s))\mathrm{d}s\middle|\mathcal{F}_t\right]$$

Lemma D.2. Let $\partial_q R$ be bounded by an integrable function h_R and R be strictly convex and essentially smooth. There exists an inverse of

$$\partial\phi_t(x_{t,T}) = \begin{pmatrix} \partial_{x_{S_2}} E\left[\int_t^T e^{-\int_t^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\, \mathcal{F}_t\right] \\ \left(\partial_{x_{F_2,T_m}} E\left[\int_t^T e^{-\int_t^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\, \mathcal{F}_t\right]\right)_{T_m \in \#F_2(t,T)} \end{pmatrix}$$

We define the inverse functions $I_{R,t}$.

---

[23] This proof is basically same as that of Fleming and Soner (2006), Chapter IV, Section IV.10, Lemma 10.5.

[24] Since $x_{S_2}(t)$ depends on $x_{F_2}(t, T_m)$, $R(x_{S_2}(t), S_3(t))$ depends on $(x_{S_2}(t), (x_{F_2}(t, T_m))_{T_m \in \#F_2(t,T)})$ as well.

Proof. $E\left[\int_t^{T_m} e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\mathrm{d}s\right]$ is a strictly convex function of x_t. Also, note that

$$\begin{aligned}
& \partial_x E\left[\int_t^T e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\mathcal{F}_t\right] \\
= \ & \partial_{x_{S_2}} E\left[\int_t^T e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\mathcal{F}_t\right] \\
& \partial_x E\left[\int_{T_m}^T e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\mathcal{F}_t\right] \\
= \ & \partial_{x_{F_2,T_m}} E\left[\int_t^T e^{-\int_0^s r(u)\mathrm{d}u} R(x_{S_2}(s), S_3(s))\middle|\mathcal{F}_t\right]
\end{aligned}$$

by the chain rule.

We have an inverse for

$$\partial\phi_t(x_{t,T})$$

from Theorem 23.5, Corollary 23.5.1, and Corollary 26.3.1 of Rockafellar (1970). □

If p is strictly concave and essentially smooth, then there exist an inverse function I_p for $p'(\cdot)$ from Theorem 23.5, Corollary 23.5.1, and Corollary 26.3.1 of Rockafellar(1970). Therefore, from equation (A.27) we can derive

$$q^*_{S_2,u}(t) = I_p\left(\frac{S_2(t) - (\lambda_{S_2,q_u,l}(t,x) - \lambda_{S_2,q_u,u}(t,x))}{S_1(t)}\right). \tag{D.1}$$

From Lemma D.2 and equations (A.24), (A.25) we have

$$(x^*_{S_2}(t), (x^*_{F_2}(t,T_m))_{T_m\in\#F_2(t,T)}) \quad = \quad I_{R,t}(x_{0,t,T}) \tag{D.2}$$

where

$$\begin{aligned}
x_{0,t,T} &= \begin{pmatrix} x_{0,t,T,S_2} \\ (x_{0,t,T,F_2,T_m})_{T_m\in\#F_2(t,T_m)} \end{pmatrix} \\
x_{0,t,T,S_2} &= -S_2(t) + E\left[e^{-\int_t^T r(u)\mathrm{d}u} S_2(T)\middle|\mathcal{F}_t\right] + \lambda_{S_2,b}(t), \\
x_{0,t,T,F_2,T_m} &= -e^{-\int_t^{T_m} r(u)\mathrm{d}u} F_2(t,T_m) + E\left[e^{-\int_t^T r(u)\mathrm{d}u} S_2(T)\middle|\mathcal{F}_t\right] \\
&\quad +\lambda_{F_2,b_l}(t,T_m) - \lambda_{F_2,b_u}(t,T_m)
\end{aligned}$$

Furthermore, we can calculate the optimal trading strategy by

$$\begin{aligned}
dx_{S_2}(t) &= (q_{S_2,b}(t) - q_{S_2,u}(t))\mathrm{d}t + \sum_{T_m\in\#F_2(0,t)} 1_{t=T_m} x_{F_2}(t,T_m), \\
&\quad 0 \le t \le T \\
dx_{F_2}(t,T_m) &= q_{F_2,b}(t,T_m)\mathrm{d}t, 0 \le t \le T, T_m \in \#F_2(t,T).
\end{aligned}$$

Thus, we derived the optimal production plan and trading strategy.

E Proof of Proposition 3.4

In this section, we derive the necessary condition for problem (6). We omit the index j and denote $u_j(\cdot), W_j(\cdot), c_{1,j}(\cdot), \theta_j(\cdot)$ to be $u(\cdot), W(\cdot), c_1(\cdot), \theta(\cdot)$ for simplicity.

Let us define

$$J_u(t_0, x; (c(\cdot), \theta(\cdot))) \quad = \quad E\left[\int_{t_0}^{T} u(t, c_1(t))\mathrm{d}t + U(W(T))\right].$$

Define the gain process $G(t)$ as

$$\begin{aligned}
& G(t) \\
= \quad & G(t_0) + \int_{t_0}^{t} \theta_{P_0}(t)P_0(t)r(t)\mathrm{d}t \\
& + \int_{t_0}^{t} \sum_{T_m \in \#F_2(t_0,T)} \theta_P(s,T_m)P(s,T_m)\{(\mu_P(s,T_m))\mathrm{d}s + \sigma_P(s,T_m)^\top dB(s)\} \\
& + \int_{t_0}^{t} \sum_{T_m \in \#F_2(t_0,T)} \theta_{F_2}(s,T_m)F_2(s,T_m)\{\mu_{F_2}(s,T_m)\mathrm{d}s + \sigma_{F_2}(s,T_m)^\top dB(s)\}
\end{aligned}$$

The wealth process $W(t)$ is

$$W(t) = W_{t_0} - \int_{t_0}^{t} c_1(s)S_1(s)\mathrm{d}s + G(t)$$

The speculator's strategy θ finances the net consumption process $c_1(t)S_1(t)$

$$W(t) = \theta_{P_0}(t)P_0(t) + \sum_{T_m \in \#F_2(t,T)} \theta_P(t,T_m)P(t,T_m) + \theta_{F_2}(t,T_m)F_2(t,T_m)$$

Thus, the wealth process follows the following stochastic differential equation;

$$\begin{aligned}
& \mathrm{d}W(t) = \mathrm{d}G(t) - c_1(t)S_1(t)\mathrm{d}t \\
= \quad & W(t)r(t)\mathrm{d}t \\
& + \sum_{T_m \in \#F_2(t,T)} W(t)w_P(t,T_m)\{(\mu_P(t,T_m) - r(t))\mathrm{d}t + \sigma_P(t,T_m)^\top dB(t)\} \\
& + \sum_{T_m \in \#F_2(t,T)} W(t)w_{F_2}(t,T_m)\{(\mu_{F_2}(t,T_m) - r(t))\mathrm{d}t + \sigma_{F_2}(t,T_m)^\top dB(t)\} \\
& - c_1(t)S_1(t)\mathrm{d}t
\end{aligned}$$

where

$$
\begin{aligned}
w(t) &= \left(w_{P_0}(t), (w_P(t,T_m))_{T_m\in\#F_2(t,T)}, (w_{F_2}(t,T_m))_{T_m\in\#F_2(t,T)}\right)^\top \\
w_{P_0}(t) &= \theta_{P_0}(t)P_0(t)/W(t), \\
w_P(t,T_m) &= \theta_P(t,T_m)P(t,T_m)/W(t), \\
w_{F_2}(t,T_m) &= \theta_{F_2}(t,T_m)F_2(t,T_m)/W(t)
\end{aligned}
$$

and used the fact that the sum of weights is 1.

$$
1 = w_{P_0}(t) + \sum_{T_m\in\#F_2(t,T)} w_P(t,T_m) + w_{F_2}(t,T_m)
$$

The dynamics of the wealth process can be expressed as

$$
\begin{aligned}
& \mathrm{d}W(t) \\
= & \left(W(t)(w(t)^\top(\mu_u(t) - r(t)) + r(t)) - c_1(t)S_1(t)\right)\mathrm{d}t \\
& + W(t)w(t)^\top\sigma_u(t)\mathrm{d}B(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_u(t) &= \left((\mu_P(t,T_m))_{T_m\in\#F_2(t,T)}, (\mu_{F_2}(t,T_m))_{T_m\in\#F_2(t,T)}\right)^\top \\
\sigma_u(t) &= \left((\sigma_P(t,T_m))_{T_m\in\#F_2(t,T)}, (\sigma_{F_2}(t,T_m))_{T_m\in\#F_2(t,T)}\right)^\top
\end{aligned}
$$

This wealth process is the state process.

The value function of the optimization problem (6) can be written as

$$
V_u(t_0,W) = \sup_{(c_1(\cdot),\theta(\cdot))\in\mathcal{A}(t_0,T)} J_u(t_0,x;(c_1(\cdot),\theta(\cdot))) \tag{E.1}
$$

$$
V_u(T,W) = U(W) \tag{E.2}
$$

where now the control is $w(\cdot)$ which replaces $\theta(\cdot)$.

Again, we apply the dynamic programming principle

$$
\begin{aligned}
& V_u(t_0,W) \\
= & \sup_{(c_1(\cdot),w(\cdot))\in\mathcal{A}(t_0,T)} E\Bigg[\int_{t_0}^{t_1} u(t, c_1(t;t_0,x,(c_1(\cdot),w(\cdot)),(c_1(t),w(t)))\mathrm{d}t \\
& + V_u(t_1, W(t_1;t_0,W,(c_1(\cdot),w(\cdot)))\Bigg], 0\le t_0\le t_1\le T.
\end{aligned} \tag{E.3}
$$

and divide the problem into subperiods which are delimited by the maturities of forward.

$$
[T_0,T_1],\cdots,[T_{M-1},T_M]
$$

We start from the last period $[T_{M-1}, T_M]$ and the corresponding optimal control problem is

$$V(t_0, W_{t_0}) = \sup_{(c_1(\cdot),w(\cdot))\in\mathcal{A}(t_0,T_M)} \mathrm{E}\left[\int_{t_0}^{T_M} u(t, c_1(t))\mathrm{d}t + U(C_1(T_M))\right] \tag{E.4}$$

where

$$\begin{aligned} & \mathcal{A}(t_0, T_M) \\ = & \Bigg\{(c_1(\cdot), w(\cdot)) \in C \times \Theta_1 : \\ & W(t) = W_{t_0} + \int_{t_0}^{t} W(s)(w(s)^\top(\mu_u(s) - r(s)) + r(s)) - c_1(s)S_1(s)\mathrm{d}s \\ & + \int_{t_0}^{t} W(s)w(s)^\top\sigma_u(s)\mathrm{d}B(s)c_1(t) \geq 0, \theta_{F_2}(t,t) = 0, t_0 \leq t \leq T_M\Bigg\} \end{aligned}$$

and Θ_1 be a space of $\{\mathcal{F}(t)\}$-progressively measurable, $\boldsymbol{R}^{2\#F_2(t_0,T)}$ valued process.

The Hamilton-Jacobi-Bellman equation for (E.4) is

$$\begin{aligned} 0 &= \partial_t V_u(t,W) \\ &\quad + \sup_{(c_1,w)\in A} G_{u,T_M}(t, W, (c,w), \partial_W V_u(t,W), \partial_{WW} V_u(t,W), V_u(t,W)) \end{aligned} \tag{E.5}$$

$$V_u(T_M, W) = U(W), W \in \mathbb{R} \tag{E.6}$$

where ∂_W and ∂_{WW} are the partial derivatives and

$$\begin{aligned} & G_{u,T_M}(t, W, (c,w), V_1, V_2) \\ = & \frac{1}{2}(V_2 W^2 w^\top \sigma_u(t)\sigma_u(t)^\top w) \\ & + V_1(Ww^\top(\mu_u(t) - r(t)) + Wr(t) - c_1 S) + u(t, c_1), \\ & \forall(t, W, (c,w), p, P) \in [T_{M-1}, T_M] \times \mathbb{R} \times A \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

We have the following optimization problem

$$\sup_{(c,w)\in A} G_{u,T_M}(t, W, (c,w), \partial_W V_u(t,W), \partial_{WW} V_u(t,W), V_u(t,W))$$

and the first order condition for this problem is

$$\begin{aligned} 0 &= \partial_{WW} V_u(t, W(t))W^2(t)\sigma_u(t)\sigma_u(t)^\top w^*(t) + \partial_W V_u(t, W(t))W(t)(\mu_u(t) - r(t)) \\ &\quad + \partial_W V_u(t, W(t))W(t)1_{1+2\#F_2(t_0,T)} r(t) \end{aligned} \tag{E.7}$$

$$0 = -\partial_W V_u(t, W(t))S_1(t) + \partial_c u(t, c^*) \tag{E.8}$$

where 1_n is $n \times 1$ column of 1.

From the Hamilton-Jacobi-Bellman equation (E.5), we have

$$\begin{aligned} & G_{u,T_M}(t, W^*(t), (c_1^*(t), w^*(t)), \partial_W V_u(t, W^*(t)), \partial_{WW} V_u(t, W^*(t)), V_u(t, W^*(t))) + \partial_t V_u(t, W^*(t)) \\ = \; & 0 \\ \geq \; & G_{u,T_M}(t, W, (c, w), \partial_W V_u(t, W), \partial_{WW} V_u(t, W), V_u(t, W)) + \partial_t V_u(t, W). \end{aligned}$$

where $(W^*(t), (c_1^*(t), w^*(t)))$ is the optimal solution. Since $V_u \in C^{1,3}([T_{M-1}, T_M] \times \mathbb{R})$ and $\partial_{tW} V_u$ being continuous, we have

$$\begin{aligned} 0 \;=\; & \partial_W G_{u,T_M}(t, W^*(t), (c_1^*(t), w^*(t)), \partial_W V_u(t, W^*(t)), \partial_{WW} V_u(t, W^*(t)), V_u(t, W^*(t))) \\ & + \partial_t V_u(t, W^*(t)). \end{aligned}$$

If we multiply optimal weights to the equation (E.7) and sum it up, we have

$$\begin{aligned} & \partial_{WW} V_u(t, W(t)) W^2(t) w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \\ = \; & -\partial_W V_u(t, W(t)) W(t) w^*(t)^\top (\mu_u(t) - r(t)) - \partial_W V_u(t, W(t)) W(t) w^*(t)^\top 1_{1+2\#F_2(t,T)} r(t) \end{aligned}$$

Therefore, if $W(t) \neq 0$,

$$\begin{aligned} & \partial_{WW} V_u(t, W(t)) W(t) w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \\ = \; & -\partial_W V_u(t, W(t)) w^*(t)^\top (\mu_u(t) - r(t)) - \partial_W V_u(t, W(t)) r(t) \qquad \text{(E.9)} \end{aligned}$$

By the definition of G_{u,T_m},

$$\begin{aligned} & \partial_{tW} V_u(t, W^*(t)) \\ & + \partial_{WW} V_u(t, W^*(t)) (W^*(t) w^*(t)^\top (\mu_u(t) - r(t)) + W^*(t) r(t) - c_1^*(t) S(t)) \\ & + (w^*(t)^\top (\mu_u(t) - r(t)) + r(t)) \partial_W V_u(t, W^*(t)) \\ & + \frac{1}{2} \left(\partial_{WWW} V_u(t, W^*(t)) W^*(t)^2 w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \right) \\ & + \left(\partial_{WW} V_u(t, W^*(t)) W^*(t) w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \right) = 0, \qquad \text{(E.10)} \end{aligned}$$

From equation (E.9), we have

$$\begin{aligned} & \partial_{tW} V_u(t, W^*(t)) \\ & + \partial_{WW} V_u(t, W^*(t)) (W^*(t) w^*(t)^\top (\mu_u(t) - r(t)) + W^*(t) r(t) - c_1^*(t) S(t)) \\ & + \frac{1}{2} \left(\partial_{WWW} V_u(t, W^*(t)) W^*(t)^2 w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \right) = 0, \qquad \text{(E.11)} \end{aligned}$$

and if we multiply it by $F_2(t)$ and use $\partial_{\theta_{F_2}} V_u(t, W^*(t)) = \partial_W V_u(t, W^*(t)) F_2(t, T_M)$

$$\begin{aligned} & \partial_{t\theta_{F_2}} V_u(t, W^*(t)) \\ & + \partial_{W\theta_{F_2}} V_u(t, W^*(t)) (W^*(t) w^*(t)^\top (\mu_u(t) - r(t)) + W^*(t) r(t) - c_1^*(t) S(t)) \\ & + \frac{1}{2} \left(\partial_{WW\theta_{F_2}} V_u(t, W^*(t)) W^*(t)^2 w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t) \right) = 0. \qquad \text{(E.12)} \end{aligned}$$

Note that $\partial_W V_u(T_M, W^*(T_M)) = \partial_W U(W^*(T_M)) = \partial_C U(C^*(T_M)S_1(T))/S_1(T) = \partial_C U_T(C^*(T_M))/S_1(T)$.

Applying Feynman-Kac formula for (E.12), we have

$$\begin{aligned}\partial_{\theta_{F_2}} V_u(t, W^*(t)) &= E_{P_N}\left[\partial_{\theta_{F_2}} V_u(T_M, W^*(T_M))\middle|\mathcal{F}_t\right] \\ &= E_{P_N}\left[\partial_C U_T(C^*(T_M))F_2(t, T_M)/S_1(T)\middle|\mathcal{F}_t\right]\end{aligned}$$

From equation (E.8),

$$\partial_\theta V_u(t, W(t)) = \partial_W V_u(t, W(t))F_2(t, T_M) = \partial_c u(t, c^*)F_2(t, T_M)/S_1(t) \tag{E.13}$$

Thus,

$$F_2(t, T_M) = E_{P_N}\left[\frac{\partial_c U_T(C_1^*)/S_1(T_M)}{\partial_c u(t, c_1^*)/S_1(t)} S_2(T_M)\middle|\mathcal{F}_t\right]$$

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