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Firm Growth by Product Innovation in the Presence of the Product Life Cycle*

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Abstract

In this paper, we present a model which enables us to look into the process of research and development (R&D) for product innovation in the presence of the product life cycle and the resultant firm or economic growth. Specifically, we describe R&D for product innovation as an activity to control the birth rate of a new product, which measures the probability of product innovation; derive the optimal birth rate of a new product, which determines the size of R&D expenditure; and examine the growth rate of the (representative) firm('s expected total revenue) along the optimal R&D plan. We then find that the growth rate of the firm converges to the optimal birth rate of a new product in the long run.

Keywords: Firm growth, Optimal birth rate of products, Product innovation, Product life cycle
JEL Classification: O31; O32; O41

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1 Introduction

The so-called endogenous (new) growth theory was initiated by Lucas [10] and Romer [14] in order that the mechanism of technical progress, which is treated as a "black box" in the exogenous (old) growth theory (e.g., Solow [16]), could be explained by the logic of economics. To expound the mechanism of technical progress, the endogenous growth theory emphasizes the role of research and development activities (R&D, henceforth). In particular, investment on R&D is thought to be the engine of improvement of technique (knowledge) in this theory. So this theory ends up with debating the "optimal" level of R&D and the resultant economic growth by taking account of the cost and benefit of R&D investment.[1] Given that the endogenous growth theory puts much emphasis on progress in production technique through R&D, it is natural that this theory takes a closer look at the "supply side" of the economy but seldom looks into the "demand side." As Solow [17] maintained, the demand side should also be taken into account in the theory of economic growth.[2]

Along the line of the endogenous growth theory, however, Grossman and Helpman [6, chaps. 3 and 4] took up the demand side by paying attention to the fact that the variety and quality of products have an influence on consumers' utility. In particular, they clarified how the optimal level of R&D for expanding the variety of products or for raising the quality of products and the resultant rate of economic growth are determined. Their approach was to some extent successful in introducing the demand side in the endogenous growth theory but they were not conscious of the concept of "product life cycle."[3]

In the field of business administration, the concept of product life cycle has been established (e.g., Day [5]). According to this this concept, each innovative product (good or service) has its life cycle and along its life cycle, it undergoes four phases: Introduction, Growth, Maturity and Decline (Figure 1). In the field of economics, especially of economic growth, however, this concept has not properly been introduced in the context of economic growth.[4]

[1]For details on the endogenous growth theory, see, for example, Aghion and Howitt [1].

[2]Indeed, Solow [17, pp. 231-232] stated as follows:

> One major weakness in the core of macroeconomics as I have represented it is the lack of real coupling between the short-run picture and the long-run picture. Since the long run and the short run merge into one another one feels they cannot be completely independent. There are some obvious, perfunctory connections: every year's realized investment gets incorporated in the long-run model. That is obvious. A more interesting question is whether a major episode in the growth of potential output can be driven from the demand side.

[3]Certainly, Grossman and Helpman [6, chap. 12] extended their "quality ladder" model [6, chap. 4] so as to discuss the "product cycle hypothesis" established by Vernon [20], which states that an innovative product is initially produced in an advanced country, in which production is comparatively costly, but once the production technique is standardized, this product begins to be manufactured in a developing country, in which production cost is relatively low. Vernon's [20] product cycle hypothesis is concerned about the "life cycle" of products in terms of the supply side but does not involve the "life cycle" of products in terms of the demand side. In this respect, the demand side was not properly treated in Grossman and Helpman's [6, chap. 12] analysis of product cycle in the context of economic growth.

[4]There were some exceptional works. For instance, Klepper [9] shed light on the concept of product life cycle and considered firms' entry and exit over this cycle, but his analysis focused on the explanation of product life cycle and did not examine the optimal

Needless to say, this stylized fact has significant influence on the long run firm or economic growth.

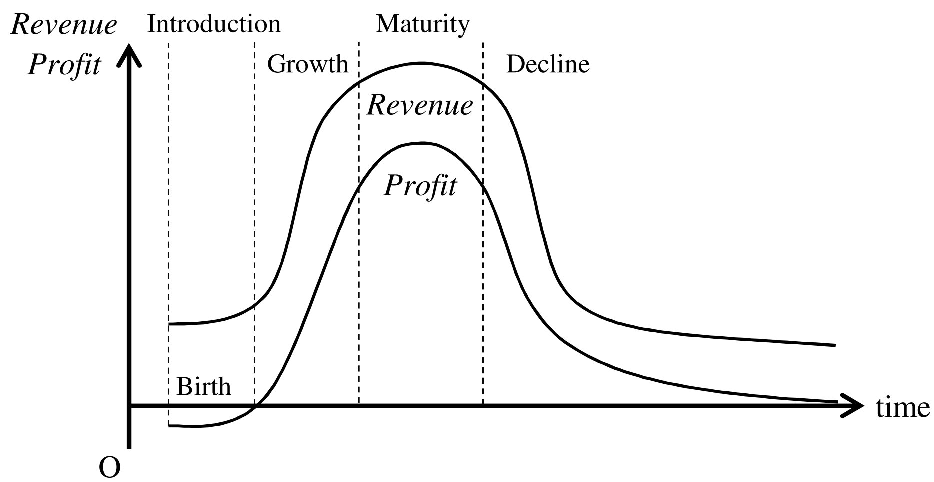

Figure 1: Product life cycle

The aim of this paper is to make a contribution to the argument of economic or firm growth by introducing the concept of product life cycle in the endogenous growth theory. As is easily seen, firms in the presence of product life cycle can only keep on growing by inventing new products and acquiring more demand. Of course, invention of a new product or product innovation cannot easily be generated and it can only occur as a result of costly R&D investment. In this paper, we attempt to present a framework which enables us to look into the process of R&D for product innovation in the presence of product life cycle.

This paper is organized as follows.

In Section 2, we shall describe the structure of our model in details. First, the concept of product life cycle shall be formalized. Second, the process of product innovation shall be described. Third, the total expected profit and the expenditure on R&D shall be calculated. Finally, the expected (net) profit maximization problem shall be presented. In the expected (net) profit maximization problem, the control variable shall be the birth rate of a new product (i.e., the probability of successful product innovation).

In Section 3, we shall describe the growth of the (representative) firm along the optimal R&D plan for product innovation. To begin, the expected (net) profit maximization problem formalized in Section 2 shall be solved and the optimal birth rate of a new product and then the optimal R&D plan and expenditure shall be derived. Next, the growth rate of the firm('s expected revenue) shall be calculated. As a consequence of the analysis in this section,

R&D along this cycle from the long run perspective On the other hand, Aoki and Yoshikawa [2, chap. 8] put much emphasis on the phenomenon of "demand saturation" but they failed to describe the Decline phase of the product life cycle (Figure 1).

it shall turn out that the growth rate of the firm converges to the optimal birth rate of a new product in the long run.

In Section 4, we shall conclude this paper.

In Appendices A-F, we shall provide the proofs of the mathematical propositions presented in this paper.

2 Model structure

We present a model of product innovation through R&D in the face of product life cycle.

2.1 Product life cycle

As mentioned in Section 1, it has been widely recognized that every innovative product has its life cycle, which generally consists of four phases: Introduction, Growth, Maturity and Decline (Figure 1). To formalize the concept of product life cycle, it is supposed that the profit on each kind of product depends upon the "age" of this product (i.e., the time span elapsing from the birth). More specifically, we assume that, at time t, product of "vintage s," which appears in the market as a consequence of product innovation at time $s \leq t$, has profit π_s^t represented in the following form:

$$\pi_s^t = \pi(t - s). \tag{1}$$

In what follows, we refer to π as the profit function.[5]

As regards the profit function π, we impose the following fairly weak assumption.

Assumption 1. $\pi : \mathbb{R}_+ \to \mathbb{R}$ *is bounded and continuous.*

The key point of Assumption 1 is that the profit function π is bounded. Note that Assumption 1 allows the profit on a product to be negative for some time. As Figure 1 indicates, the profit on a product can be negative soon after its birth.

2.2 Product innovation

We consider the process of product innovation (invention of a new product) as an successful outcome of R&D. To focus on the relationship between R&D and product innovation, we restrict the role of R&D to raising the

[5]Strictly speaking, the profit function π should be interpreted as the time path of *expected* profit from each product. Psychological factors such as "animal spirits" (Keynes [8, chap. 12]) and "entrepreneurship" (Schumpeter [15, chap. 2]) can be thought to be reflected in this function.

probability of product innovation (or of invention of a new product). In particular, this type of R&D may also be called "R&D for product innovation" henceforth.

Generally, intensive R&D increases the probability of product innovation, but it does not always result in success. To reflect this reality, we describe product innovation as a random event.

Furthermore, a new product is generally thought to be invented by "new combination" of the existing products or of know-how on them (Schumpeter [15, chap. 2]). Thus, it is natural to assume that the more the number of product types the firm has, the more likely it is to invent a new product.[6] Since the existing product types are the outcomes of past product innovation, the above assumption may imply that product innovation occurs in the process of leaning by doing (Arrow [3]).

Specifically, we suppose that, when the firm under consideration has N kinds of products at time t, it succeeds in giving birth to a new product during an infinitesimal time Δt with the probability $\lambda(t)N\Delta t$, where $\lambda(t)$ measures the intensity of product innovation. In what follows, $\lambda(t)$ is called the *birth rate of a new product*. Since new product innovation is given rise to in the learning-by-doing process, it is assumed that the probability of new product innovation is dependent upon the number of the existing products, which were born as a consequence of successful product innovation.

Let $P(N,t)$ be the probability that the firm has N distinct types of products at time t. Then, since a new product is invented during an infinitesimal time Δt with the probability $\lambda(t)N\Delta t$, the change in $P(N,t)$ in the time interval $(t,\ t+\Delta t)$ can be represented as follows:

$$P(N,t+\Delta t) - P(N,t) = \lambda(t)(N-1)P(N-1,t)\Delta t - \lambda(t)NP(N,t)\Delta t + o(\Delta t).$$

Dividing both sides by Δt and letting $\Delta t \to 0$, we obtain

$$\dot{P}(N,t) = \lambda(t)(N-1)P(N-1,t) - \lambda(t)NP(N,t). \tag{2}$$

By following a method similar to those of Aoki and Yoshikawa [2, chap. 8] and of Murakami [13], we can obtain the solution of $P(N,t)$ as follows.

[6] The opposite hypothesis may be argued: the more inventions are made, the less room for new inventions in the future there is, because useful ideas are limited. This effect is often called the "crowding out" effect (e.g., Barro and Sala-i-martín [4]). The crowding-out effect must be taken into consideration, but if it is regarded as a cost of R&D for product innovation, it can be included in the cost function concerning R&D. As we shall explain below, this effect may be allowed for in our R&D expenditure function.

Lemma 1. *If $P(N_0, 0) = 1$,*[7] *equation (2) can be solved as follows:*

$$P(N,t) = \frac{(N-1)!}{(N_0-1)!(N-N_0)!}\Big[\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^{N_0}\Big[1-\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^{N-N_0}, \quad \text{for } N \geq N_0. \tag{3}$$

Proof. See Appendix A. □

2.3 Expected total profit

We are now in the position to calculate the firm's expected total profit at time t. Since the probability that the firm has N kinds of products and a new product is invented in the time interval $(t,\ t+\Delta t)$ is equal to $\lambda(t)NP(N,t)\Delta t$, the expected total profit at time t, $\Pi(t)$, can be calculated as follows:

$$\Pi(t) = \sum_{i=1}^{N_0} \pi(t-s_i) + \sum_{N=N_0}^{\infty} \int_0^t \lambda(s)NP(N,s)\pi(t-s)ds, \tag{4}$$

where $s_i \leq 0,\ i = 1,\ ...,\ N_0$, is the time when each product existing at time 0 was invented. It is possible to represent $\Pi(t)$ in the form in the following lemma.

Lemma 2. *Equation (4) can be represented as follows:*

$$\Pi(t) = \sum_{i=1}^{N_0} \pi(t-s_i) + N_0 \int_0^t \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds \tag{5}$$

Proof. See Appendix B. □

Define the expected number of product types the firm owns at time t as $\Lambda(t)$:

$$\Lambda(t) = \sum_{N=N_0}^{\infty} NP(N,t). \tag{6}$$

From Lemma 1, we can calculate Λ as follows.

Lemma 3. *Equation (6) can be represented as follows:*

$$\Lambda(t) = N_0 \exp\Big(\int_0^t \lambda(\tau)d\tau\Big). \tag{7}$$

[7] $P(N_0, 0) = 1$ says that the number of product types at time 0 is N_0 (with certainty).

Proof. See Appendix C. □

Lemma 3 suggests that the expected number of product types Λ is proportionate to the number of product types at the initial time 0, N_0, and that the birth rate of a new product λ is the growth rate of Λ.

Since a new product appears as a outcome of successful product innovation, the number of product types is the number of past successful experiences. In this respect, the expected number of product types Λ may also be labeled as the *expected stock of experience in product innovation*.[8]

By Lemmas 2 and 3, equation (5) can be rewritten as follows:

$$\Pi(t) = \sum_{i=1}^{N_0} \pi(t - s_i) + \int_0^t \lambda(s)\Lambda(s)\pi(t - s)ds. \tag{8}$$

Equation (8) implies that the expected total profit Π is enhanced by an increase in the expected stock of experience in product innovation Λ. The relationship between Π and Λ may be similar to that between the production output Y and the stock of capital K. In the latter relationship, an increase in K leads to an expansion of Y. Also, equation (8) means that an increase in the birth rate of a new product (or the growth rate of Λ) λ contributes to a rise in the expected total profit. If the expected stock of experience in product innovation Λ is identified with the stock of capital K, the birth rate of a new product λ is regarded as the rate of capital accumulation and an activity to determine λ is viewed as "investment."

2.4 Expected R&D expenditure

We consider how the firm's expenditure on R&D is determined.

Since the objective of R&D for product innovation is to enhance the probability of product innovation and the expected total profit gained by successful product innovation is determined by the birth rate of a new product (see Lemma 2), we may define an R&D plan as the activity that controls the birth rate of a new product λ. Specifically, we define an *R&D plan* as a time path of the birth rate of a new product $\{\lambda(t)\}_{t=0}^{\infty}$.

By defining an R&D plan as above, it is natural to reckon that the firm has to pay more cost to increase the birth rate of a new product λ.[9] So we may assume that, for each exiting product type, the expenditure on R&D

[8] Grossman and Helpman [6, p. 58] also defined the stock of knowledge capital as the number of product varieties in their *variety expansion* model.

[9] As we have mentioned, the "crowding out" effect of product innovation can be included in the R&D expenditure function.

associated with the (target) birth rate of a new product λ, φ, is given by

$$\varphi(t) = \varphi(\lambda(t)), \tag{9}$$

where φ is the cost for setting the birth rate of a new product as λ. In what follows, φ is called the R&D expenditure function.[10]

As regards the R&D expenditure function, φ, we assume that it is nonnegative, strictly increasing and strictly convex.

Assumption 2. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is twice continuously differentiable with*

$$\varphi(0) = 0,\ \ \varphi'(\lambda) > 0,\ \ \varphi''(\lambda) > 0, \quad \text{for } \lambda \geq 0. \tag{10}$$

Since a new product is given birth to by the existing products, the expected total expenditure at time t along R&D plan $\{\lambda(t)\}_{t=0}^{\infty}, \Phi$, is equal to the expenditure associated with the birth rate of a new product λ ($\varphi(\lambda)$) times the expected number of product types $\Lambda(t)$. Then, it follows from Proposition 3 that the expected total expenditure at time t along R&D plan $\{\lambda(t)\}_{t=0}^{\infty}, \Phi$, can thus be represented by

$$\Phi(t) = \sum_{N=N_0}^{\infty} \varphi(\lambda(t)) N P(N,t) = \varphi(\lambda(t))\Lambda(t). \tag{11}$$

We take a closer look at the R&D expenditure function φ and provide another interpretation for it. As we have mentioned above, the relationship between the expected stock of experience in product innovation Λ and the birth rate of a new product λ is similar to that between the stock of capital K and the rate of capital accumulation z. So if Λ and λ may be regarded as the counterparts of K and z, respectively, the expected total expenditure Φ may be interpreted as the counterpart of the effective cost (including adjustment cost) of investment. Since the effective cost function (or the adjustment cost function) of investment is usually assumed to satisfy the same properties as those of our R&D expenditure function φ, i.e., strict increasingness and strict convexity (e.g., Uzawa [19]), the R&D expenditure function φ may be seen as the effective (adjustment) cost function associated with R&D investment.

[10] Like the profit function π, the R&D expenditure function may be considered to include psychological costs or burdens affected by animal spirits or entrepreneurship.

2.5 R&D planning problem

We are now ready to formalize the firm's expected (net) profit maximization problem associated with R&D plan $\{\lambda(t)\}_{t=0}^{\infty}$. Along R&D plan $\{\lambda(t)\}_{t=0}^{\infty}$, the expected net profit at time t obtained after subtracting the expected total expenditure on R&D, $\Phi(t)$, is given by $\Pi(t) - \Phi(t)$. Then, the expected net profit maximization problem can thus be formulated as follows:

$$\max_{\{\lambda(t)\geq 0\}_{t=0}^{\infty}} \int_0^{\infty} [\Pi(t) - \Phi(t)] e^{-\rho t} dt \quad \text{s.t. } (8)\text{–}(11),$$

or

$$\max_{\{\lambda(t)\geq 0\}_{t=0}^{\infty}} \int_0^{\infty} \Big[\sum_{i=1}^{N_0} \pi(t-s_i) + N_0 \int_0^t \lambda(s) \exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds - N_0\varphi(\lambda(t)) \exp\Big(\int_0^t \lambda(\tau)d\tau\Big)\Big] e^{-\rho t} dt, \quad \text{(P)}$$

where $\rho > 0$ is the positive constant that stands for the interest rate. In what follows, the above problem is labeled as "Problem (P)." The optimal R&D plan can be derived as a solution of Problem (P), $\{\lambda(t)\}_{t=0}^{\infty}$.

Since the times when products existing at the planning time 0 were invented, s_i, $i = 1, ..., N_0$ have no influence on a solution of Problem (P), we may assume without loss of generality that $s_i = 0$ for all $i = 1, ..., N_0$, i.e., that the original N_0 products are born at time 0. Then, Problem (P) can reformulated as follows:

$$\max_{\{\lambda(t)\geq 0\}_{t=0}^{\infty}} \int_0^{\infty} N_0 \Big[\pi(t) + \int_0^t \lambda(s) \exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds - \varphi(\lambda(t)) \exp\Big(\int_0^t \lambda(\tau)d\tau\Big)\Big] e^{-\rho t} dt.$$

Then, we find that the maximand of Problem (P) is proportional to the original number of product types N_0. Thus, we may also assume without loss of generality that the original number of product types N_0 is unity so that Problem (P) can be reduced to

$$\max_{\{\lambda(t)\geq 0\}_{t=0}^{\infty}} \int_0^{\infty} \Big[\pi(t) + \int_0^t \lambda(s) \exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds - \varphi(\lambda(t)) \exp\Big(\int_0^t \lambda(\tau)d\tau\Big)\Big] e^{-\rho t} dt. \quad \text{(P)}$$

In what follows, Problem (P), redefined above, is investigated.

3 Firm growth along optimal R&D plan

We shall derive the optimal birth rate of a new product and the resultant optimal R&D plan for product innovation and calculate the growth rate of the firm('s expected revenue) along the optimal R&D plan.

3.1 Optimal R&D plan

To characterize the optimal R&D plan, we proceed to derive the condition for optimality concerning Problem (P). Since it is difficult to apply the maximum principle, we employ the variational method.[11]

Let

$$V = \int_0^\infty \Big[\pi(t) + \int_0^t \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds - \varphi(\lambda(t))\exp\Big(\int_0^t \lambda(\tau)d\tau\Big)\Big]e^{-\rho t}dt. \tag{12}$$

Suppose that the firm alters the original R&D plan so that the birth rate of a new product λ is increased by the amount of $\Delta\lambda$ in the time interval $(t,\ t+\Delta t)$, where $\Delta\lambda$ and Δt are both infinitesimal values. Let ΔV be the resultant change in V by this alteration of R&D plan. Then, ΔV can be calculated as follows:

$$\Delta V = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5, \tag{13}$$

where

$$\begin{aligned}
\Delta_1 &= \int_t^{t+\Delta t} e^{-\rho u}\int_t^u \Big[(\lambda(s)+\Delta\lambda)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)e^{\Delta\lambda(s-t)} - \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\Big]\pi(u-s)dsdu,\\
\Delta_2 &= \int_{t+\Delta t}^\infty e^{-\rho u}\int_t^{t+\Delta t} \Big[(\lambda(s)+\Delta\lambda)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)e^{\Delta\lambda(s-t)} - \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\Big]\pi(u-s)dsdu,\\
\Delta_3 &= \int_{t+\Delta t}^\infty e^{-\rho u}\int_{t+\Delta t}^u \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)(e^{\Delta\lambda\Delta t}-1)\pi(u-s)dsdu,\\
\Delta_4 &= -\int_t^{t+\Delta t} e^{-\rho u}\Big[\varphi(\lambda(u)+\Delta\lambda)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)e^{\Delta\lambda(u-s)} - \varphi(\lambda(u))\exp\Big(\int_0^u \lambda(\tau)d\tau\Big)\Big]dsdu,\\
\Delta_5 &= -\int_{t+\Delta t}^\infty e^{-\rho u}\varphi(\lambda(u))\exp\Big(\int_0^u \lambda(\tau)d\tau\Big)(e^{\Delta\lambda\Delta t}-1)dsdu.
\end{aligned} \tag{14}$$

The sum of Δ_1, Δ_2 and Δ_3 is the gain obtained by the change in R&D plan, while that of Δ_4 and Δ_5 is the loss caused by this change.

According to the variational method, the change in V, ΔV, is required to be 0 along an optimal R&D plan (provided that $\{\lambda(t)\}_{t=0}^\infty$ is an inner solution). This requirement can be interpreted from an economic point of

[11]The way to derive the condition for optimality is basically the same as that adopted by Judd [7] or Murakami [13]. Our Problem (P) is different from the problem analyzed by Judd [7] and may be regarded as a generalized version of that analyzed by Murakami [13].

view as follows; along an optimal R&D plan, the marginal gain (the sum of Δ_1, Δ_2 and Δ_3) must be equal to the marginal loss (the sum of Δ_4 and Δ_5). By the variational method, we can first obtain the following result about an inner solution of Problem (P).

Lemma 4. *Let Assumptions 1 and 2 hold. Assume that the following condition is also imposed in Problem (P):*[12]

$$\lambda(t) < \rho, \quad \text{for } t \geq 0. \tag{15}$$

If there exists a solution of Problem (P), $\{\lambda^*(t)\}_{t=0}^{\infty}$ *with* $\lambda^*(t) > 0$ *for all* $t \geq 0$*, then the following condition is satisfied for all* $t \geq 0$*:*

$$\dot{\lambda}^*(t) = \frac{1}{\varphi''(\lambda^*(t))}[\varphi'(\lambda^*(t))(\rho - \lambda^*(t)) - (r - \varphi(\lambda^*(t)))]. \tag{16}$$

where[13]

$$r = \rho \int_0^{\infty} \pi(t) e^{-\rho t} dt. \tag{17}$$

Proof. See Appendix D. □

Lemma 4 states that condition (16) is what is called the Euler condition in Problem (P). This is a necessary condition that a solution of Problem (P) or an optimal R&D plan must satisfy.

We take a close look at the positive constant r defined in (17). For this purpose, we rewrite (17) as follows:

$$r = \rho \int_t^{\infty} \pi(\tau - t) e^{-\rho(\tau - t)} d\tau. \tag{18}$$

Noting that $\pi(\tau - t)$ is the (expected) instantaneous profit obtained at time τ from product of vintage t, which is born by (expected) product innovation at time t, $\int_t^{\infty} \pi(\tau - t) e^{-\rho(\tau - t)} dt$ in (18) is the discounted present value (evaluated at time t) of the (expected) profit induced by (expected) product innovation at time t. Then, we find

[12] Condition (15) is necessary (but not sufficient) for the maximand of Problem (P) to be finite. It may also be interpreted as the transversality condition of Problem (P).

[13] It follows from Assumption 1 that the finiteness of ρ is readily confirmed:

$$|r| \leq \rho \int_0^{\infty} |\pi(\tau)| e^{-\rho \tau} d\tau \leq \rho \int_0^{\infty} k e^{-\rho \tau} d\tau = k,$$

where k is the supremum of $|\pi|$ (Assumption 1).

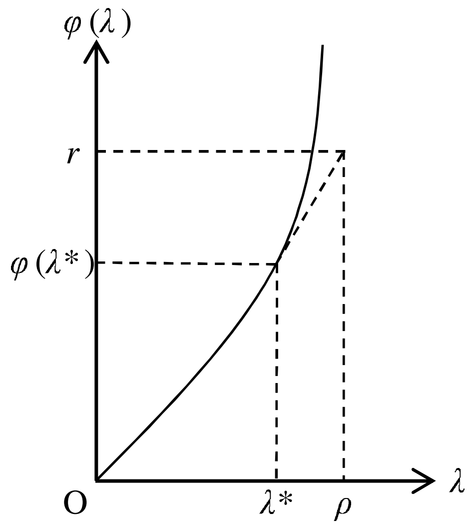

Figure 2: Optimal birth rate of a new product

from (18) that r is the *average (expected) profit* through the life cycle of product of vintage t. In this sense, the constant r can be viewed as the *expected profit on product innovation.* Furthermore, in relation to the expected stock of experience in product innovation Λ, this figure r corresponds to the *profit rate of expected stock of experience in product innovation.*[14]

Fortunately, we can prove that if a solution of Problem (P),$\{\lambda^*(t)\}_{t=0}^{\infty}$, exists, $\lambda^*(t)$ is constant over time.

Proposition 1. *Let Assumptions 1 and 2 hold. Assume that condition (15) is also imposed in Problem (P). If there exists a solution of Problem (P), $\{\lambda^*(t)\}_{t=0}^{\infty}$, then $\lambda^*(t)$ is equal to the constant λ^* for all $t \geq 0$ such that*

$$\begin{cases} [r - \varphi(\lambda^*)]/(\rho - \lambda^*) = \varphi'(\lambda^*) \ \text{ if } r/\rho > \varphi'(0) \\ \lambda^* = 0 \ \text{ if } r/\rho \leq \varphi'(0) \end{cases}, \tag{19}$$

where r is defined in (17).

Proof. See Appendix E. □

Figure 2 provides a graphical illustration as to how the optimal birth rate of a new product λ^* is determined.

Proposition 1 may imply the following fact. The optimal birth rate of a new product λ^* is positive if

$$\frac{r}{\rho} > \varphi'(0),$$

[14]The profit rate of expected stock of experience in product innovation r may be said to have a meaning similar to Keynes' [8] marginal efficiency of investment.

which can by (17) be rewritten as

$$\int_0^\infty \pi(t)e^{-\rho t}dt > \varphi'(0), \tag{20}$$

whilst the optimal birth rate of a new product λ^* is 0 if

$$\frac{r}{\rho} \le \varphi'(0),$$

or

$$\int_0^\infty \pi(t)e^{-\rho t}dt \le \varphi'(0). \tag{21}$$

The left hand side of (20) or (21) is the discounted present value of the expected profit on a new product invented by product innovation. Also, since this value is gained when new product is (marginally) born by product innovation, it may be regarded as the "marginal benefit" brought as a result R&D for product innovation. The right hand side of (20) or (21) is, on the other hand, the marginal expenditure on R&D for product innovation evaluated at $\lambda = 0$. Conditions (20) and (21) thus mean that the firm does not embark on R&D for product innovation unless the "marginal benefit" of R&D matches the "marginal cost" of it.

Interestingly, if λ and φ are read as the rate of capital accumulation and the effective cost (including adjustment cost) function of investment, respectively, the condition for optimality in Problem (M), (19), is identical with the condition for optimality in the optimal investment problem in the existence of adjustment cost of investment in Uzawa [19]. The profit rate of expected stock of experience in product innovation r corresponds to the profit rate of capital in Uzawa's [19, p. 643] condition. In this sense, our optimal planning problem concerning R&D for product innovation is closely related to the optimal capital accumulation problem in the existence of adjustment cost of investment. Furthermore, since the ratio of the profit rate of capital to the interest rate is generally equal to Tobin's [18] (average) q, a similar q ratio can be defined as $q = r/\rho$ in our framework of R&D investment.[15]

3.2 Firm growth

We turn our attention to the growth of the firm through product innovation. To discuss firm growth, the growth rate of the firm is measured in terms of the (expected) total revenue.[16]

[15] As Yoshikawa [21] clarified, Tobin's [18] q theory of investment can be derived from Uzawa's [19] adjustment cost theory of investment. In this respect, it is natural that some ratio, conceptually similar to Tobin's q, can be defined in our framework because our framework has much in common with Uzawa's [19] framework. For more details, see Murakami [11] [12].

[16] We may discuss the growth rate of the firm's (expected) total profit, but the (expected) total profit Π may be negative under Assumption 1. So we may avoid the difficulty arising from the zero profit by measuring the growth of the firm in terms of its total

As in the case of the profit π, we assume that the revenue from each product varies with passage of time in order to reflect "product life cycle," which is illustrated in Figure 1. Specifically, we suppose that the revenue of product of vintage s earned at time t, y_s^t, is a function of the "age" of the product, $t-s$, in the following form:

$$y_s^t = y(t-s). \tag{22}$$

In what follows, we refer to y as the revenue function. As regards the revenue function y, we impose the following assumption.

Assumption 3. $y : \mathbb{R}_+ \to \mathbb{R}_{++}$ *is bounded and differentiable.*

It is assumed in Assumption 3 that the revenue function y always takes on a positive value so that the complexity would not arise in the calculation in the growth rate of the firm.

Therefore, it follows from (7) and (22) that the expected total revenue along the optimal R&D plan $\{\lambda^*(t)\}_{t=0}^{\infty}$, which is defined in Proposition 1, at time $t, Y(t)$, can be calculated as follows:

$$\begin{aligned} Y(t) &= \sum_{i=1}^{N_0} y(t-s_i) + \sum_{N=N_0}^{\infty} \int_0^t \lambda^*(s) N P(N,s) y(t-s) ds \\ &= \sum_{i=1}^{N_0} y(t-s_i) + \sum_{N=N_0}^{\infty} \int_0^t \lambda^*(t-\tau) N P(N,t-\tau) y(\tau) d\tau. \end{aligned}$$

Since Proposition 1 implies that $\lambda^*(t)$ is constant over time, we can know from Lemma 1 that

$$Y(t) = \sum_{i=1}^{N_0} y(t-s_i) + N_0 \lambda^* e^{\lambda^* t} \int_0^t y(\tau) e^{-\lambda^* \tau} d\tau. \tag{23}$$

By Assumption 3, $Y(t)$ is positive for all $t \geq 0$. Then, the growth rate of the expected total revenue denoted by g can be derived as follows:

$$g(t) = \frac{\dot{Y}(t)}{Y(t)} = \lambda^* + \frac{\lambda^*[N_0 y(t) - \sum_{i=1}^{N_0} y(t-s_i)] + \sum_{i=1}^{N_0} \dot{y}(t-s_i)}{Y(t)}. \tag{24}$$

In what follows, we refer to g as the growth rate of the firm.

In the case where the optimal birth rate of a new product λ^* exists and is positive, we can prove that the

revenue, which is by definition positive.

asymptotic value of the growth rate of the firm g is equal to λ^*.

Proposition 2. *Let Assumptions 1-3 hold. Assume that there exists a positive solution of Problem (P), λ^*, which is characterized by (19). Then, $g(t)$, defined in (24), converges to λ^* as $t \to \infty$.*

Proof. See Appendix F. □

Proposition 2 indicates the fact that the growth rate of the firm (along the optimal R&D plan) is determined by the optimal birth rate of a new product λ^* in the long run, provided that λ^* is positive. Since the birth rate of a new product λ is determined as a result of the firm's behavior towards R&D for product innovation, the growth of the firm is driven by R&D for product innovation.

4 Conclusion

The analysis of this paper is first summarized.

In Section 2, we have presented a model of firm growth through product innovation that emphasizes the existence of product life cycle. In our model, the firm cannot enjoy growth for a long while due to the presence of product life cycle and so it has to invest on R&D, which is risky but enhances the possibility of its growth, and give birth to a new product (or service) to move forward. The birth rate of a new product is the key variable that the firm controls through its decision makings on R&D investment.

In Section 3, we have characterized the optimal R&D plan for product innovation and the resultant growth rate of the firm. We have demonstrated that the growth rate of the firm converges to the (optimal) birth rate of a new product. In other words, we have found that firm or economic growth is determined by the intensity of product innovation or by the activeness of R&D in the long run.

Finally, we shall briefly mention some possible future extensions of our framework. In the present study, the government is assumed to have no role in decision makings on R&D. In reality, however, corporate taxes and subsidies have much influence on R&D policies. In this respect, it may be worthwhile to analyze the effects of these factors in our framework.

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Appendix

A Proof of Lemma 1

Proof. Define the probability generating function G as follows:

$$G(z,t) = \sum_{N=0}^{\infty} P(N,t) z^N. \tag{25}$$

Differentiating both sides of (25) with respect to z, we have

$$\frac{\partial G}{\partial z} = \sum_{N=1}^{\infty} N P(N,t) z^{N-1}, \tag{26}$$

Differentiating both sides of (25) with respect to t and taking account of (2) and (26), we obtain

$$\frac{\partial G}{\partial t} = \lambda(t)\Big[\sum_{N=2}^{\infty}(N-1)P(N-1,t)z^N - \sum_{N=1}^{\infty} NP(N,t)z^N\Big] = \lambda(t)z(z-1)\frac{\partial G}{\partial z}. \tag{27}$$

Next, consider the following differential equations:

$$\frac{dz}{d\sigma} = -\lambda(t)z(z-1), \tag{28}$$

$$\frac{dt}{d\sigma} = 1, \tag{29}$$

with $t|_{\sigma=0} = 0$. Noting that $t = \sigma$, it follows from (28) that

$$-\int_0^t \lambda(s)ds = \log\Big(\frac{z}{1-z}\Big)\Big(\frac{1-m}{m}\Big),$$

where $m = z|_{\sigma=0}$. Hence, we obtain

$$m = \frac{z\exp(-\int_0^t \lambda(s)ds)}{1-[1-\exp(-\int_0^t \lambda(s)ds)]z}. \tag{30}$$

One can find from (25)-(29) that the derivative of G with respect to σ can be represented as follows:

$$\frac{dG}{d\sigma} = \frac{\partial G}{\partial z}\frac{dz}{d\sigma} + \frac{\partial G}{\partial t}\frac{dt}{d\sigma} = -\lambda(t)z(z-1)\frac{\partial G}{\partial z} + \frac{\partial G}{\partial t} = 0$$

Then, G is constant with respect to σ. Noting that $\sigma = t$, we find that G is also constant with respect to t. Since we have $G(z,0) = z^{N_0}$ because of $P(N_0, 0) = 1$ (see (25)) and $z^{N_0} = m$ for $\sigma = t = 0$, we find from (30) that:

$$G(z,t) = G(z,0) = m^{N_0} = \frac{[\exp(-\int_0^t \lambda(s)ds)]^{N_0} z^{N_0}}{\{1-[1-\exp(-\int_0^t \lambda(s)ds)]z\}^{N_0}},$$

or[17]

$$\begin{aligned} G(z,t) &= \Big[\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^{N_0} \sum_{n=0}^{\infty} \frac{(N_0+n-1)!}{(N_0-1)!n!}\Big[1-\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^n z^{n+N_0} \\ &= \sum_{N=N_0}^{\infty} \frac{(N-1)!}{(N_0-1)!(N-N_0)!}\Big[\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^{N_0}\Big[1-\exp\Big(-\int_0^t \lambda(s)ds\Big)\Big]^{N-N_0} z^N. \end{aligned} \tag{31}$$

We can obtain (3) by comparing (25) and (31) with each other. □

B Proof of Lemma 2

Proof. Define $\Pi_2(t)$ as follows:

$$\Pi_2(t) = \sum_{N=N_0}^{\infty} \int_0^t \lambda(s)NP(N,s)\pi(t-s)ds.$$

[17] Note that, for $|x| < 1$,

$$\frac{1}{(1-x)^{N_0}} = \sum_{n=0}^{\infty} \frac{(N_0+n-1)!}{(N_0-1)!n!}x^n.$$

Letting $x(t) = \exp(-\int_0^t \lambda(\tau)d\tau)$, it follows from (3) that Π_2 can be written as follows:

$$\Pi_2(t) = N_0 \int_0^t \lambda(s)\pi(t-s)x^{N_0}(s) \sum_{N=N_0}^{\infty} \frac{N!}{N_0!(N-N_0)!}[1-x(s)]^{N-N_0}ds. \tag{32}$$

Define S_{N_0} as follows:

$$S_{N_0} = \sum_{N=N_0}^{\infty} \frac{N!}{N_0!(N-N_0)!}(1-x)^{N-N_0} = \sum_{N=0}^{\infty} \frac{(N+N_0)!}{N_0!(N-N_0)!}(1-x)^N. \tag{33}$$

Multiplying both sides of (33) by $1-x$ and noting that $x \in (0,1)$, we have

$$(1-x)S_{N_0} = \sum_{N=0}^{\infty} \frac{(N+N_0)!}{N_0!(N-N_0)!}(1-x)^N. \tag{34}$$

Subtracting (34) from (33) yields

$$xS_{N_0} = 1 + \sum_{N=1}^{\infty} \frac{(N+N_0-1)!}{(N_0-1)!N!}(1-x)^N = \sum_{N=0}^{\infty} \frac{(N+N_0-1)!}{(N_0-1)!N!}(1-x)^N = S_{N_0-1}$$

or

$$S_{N_0} = x^{-1}S_{N_0-1}.$$

Hence, we obtain

$$S_{N_0} = x^{-N_0}S_0 = x^{-(N_0+1)}. \tag{35}$$

Finally, substituting (33) and (35) in (32) gives

$$\Pi_2(t) = N_0 \int_0^t \lambda(s)\pi(t-s)x^{-1}ds = N_0 \int_0^t \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(t-s)ds.$$

$\square$

C Proof of Lemma 3

Proof. Define $x(t)$ as in Appendix B. We find from (1) that

$$\Lambda(t) = N_0 \sum_{N=N_0}^{\infty} \frac{N!}{N_0!(N-N_0)!} x(t)^{N_0}[1-x(t)]^{N-N_0}.$$

Then, we know from (33) and (35) that

$$\Lambda(t) = N_0 x(t)^{N_0} S_{N_0} = N_0 x(t)^{-1} = N_0 \exp\Big(\int_0^t \lambda(\tau)d\tau\Big).$$

$\square$

D Proof of Lemma 4

Proof. It follows from the mean value theorem that for some $\theta_i, \theta_i', \theta_i'' \in (0,1), i = 1,2,3,4,5$, equations in (14) can be written as follows:

$$\Delta_1 = e^{-\rho(t+\theta_1'\Delta t)}\Big[\lambda(t+\theta_1\theta_1'\Delta t)\exp\Big(\int_0^{t+\theta_1\theta_1'} \Delta t\lambda(\tau)d\tau\Big)(e^{\theta_1\theta_1'\theta_1''\Delta\lambda\Delta t}\Delta t) + \exp\Big(\int_0^{t+\theta_1\theta_1'\Delta t} \lambda(\tau)d\tau\Big)e^{\theta_1\theta_1'\Delta\lambda\Delta t}\Big]$$
$$\times\, \pi(t+(1-\theta_1)\theta_1'\Delta t)\Delta\lambda(\Delta t)^2,$$
$$\Delta_2 = \Delta\lambda\Delta t\int_{t+\Delta t}^{\infty} e^{-\rho u}\Big[(\lambda(t+\theta_2\Delta t)\exp\Big(\int_0^{t+\theta_2\Delta t}\lambda(\tau)d\tau\Big)e^{\theta_2\theta_2'\Delta\lambda\Delta t}\Delta t + \Big(\int_0^{t+\theta_2\Delta t}\lambda(\tau)d\tau\Big)\Big]\pi(u-t-\theta\Delta t)du,$$
$$\Delta_3 = e^{\theta_3\Delta\lambda\Delta t}\Delta\lambda\Delta t\int_{t+\Delta t}^{\infty} e^{-\rho u}\int_{t+\Delta t}^{u}\lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(u-s)dsdu,$$
$$\Delta_4 = -e^{-\rho(t+\theta_4\Delta t)}\Big(\int_0^{t+\theta_4\Delta t}\lambda(\tau)d\tau\Big)[\varphi(\lambda(t+\theta_4\Delta t))e^{\theta_4'\Delta\lambda\Delta t}\Delta t + \varphi'(\lambda(t+\theta_4\Delta t)+\theta_4''\Delta\lambda) + \varphi'(\lambda(t+\theta_4\Delta t)+\theta_4'\Delta\lambda)]\Delta\lambda\Delta t,$$
$$\Delta_5 = -e^{\theta_5\Delta\lambda\Delta t}\Delta\lambda\Delta t\int_{t+\Delta t}^{\infty} e^{-\rho u}\varphi(\lambda(u))\Big(\int_0^u \lambda(\tau)d\tau\Big)du.$$

Since $\Delta\lambda$ and Δt are arbitrary infinitesimals, we know from (13) that along an optimal R&D plan $\{\lambda^*(t)\}_{t=0}^{\infty}$, the following condition must be satisfied for all $t \geq 0$:

$$\lim_{\Delta\lambda\to 0 \Delta t\to 0}\frac{\Delta V}{\Delta\lambda\Delta t} = \sum_{i=1}^{5}\lim_{\Delta\lambda\to 0,\Delta t\to 0}\frac{\Delta_i}{\Delta\lambda\Delta t} = 0,$$

or

$$\int_t^\infty \Big[\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)\pi(u-t) + \int_t^u \lambda^*(s)\exp\Big(\int_0^u \lambda^*(\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda^*(u))\exp\Big(\int_0^u \lambda^*(\tau)d\tau\Big)\Big]e^{-\rho(u-t)}du$$
$$= \varphi'(\lambda^*(t))\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big). \tag{36}$$

Condition (36) is thus the condition for optimality in Problem (P) in the case of an inner solution. By differentiating both sides of (36) with respect to t, we can obtain (16) as the condition for optimality in Problem (P). □

E Proof of Proposition 1

Proof. Define $V^*(t)$ as follows:

$$\begin{aligned} V^*(t) = \max_{\{\lambda(u)\geq 0\}_{u=t}^\infty} \int_t^\infty \Big[\Lambda^*_{\text{tran}}(t)\pi(u-t-s)ds + \int_t^u \lambda(s)\Lambda^*_{\text{tran}}(s)\pi(u-s) - \varphi(\lambda(u))\Lambda^*_{\text{tran}}(u)\Big]e^{-r(u-t)}du \\ \text{s.t. } \Lambda^*_{\text{tran}}(u) = \exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)\exp\Big(\int_t^u \lambda(\tau)d\tau\Big). \end{aligned} \tag{37}$$

Let $\{\lambda^*(t)\}_{t=0}^\infty$ be a solution of Problem (P). Then, it follows from the optimality of $\{\lambda^*(u)\}_{u=0}^\infty$ that

$$\begin{aligned} V^*(t) &= \exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big) \\ &\times \int_t^\infty \Big[\pi(u-t) + \int_t^u \lambda^*(s)\exp\Big(\int_t^s \lambda^*(\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda^*(u))\exp\Big(\int_t^u \lambda^*(\tau)d\tau\Big)\Big]e^{-r(u-t)}du. \\ &= \exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big) \\ &\times \int_0^\infty \Big[\pi(u) + \int_0^u \lambda^*(t+s)\exp\Big(\int_0^s \lambda^*(t+\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda^*(t+u))\exp\Big(\int_0^u \lambda^*(t+\tau)d\tau\Big)\Big]e^{-ru}du \end{aligned} \tag{38}$$

We also know from the optimality of $\{\lambda^*(u)\}_{u=0}^\infty$ that

$$\begin{aligned} V^*(t) &= \exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big) \\ &\times \max_{\{\lambda(t+u)\geq 0\}_{u=0}^\infty} \int_0^\infty \Big[\pi(u) + \int_0^u \lambda(t+s)\exp\Big(\int_0^s \lambda(t+\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda(t+u))\exp\Big(\int_0^u \lambda(t+\tau)d\tau\Big)\Big]e^{-ru}du, \end{aligned}$$

or

$$\frac{V^*(t)}{\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)} = \max_{\{\lambda(t+u)\geq 0\}_{u=0}^{\infty}} \int_0^{\infty} \Big[\pi(u) + \int_0^u \lambda(t+s)\exp\Big(\int_0^s \lambda(t+\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda(t+u))\exp\Big(\int_0^u \lambda(t+\tau)d\tau\Big)\Big]e^{-ru}du.$$

Since the value of t is irrelevant to that of the right hand side, we see that

$$\frac{V^*(t)}{\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)} = \max_{\{\lambda(u)\geq 0\}_{u=0}^{\infty}} \int_0^{\infty} \Big[\pi(u) + \int_0^u \lambda(s)\exp\Big(\int_0^s \lambda(\tau)d\tau\Big)\pi(u-s) - \varphi(\lambda(u))\exp\Big(\int_0^u \lambda(\tau)d\tau\Big)\Big]e^{-ru}du.$$

Noticing the definition of $V^*(t)$ given in (37), we arrive at

$$\frac{V^*(t)}{\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)} = V^*(0).$$

or

$$V^*(t) = V^*(0)\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big). \tag{39}$$

Comparing (36) and (39) with each other yields

$$\varphi'(\lambda^*(t))\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big) = V^*(t) = V^*(0)\exp\Big(\int_0^t \lambda^*(\tau)d\tau\Big)$$

or

$$\varphi'(\lambda^*(t)) = V^*(0). \tag{40}$$

Since t is arbitrarily chosen, equation (40) holds for every $t \geq 0$. Then, by condition (10) in Assumption 2, we find that $\lambda^*(t)$ is some constant λ^* for all $t \geq 0$. Since the solution of Problem (P) $\lambda^*(t)$ is equal to the constant λ^* for all $t \geq 0$, if λ^* is an inner solution with $\lambda^* > 0$, the value of λ^* is obtained by putting $\dot{\lambda}^*(t) = 0$ in (16):

$$\frac{r - \varphi(\lambda^*)}{\rho - \lambda^*} = \varphi'(\lambda^*). \tag{41}$$

Define

$$f(\lambda) = [r - \varphi(\lambda)] - \varphi'(\lambda)(\rho - \lambda).$$

Then, we have

$$f'(\lambda) = -\varphi''(\lambda)(\rho - \lambda) < 0, \quad \text{for } \lambda \in [0, \rho).$$

The last inequality follows from Assumption 2. Hence, for some positive $\lambda^* < \rho$[18] to satisfy $f(\lambda^*) = 0$ or (41), it is necessary (but not always sufficient) that

$$f(0) = r - \rho\varphi'(0) > 0.$$

Otherwise, there is no positive λ^* that fulfills (41). □

F Proof of Proposition 2

Proof. First, since $y(t)$ is positive for all $t \geq 0$ by Assumption 3 and λ^* is positive, we know from (23) that, for $t \geq 1$

$$Y(t) > N_0 \lambda^* e^{\lambda^* t} \int_0^t y(\tau) e^{-\lambda^* \tau} d\tau \geq N_0 \lambda^* e^{\lambda^* t} \int_0^1 y(\tau) e^{-\lambda^* \tau} d\tau.$$

Hence, we have

$$Y(t) > N_0 \lambda^* e^{\lambda^* t} \int_0^1 y(\tau) e^{-\lambda^* \tau} d\tau \to \infty \quad \text{as } t \to \infty. \tag{42}$$

Next, since $y(t)$ is bounded and differentiable for all $t \geq 0$ by Assumption 3, $\dot{y}(t)$ is bounded for all $t \geq 0$. Then, we find from (24) and (42) that

$$|g(t) - \lambda^*| \leq \frac{\lambda^* [N_0 |y(t)| + \sum_{i=1}^{N_0} |y(t - s_i)|] + \sum_{i=1}^{N_0} |\dot{y}(t - s_i)|}{Y(t)} \to 0, \quad \text{as } t \to \infty.$$

□

References

[1] Aghion, P., Howitt, P., 1998. Endogenous Growth Theory. MIT Press, Cambridge, MA.

[2] Aoki, M., Yoshikawa, H., 2007. Reconstructing Macroeconomics: A Perspective from Statistical Physics and Combinational Stochastic Processes. Cambridge University Press, Cambridge.

[18] Note that condition (15) is also imposed.

[3] Arrow, K. J., 1962. The economic implications of learning by doing. Review of Economic Studies 29 (3), 155-173.

[4] Barro, R. J., Sala-i-Martín, X., 1995. Economic Growth. McGraw-Hill, New York.

[5] Day, G. S., 1981. The product life cycle: Analysis and applications issues. Journal of Marketing 45, 60-67.

[6] Grossman, G. M., Helpmen, E., 1991. Innovation and Growth in the Global Economy. MIT Press, Cambridge, MA.

[7] Judd, K. L., 1985. On the performance of patents. Econometrica 53 (3), 567-586.

[8] Keynes, J. M., 1936. The General Theory of Employment, Interest and Money. Macmillan, London.

[9] Klepper, S., 1996. Entry, exit, growth, and innovation over the product life cycle. American Economic Review 86 (3), 562-583.

[10] Lucas, R. E., 1988. On the mechanics of economic development. Journal of Monetary Economics 22 (1), 3-42.

[11] Murakami, H., 2015. Wage flexibility and economic stability in a non-Walrasian model of economic growth. Structural Change and Economic Dynamics 32, 25-41.

[12] Murakami, H., 2016. A non-Walrasian microeconomic foundation of the "profit principle" of investment. In: Matsumoto A., Szidarovszky, F., Asada, T. (Eds.), Essays in Economic Dynamics: Theory, Simulation Analysis, and Methodological Study. Springer, Tokyo, forthcoming.

[13] Murakami, H., 2016. Economic growth with demand saturation and "endogenous" demand creation. mimeo.

[14] Romer, P. M., 1990. Endogenous technological change. Journal of Political Economy 98 (5), 71-102.

[15] Schumpeter, J. A., 1934. The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle. Harvard University Press, Cambridge, MA.

[16] Solow, R. M. 1956. A contribution to the theory of economic growth. Quarterly Journal of Economics 70 (1), 65-94.

[17] Solow, R. M., 1997. Is there a core of usable macroeconomics we should all believe in? American Economic Review 87 (2), 230-232.

[18] Tobin, J., 1969. A general equilibrium approach to monetary theory. Journal of Money, Credit and Banking 1 (1), 15-29.

[19] Uzawa, H., 1969. Time preference and the Penrose effect in a two-class model of economic growth. Journal of Political Economy 77 (4), 628-652.

[20] Vernon, R., 1966. International investment and international trade in the product cycle. Quarterly Journal of Economics 80 (2), 190-207.

[21] Yoshikawa, H., 1980. On the "q" theory of investment. American Economic Review, 70 (4), 739-743.