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Structural Estimation of the Scoring Auction Model

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Abstract

This paper offers an analytical framework for the scoring auction. We first characterize a symmetric monotone equilibrium in the scoring auction. We then propose a semiparametric procedure to identify the joint distribution of the bidder's multidimensional signal from scoring auction data. Our approach allows for a broad class of scoring rules in settings with multidimensional signals. Finally, using our analytical framework, we conduct an empirical experiment to estimate the impacts of the change of auction formats and scoring rules. The data on scoring auctions are from public procurement auctions for construction projects in Japan.

Keywords: Scoring auctions, Structural estimation, Procurement

JEL classification: C13, D44, L70

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1 Introduction

The scoring auction is a form of multidimensional bidding in which each bidder is asked to bid price and non-price attributes. A publicly announced scoring rule maps the multidimensional bid into the score, and the bidder who obtains the highest or lowest score is awarded. In procurement settings, for instance, the non-price attributes (quality bid) include service life, delivery date, and the extent of production processes's environmental burden. By encouraging a more complete comparison of the attributes of bidders and their proposals, the scoring auction allows the auctioneer to obtain greater welfare without reducing the bidders' profits than with price-only auctions (e.g., Bichler, 2000; Milgrom, 2004; Asker and Cantillon, 2008; and Lewis and Bajari, 2011).

In this paper, we offer a framework to analyze the scoring auction both theoretically and econometrically. Specifically, we construct a scoring auction model in which there are $n \geqq 2$ risk-neutral suppliers, each of which submits an L-dimensional bid with $L \geqq 2$ for a procurement contract after drawing a K-dimensional signal. The publicly known scoring rule maps each bidder's bid into a score, and the lowest-score bidder wins the contract. The bidder cost function is known up to the K-dimensional signal. We then examine identification of the scoring aumodel.

The contribution of our paper is threefold. First, we characterize the equilibrium in the scoring auction with minimal constraints on primitives. Since the seminal work by Che (1993), scoring auction analyses rely either on a specific form of the scoring rule – quasilinear (QL) – or on a strong restriction on the bidder's cost function – additively separable in quality and type, etc.[1] Our approach is free from those restrictions.

We show that the sorting condition (i.e., a slightly stronger condition than the single-crossing property) is sufficient to characterize a monotone equilibrium in setting with multidimensional types. The condition is commonly used in the auction literature to guarantee the existence of a strictly monotone equilibrium.[2] We then demonstrate that each bidder's

[1]See, e.g., Albano, Dini and Zampino (2009) and Hanazono, Nakabayashi and Tsuruoka (2013) for examples of non-QL scoring rules.

[2]McAdams (2003) is the first study that shows the equilibrium existence in games with incomplete information where types and actions are multidimensional. More recently, Reny (2011) shows the equilibrium existence in settings with multidimensional types and actions. Other studies on equilibrium existence frequently referred to in the auction literature include Lebrun (1996) and Athey (2001).

L-dimensional system of best-response functions characterizes a unique symmetric pure monotone equilibrium. Moreover, we propose a set of sufficient primitive conditions under which the sorting condition is satisfied in the scoring auction.

Second, we examine identification of the scoring auction model. To the best of our knowledge, our analysis is the first to propose a sufficient condition for the identification of the bidder's multidimensional signal from the scoring auction data. Our approach exploits the global invertibility of the bidder's system of best-response functions with respect to the multidimensional signal.

To make our argument precise, let $\boldsymbol{\theta}$, $\mathbf{q}^*$, and $\mathbf{b}^*$ denote the bidder's K-dimensional latent signal, the observed $(L-1)$-dimensional quality bid, and an L-dimensional vector that uniquely represents the remaining observables, such as the price bid and the distribution of bidders' scores. Let $C(\mathbf{q}, \boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta})$ with $\ell = 1, \ldots, L-1$ denote the bidder's cost function and its marginal cost with respect to the ℓth dimensional quality, q^ℓ, and define $A(\boldsymbol{\theta}; \mathbf{q}) := (C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \ldots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^{\mathrm{T}}$. We show that the bidder's system of best-response functions can be rearranged to an L-dimensional nonlinear system, $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$. We then propose a sufficient condition for the bidder's cost function under which $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible with respect to $\boldsymbol{\theta}$ for any $\mathbf{q}$. The condition is fairly natural – i.e., the cost function exhibits a relatively strong additive separable form, which makes each dimension of $\boldsymbol{\theta}$ have a unique impact on $A(\boldsymbol{\theta}; \mathbf{q})$. Finally, we apply the global inverse function theorem to show that the nonlinear system has a unique solution for $\boldsymbol{\theta}$.

Given that the approach relies on the bidder's best response, our procedure is a natural extension of the structural estimation method of auctions by Guerre, Perrigne and Vuong (2000). At the same time, given that the approach relies on the invertibility of the system of (best-response) functions, our procedure is related in spirit to the literature on identification for simultaneous equation systems, which exploits the global invertibility of real functions (e.g., Matzkin (2008)). Global invertibility – as in Beckert and Blundell (2008) and Berry, Gandhi and Haile (2013) – plays the central role in demand estimation analyses, as well. Both papers seek to provide economically interpretable sufficient conditions (e.g., connected substitutes) for the global invertibility of a demand system. In our analysis, the sufficient condition for the invertibility of the system of best-response functions is interpreted as relatively strong separability in the cost function.

Our final contribution is to provide an empirical analysis based on the structural method. We estimate the bidder's multidimensional signal using the scoring auction data. The data are from bid results of public procurement auctions for construction projects in Japan, where the scoring rule is not QL.[3] We conduct a series of counterfactual analyses to measure the impact of the change in auction formats (i.e., first-score (FS) vs second-score (SS) auctions) or scoring rules on utilities of both the procurement buyer and suppliers. Moreover, we estimate the buyer's gain from using scoring auctions instead of conventional price-only auctions.

We find that the government (auctioneer) would gain from using an alternative scoring-auction format; the buyer would obtain approximately .7 percent greater utility, on average, from using the SS rather than the observed FS auction. As for the impact of the change in the scoring rule, we find that by redesigning the scoring rule on the basis of the QL function, the buyer improves utility by approximately .7 percent. To measure the buyer's gain from the use of the scoring rather than the price-only auction, we simulate a series of counterfactual price-only auctions with different quality standards, based on which each bidder submits a price bid only. We find that the buyer obtains lower gains from the price-only auctions, which, however, vary from approximately 1.0 to 8.7 percent depending on the quality standard. The results suggest that a procurer can obtain an almost equivalent (slightly lower) gain with the use of a price-only auction with a well-designed fixed quality standard.

The reminder of this article is organized as follows. Section 2 describes the model of scoring auctions. Section 3 gives the equilibrium analysis. Section 4 discusses the identification of the distribution of bidders' cost schedule parameters. Section 5 conducts empirical examinations using our structural estimation method, and Section 6 concludes.

1.1 Related literature

Che (1993) gives the seminal analysis on scoring auctions, followed by Branco (1997), who relaxes the assumption that the bidder's private signal be independent, and by Asker

[3]The score is given by the weighted sum of non-price attributes divided by price. See Section 5 for more details.

and Cantillon (2008), who extend Che (1993) to settings in which bidders' signals are multidimensional. More recently, Wang and Liu (2014), Hanazono et al. (2013), and Dastidar (2014) provide theoretical analyses in the absence of the QL assumption. Among these, Hanazono et al. (2013) establish the most general framework for analyzing the scoring auction. They also show that their model applies to the previous analyses on multidimensional bidding, such as Thiel (1988), who analyzes the multidimensional quality competition with fixed payment, and Bajari, Houghton and Tadelis (2014), who analyze the multidimensional unit-price auction by using the scoring auction model.[4]

A large body of literature examines auctions in which the price is not the sole determinant of the winner. In this literature, a set of papers analyzes multidimensional bidding where non-price attributes are bidder characteristics – what bidders cannot choose at bidding (e.g., Marion, 2007; Krasnokutskaya and Seim, 2011; Krasnokutskaya, Song and Tang, 2013; and Mares and Swinkels, 2014) – unlike in the scoring auction. Asker and Cantillon (2008) discuss multidimensional bidding where bidders choose non-price attributes, but the auctioneer keeps the scoring rule secret at bidding (i.e., menu auctions and beauty contests). Studies on optimal design in multidimensional bidding are seen in Che (1993), Asker and Cantillon (2010), etc.

Our paper is also related to the literature on identification of the auction model, such as Athey and Haile (2002) and Athey and Haile (2007), and the literature on the structural estimation method of auctions, including Paarsch (1992); Laffont, Ossard and Vuong (1995); Guerre et al. (2000); and Krasnokutskaya (2011).[5] Given the multidimensionality in bidder private information, the structural estimation of the scoring auction model has a challenge similar to that of the structural estimation of auctions with risk-averse bidders, as in Guerre, Perrigne and Vuong (2009), Campo, Guerre, Perrigne and Vuong (2011), etc.[6] Similar to these analyses, our approach exploits a parametric assumption on the cost function to address this challenge.

As for empirical analyses, Lewis and Bajari (2011) is the first structural analysis on scoring auctions. Nakabayashi (2013) and Takahashi (2014) also perform structural analy-

[4]Athey and Levin (2001) also study the multidimensional unit-price auction.

[5]See Paarsch and Hong (2006) and Athey and Haile (2007) for book-length surveys. For a more recent survey, see, e.g., Hickman, Hubbard and Sağlam (2012).

[6]More recent papers include Campo (2012) and Fang and Tang (2014).

ses, and Iimi (2013) and Koning and van de Meerendonk (2014) offer analyses based on the reduced-form approach. So far, structural analyses have relied on restriction either on the scoring rule (i.e., the QL form, as in Lewis and Bajari, 2011) or on the cost function (i.e., an inverse L-shape, as in Nakabayashi (2013), or an additively separable form, as in Takahashi (2014)). Neither of these analyses discusses conditions under which the multidimensional type is identifiable in the scoring auction.

2 The Model of Scoring Auctions

A buyer would like to procure an item through competitive bidding by $n \geqq 2$ risk-neutral ex ante symmetric suppliers. Based on the knowledge of n, each supplier submits a price bid $p \geqq 0$, as well as an $(L-1)$-dimensional quality bid $\mathbf{q} = (q^1, \ldots, q^{L-1}) \in [\underline{q}^1, \bar{q}^1] \times \cdots \times [\underline{q}^{L-1}, \bar{q}^{L-1}] \equiv \mathcal{Q} \subset \mathbb{R}^{L-1}$, or stays out of the bidding. A monotone scoring function, $S(p, \mathbf{q}) : \mathbb{R}^L \to \mathbb{R}$, is common knowledge, mapping the L-dimensional bid into a score. Let $\mathcal{S} := \{S(p, \mathbf{q}) | p \geqq 0, \mathbf{q} \in \mathcal{Q}\}$ denote the feasible set of score.

Let $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [\underline{\theta}^0, \bar{\theta}^0] \times \cdots \times [\underline{\theta}^{K-1}, \bar{\theta}^{K-1}]$ denote a K-dimensional private signal that each supplier obtains prior to bidding. Let $C(\mathbf{q}, \boldsymbol{\theta})$ denote the supplier's cost function defined on $\mathcal{Q} \times \boldsymbol{\Theta}$. Let $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta})$ and $C_{\theta^k}(\mathbf{q}, \boldsymbol{\theta})$ denote the partial derivative of $C(\mathbf{q}, \boldsymbol{\theta})$ with respect to q^ℓ and θ^k with $\ell = 1, \ldots, L-1$ and $k = 0, \ldots, K-1$.

Two scoring-auction formats are considered in the analysis: the first-score (FS) auction and the second-score (SS) auctions. Let $(p_i, \mathbf{q}_i)$ denote bidder i's multidimensional bid in the scoring auction with $i = 1, \ldots n$, and let $(p^{post}, \mathbf{q}^{post})$ be the contracted payment and quality. The bidder with the lowest $S(p, \mathbf{q})$ wins, and only the winner performs the contract and receives a payment. Let $s_{(j)}$ denote the jth-lowest score in the auction with $j = 1, \ldots, n$.

In the FS auction, the contract payment and quality are equal to the winning bidder's bid – i.e., $S(p^{post}, \mathbf{q}^{post}) = s_{(1)}$. In the SS auction, the successful bidder can choose $(p^{post}, \mathbf{q}^{post})$ ex post such that $S(p^{post}, \mathbf{q}^{post}) = s_{(2)}$.

Then, bidder i's problems in the FS and SS auctions are given as follows:

$$\max_{p_i,\mathbf{q}_i} \left[p^{post} - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i)\right] \Pr\{\text{win}|S(p_i, \mathbf{q}_i)\}, \tag{FS}$$

subject to $(p^{post}, \mathbf{q}^{post}) = (p_i, \mathbf{q}_i)$ if i wins.

$$\max_{p_i,\mathbf{q}_i} E_{s_{(2)}}\left[\max_{p^{post},\mathbf{q}^{post}} \left\{p^{post} - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i)\middle| S(p^{post}, \mathbf{q}^{post}) = s_{(2)}\right\}\middle| \text{win}\right] \Pr\{\text{win}|S(p_i, \mathbf{q}_i)\}. \tag{SS}$$

Throughout the paper, we impose the following four assumptions:

Assumption 1 (Regularity)**.**

(i) The scoring rule, $S(p, \mathbf{q})$, is sufficiently smooth with $S_p(p, \mathbf{q}) > 0$ and $S_{q^\ell}(p, \mathbf{q}) < 0$ for all $\ell = 1, \dots, L-1$. In addition, for any $s \in \mathcal{S}$, $S_{q^\ell}(p, \mathbf{q})/S_p(p, \mathbf{q})$ subject to $S(p, \mathbf{q}) = s$ is weakly increasing in q^ℓ.

(ii) The bidder's cost function, $C(\mathbf{q}, \boldsymbol{\theta})$, is strictly positive, weakly convex, and twice-continuously differentiable for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. In addition, $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta}) \geqq 0$, $C_{\theta^k}(\mathbf{q}, \boldsymbol{\theta}) > 0$, $C_{q^\ell\theta^k}(\mathbf{q}, \boldsymbol{\theta}) \geqq 0$, and all are bounded for all $\ell = 1, \dots, L-1$ and $k = 0, \dots, K-1$.

(iii) The signal, $\boldsymbol{\theta}$, is distributed independently and identically according to a publicly known joint density, $f(\boldsymbol{\theta})$, which is continuous, positive measure, and bounded for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Assumption 1 is a set of regularity conditions imposed on the score and the cost functions. Several remarks are in order. First, condition (i) is a technical assumption and can be relaxed.[7] This condition implies that the iso-score curve depicted on the $q^\ell - p$ plain is weakly concave for all $\ell = 1, \dots, L-1$. Second, condition (ii) implies that the bidder's total and marginal costs are smooth and are increasing in $\boldsymbol{\theta}$. Note that the scoring auction model is invariant with any bijective mapping of $\boldsymbol{\theta}$. Hence, even if C_{θ^k}, $C_{q^\ell\theta^k}$, or both are negative for some $k = 0, \dots, K-1$, but for all $\ell = 1, \dots, L-1$ and for all $\boldsymbol{\theta}$ and $\mathbf{q}$, condition (ii) can hold by redefining the signal and the cost function.[8]

[7]See the discussion following Assumption 3.
[8]See the first remark on Assumption 4 for more details.

Third, by condition (iii), we assume independent private values. While assuming independence of $\boldsymbol{\theta}$ across suppliers, our model allows for correlation of $\boldsymbol{\theta}$ across dimension, – i.e., θ^k and $\theta^{k'}$ with $k \neq k'$ can be correlated with each other.

Finally, the scoring function is strictly monotone in p for any $\mathbf{q}$. This implies that the following function is well defined for all $s = S(p, \mathbf{q}) \in \mathcal{S}$:

$$P(s, \mathbf{q}) = p, \tag{1}$$

for any $\mathbf{q} \in \mathcal{Q}$. Moreover, because $S(p, \mathbf{q})$ is sufficiently smooth, $P(s, \mathbf{q})$ is also sufficiently smooth with respect to s and $\mathbf{q}$.[9] In fact, the partial derivatives of $P(s, \mathbf{q})$ with respect to s and q^ℓ are given by $P_s(s, \mathbf{q}) = 1/S_p(p, \mathbf{q})$ and $P_{q^\ell}(s, \mathbf{q}) = -S_{q^\ell}(p, \mathbf{q})/S_p(p, \mathbf{q})$, respectively, for all $\ell = 1, \ldots, L-1$. As discussed in the next section, the monotonicity of the scoring function implies that the choice problem regarding p in a multidimensional bid is replaceable with the choice problem regarding s.

Assumption 2 (Interior Solution). *For all $s \in \mathcal{S}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and $\mathbf{q} \in \mathcal{Q}$, $P(s, \mathbf{q}) - C(\mathbf{q}, \boldsymbol{\theta})$ is weakly concave. Moreover, for all $s \in \mathcal{S}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, there exists a closed interval $[\mathbf{q}_{\#}(s, \boldsymbol{\theta}), \mathbf{q}^{\#}(s, \boldsymbol{\theta})] \in \mathcal{Q}$ such that $P(s, \mathbf{q}) - C(\mathbf{q}, \boldsymbol{\theta})$ is i) strictly concave, ii) strictly increasing at $\mathbf{q}_{\#}(s, \boldsymbol{\theta})$, and iii) strictly decreasing at $\mathbf{q}^{\#}(s, \boldsymbol{\theta})$ with respect to $\mathbf{q}$.*[10]

Assumption 2 specifies the scope of our analysis: an interior solution. That is, we do not consider the case in which the auctioneer uses a quality bound in the scoring auction. Later, we will discuss that the interior solution assumption is a necessary condition for identifying bidders' K-dimensional signals. Note that the scoring auction literature regularly adopts this assumption (e.g., Asker and Cantillon, 2008).[11]

[9]Function $P(s, \mathbf{q})$ is discussed in Asker and Cantillon (2008), as denoted by $\Psi(\mathbf{Q}, t^w)$, where $\mathbf{Q}$ and t^w correspond to $\mathbf{q}$ and s in our model. Hanazono et al. (2013) show that $P(s, \mathbf{q})$ is feasible even if there is a binding reserve price.

[10]Note that we use "$\leq$" (or "$\geq$") as a vector inequality such that $\boldsymbol{\theta}_i \leq \hat{\boldsymbol{\theta}}_i$ if and only if $\theta_i^k \leqq \hat{\theta}_i^k$ for $k = 0, \ldots, K-1$ and $\boldsymbol{\theta}_i \neq \hat{\boldsymbol{\theta}}_i$.

[11]Hanazono et al. (2013) consider the case of corner solutions – i.e., binding price bounds or quality bounds. They show that by allowing corner solutions, the model of scoring auctions can apply to a broader settings in multidimensional bidding, such as the fixed price–best quality-proposal competition analyzed by Thiel (1988) and the (multidimensional) unit-price auction for incomplete contracts analyzed by Bajari et al. (2014). See Hanazono et al. (2013) for more details.

Due to Assumption 2, the relevant choice set of $\mathbf{q}$ for bidder type $\boldsymbol{\theta}$ is given in the following compact set:

$$\mathbf{Q}(\boldsymbol{\theta}) := \bigcup_{s \in \mathcal{S}} \left[\mathbf{q}_{\#}(s, \boldsymbol{\theta}), \mathbf{q}^{\#}(s, \boldsymbol{\theta})\right] \subset \mathcal{Q}.$$

This implies that, for any s, the bidder's optimal $\mathbf{q}$ exists in the interior of the domain of $C(\mathbf{q}, \cdot)$.

Assumption 3 (Sorting)**.**
For all $k = 0, \ldots, K-1$ and $\ell, m = 1, \ldots, L-1$,

(i) $\dfrac{\partial}{\partial q^m} \dfrac{C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta})}{C_{\theta^k}(\mathbf{q}, \boldsymbol{\theta})} \geqq 0,$

(ii) $\dfrac{\partial}{\partial q^\ell} \dfrac{P(s, \mathbf{q})}{P_s(s, \mathbf{q})} \geqq 0.$

for any $\mathbf{q}$ and $\boldsymbol{\theta}$ in the interiors of $\mathbf{Q}(\boldsymbol{\theta})$ and $\boldsymbol{\Theta}$.

Assumption 3 is an extension of Assumption 3 in Hanazono et al. (2013) to settings with multidimensional types and quality. Assumption 3 is a sufficient primitive condition for the *sorting condition* or, equivalently, the bidder's objective function exhibiting strictly increasing differences (Lemma 1). As known in the existing literature such as McAdams (2003), this condition ensures the existence of a pure monotone equilibrium in the Bayesian game. In Section 3.3, we show that the sorting condition guarantees not only the existence, but also the uniqueness of a symmetric pure monotone equilibrium in the FS auction. Note that this assumption is *not* required for an equilibrium in the SS auction and in the FS auction with $P_{sq^\ell} \geqq 0$ for all $\ell = 1, \ldots, L-1$ (e.g., the quasilinear (QL) scoring rule). On the other hand, we need to assume a litter more complex condition: $(C_{q^\ell q^m} - P_{q^\ell q^m})C_{\theta^k} \geqq C_{q^\ell} C_{q^m \theta^k}$, in replace with (i) if we relax Assumption 1-(i), – i.e., the score function is concave.[12]

[12]In fact, Hanazono et al. (2013) assume $(C_{qq} - P_{qq})C_\theta \geqq C_q C_{q\theta}$ in C-3 of Proposition 1 with single dimensional θ and q absent the concavity assumption of the score function. Condition $P_{sq^\ell} \leqq 0$ corresponds to C-2 of Proposition 1 in Hanazono et al. (2013).

Example: We briefly discuss a set of scoring rules that satisfy Assumptions 1 through 3. Let $V(\mathbf{q})$ be smooth, $V_{\mathbf{q}}$ be strictly positive, and $V_{\mathbf{qq}}$ be negative definite. Then, consider the following two scoring functions:

1. Quasilinear (QL): $S(p, \mathbf{q}) = \mu_1(p - V(\mathbf{q}))$,

2. Price-per-Quality Ratio (PQR): $S(p, \mathbf{q}) = \mu_2(p/V(\mathbf{q}))$,

where $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are some strictly increasing functions.

Note that, without any loss, we take μ_j as the identity function with $j = 1, 2$ because the allocation, payment, and payoffs are all invariant to μ_j. On the other hand, there is no monotone function that makes these two scoring rules isomorphic in the sense that two scoring rules generate the same outcome. Hence, it is worth discussing these two scoring rules, separately.

Now, let $s = S(p, \mathbf{q})$. Then, $P(s, \mathbf{q})$ is given by

$$P(s, \mathbf{q}) = s + V(\mathbf{q}), \tag{QL}$$

$$P(s, \mathbf{q}) = sV(\mathbf{q}), \tag{PQR}$$

under these scoring functions. Note, also, that P_s is equal to 1 (QL) or $V(\mathbf{q})$ (PQR). Hence, Assumption 3 holds in both cases.[13]

Assumption 4 (Identification)**.** *For all $\boldsymbol{\theta}$ in the interior of $\boldsymbol{\Theta}$ and for all $\mathbf{q}$ in $\mathcal{Q}$, there exists a K-dimensional nonsingular matrix, $\boldsymbol{\Gamma}$, with which $\tilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) := C(\mathbf{q}, \boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}) = C(\mathbf{q}, \boldsymbol{\theta})$*

[13] With monotone function μ_j, both are $\mu_1^{-1}(s)$ and $\mu_2^{-1}(s)V(\mathbf{q})$, respectively. The assumption is satisfied in both cases.

satisfies

$$
\begin{aligned}
|\tilde{C}_{\tilde{\theta}^0}(\mathbf{q}, \tilde{\boldsymbol{\theta}})| &> \sum_{k=1}^{K-1} |\tilde{C}_{\tilde{\theta}^k}(\mathbf{q}, \tilde{\boldsymbol{\theta}})|, \\
|\tilde{C}_{q^1\tilde{\theta}^1}(\mathbf{q}, \tilde{\boldsymbol{\theta}})| &> \sum_{k\in\{0,\dots,K-1|k\neq 1\}} |\tilde{C}_{q^1\tilde{\theta}^k}(\mathbf{q}, \tilde{\boldsymbol{\theta}})|, \\
&\vdots \\
|\tilde{C}_{q^\ell\tilde{\theta}^\ell}(\mathbf{q}, \tilde{\boldsymbol{\theta}})| &> \sum_{k\in\{0,\dots,K-1|k\neq \ell\}} |\tilde{C}_{q^\ell\tilde{\theta}^k}(\mathbf{q}, \tilde{\boldsymbol{\theta}})|, \\
&\vdots \\
|\tilde{C}_{q^{L-1}\tilde{\theta}^{L-1}}(\mathbf{q}, \tilde{\boldsymbol{\theta}})| &> \sum_{k\in\{0,\dots,K-1|k\neq K-1\}} |\tilde{C}_{q^{K-1}\tilde{\theta}^k}(\mathbf{q}, \tilde{\boldsymbol{\theta}})|.
\end{aligned}
$$

Two remarks are in order. First, Assumption 4 is satisfied if the cost function exhibits an additively separable form for each dimension of $\mathbf{q}$ such that $C_{q^\ell\theta^k} = 0$ for $\ell = 1, \dots, L-1$ and $k = 0, \dots, K-1$ with $k \neq \ell$. Moreover, it allows for many non-additively-separable cost functions as long as θ^ℓ has a stronger impact on the ℓth dimensional marginal cost, $C_{q^\ell}(\cdot)$, than the aggregate sum of all other dimensions of $\boldsymbol{\theta}$ and θ^0 has a stronger impact on the total cost, $C(\cdot)$, than the sum of all other dimensions of $\boldsymbol{\theta}$, under some bijective transformation of $\boldsymbol{\theta}$.

Second, let $A(\boldsymbol{\theta}; \mathbf{q}) := (C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \dots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^T \in \mathbb{R}^L$. Then, this assumption is equivalent to $A(\boldsymbol{\theta}; \mathbf{q})$ having a full column rank Jacobian matrix (Lemma 4).[14] We then show that the full column rank Jacobian is sufficient for the global invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$ with respect to $\boldsymbol{\theta}$ if the dimension of $\boldsymbol{\theta}$ is no greater than the dimension of the multidimensional bid (Proposition 3).

[14] If $\boldsymbol{\Gamma}$ is restricted to be a diagonal matrix, the Jacobian of $A(\boldsymbol{\theta}; \mathbf{q})$ is the quasi-dominant diagonal (q.d.d.) matrix discussed in McKenzie (1960). Because Assumption 4 is weaker than the condition for the Jacobian matrix to be q.d.d, Assumption 4 is not only sufficient but also necessary for a full-rank Jacobian of $A(\boldsymbol{\theta}; \mathbf{q})$ in our setting (i.e., $A(\cdot)$ is differentiable).

Example: Consider the following cost function:

$$C(\mathbf{q},\boldsymbol{\theta}) = \begin{cases} (q+\alpha^0\theta^0+\theta^1)^\beta + \theta^0 + \alpha^1\theta^1 & \text{if } q+\alpha^0\theta^0+\theta^1 \geqq 0 \\ \theta^0 + \alpha^1\theta^1 & \text{otherwise} \end{cases} \tag{2}$$

with $q > 0$, $\theta^0 > 0$, and $\theta^1 > 0$, where parameters $\alpha^j > 0$ with $j = 0, 1$ and $\beta > 1$ are all finite.[15]

With $\alpha^0\alpha^1 \neq 1$, let us define

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 \\ \alpha^0 & \Phi \end{bmatrix},$$

where $\Phi = (1-\alpha^0\alpha^1)/(\alpha^1 + \beta D^{\beta-1} + \epsilon)$, $D = q + \alpha^0\theta^0 + \theta^1$, and $\epsilon > 0$. Then, the Jacobian matrix of $\tilde{A}(\tilde{\boldsymbol{\theta}};\mathbf{q}) = (\tilde{C}(\mathbf{q},\tilde{\boldsymbol{\theta}}), \tilde{C}_{q^1}(\mathbf{q},\tilde{\boldsymbol{\theta}}), \ldots, \tilde{C}_{q^{L-1}}(\mathbf{q},\tilde{\boldsymbol{\theta}}))$ is given by:

$$\tilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}};\mathbf{q}) \equiv \mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q})\boldsymbol{\Gamma}^{-1} = \begin{bmatrix} \alpha^1 + \beta D^{\beta-1} + \epsilon & \alpha^1 + \beta D^{\beta-1} \\ 0 & \beta(\beta-1)D^{\beta-2} \end{bmatrix}.$$

Because the diagonal element is greater than the off-diagonal element for each row, $C(\mathbf{q},\boldsymbol{\theta})$ satisfies Assumption 4.

If $\alpha^0\alpha^1 = 1$, on the other hand, we do not find $\boldsymbol{\Gamma}$ that makes $\tilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}$ strictly diagonally dominant matrix. In fact, θ^0 and θ^1 are linearly related in the cost function as:

$$C(q,\boldsymbol{\theta}) = (q + \alpha^0\theta^{\#})^\beta + \theta^{\#},$$

where $\theta^{\#} = \theta^0 + \theta^1/\alpha^0$. As will be discussed in Section 4, this leads to nonidentification of $\boldsymbol{\theta}$ because $A(\boldsymbol{\theta};\mathbf{q})$ is not invertible.

[15] In Section 5, we conduct an empirical study. We use the same cost function with $\alpha^0 = \alpha^1 = 0$.

3 Equilibrium Analysis

In this section, we consider an alternative game (called the *score-bidding game*) to replicate the equilibrium of the scoring auction. Given that the outcome of the scoring auction consists of each bidder's score and the contract price and quality, the score-bidding game is outcome-equivalent to the original scoring auction. The approach is similar to that in the existing literature on scoring auctions such as Che (1993), Asker and Cantillon (2008), and especially Hanazono et al. (2013). Many of the arguments in this section are similar to Hanazono et al.'s (2013), except for multidimensional type and quality being taken into account. Our novel approach is seen in the analysis of the FS auction, illustrated in subsection 3.3.[16]

3.1 An Outcome-equivalent Score-bidding game (Hanazono et al. (2013))

Consider a score-bidding game in which each supplier submits a score, $s \in \mathcal{S}$, or stays out. The supplier receives zero payoff if staying out, and the score for the bidder who stays out is equal to ∞. The lowest-score bidder wins. Only the winner chooses a quality vector, $\mathbf{q}^{post}$. Based on $\mathbf{q}^{post}$, the winner performs the project work and receives a payment. Let $s_{(j)}$ denote the jth lowest score in the auction. In the FS auction, the payment is equal to $P(s_{(1)}, \mathbf{q}^{post})$. In the SS auction, the payment is equal to $P(s_{(2)}, \mathbf{q}^{post})$. Clearly, the outcome of this score-bidding game – i.e., each bidder's score and the contract price and quality – is the same as in the original scoring auction.

Using $P(s, \mathbf{q})$ in expression (1), bidder i's problems (FS) and (SS) are replicated with the score-bidding game as

$$\max_{s_i \in \mathcal{S}} \left[\max_{\mathbf{q}^{post} \in \mathbf{Q}(\boldsymbol{\theta}_i)} \left\{ P(s_i, \mathbf{q}^{post}) - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i) \right| s_i \right\} \right] \Pr\{\text{win}|s_i\}, \tag{FS'}$$

$$\max_{s_i \in \mathcal{S}} E_{s_{(2)}} \left[\max_{\mathbf{q}^{post} \in \mathbf{Q}(\boldsymbol{\theta}_i)} \left\{ P(s_{(2)}, \mathbf{q}^{post}) - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i) \right| s_{(2)} \right\} \left| \text{win} \right] \Pr\{\text{win}|s_i\}. \tag{SS'}$$

[16] Hanazono et al. (2013) analyze a multidimensional environment in which the bidder's strategic behavior can depend only on a single parameter. To do so, they impose a substantial restriction on the cost function (such as homogeneous of degree one.). On the other hand, Hanazono et al. (2013) consider the case in which a reserve price or quality bound binds in equilibrium, both of which are beyond the scope in this paper.

To consider the maximization problem in the square brackets, define

$$\mathbf{q}(s, \boldsymbol{\theta}) = \arg\max_{\mathbf{q}^{post}} \left\{ P(s, \mathbf{q}^{post}) - C(\mathbf{q}^{post}, \boldsymbol{\theta}) \middle| \, s \right\},$$

for some $s \in \mathcal{S}$. Here, s is equal to s_i in the FS auction and equal to $s_{(2)}$ in the SS auction. Assumption 2 ensures the existence and uniqueness of $\mathbf{q}(s, \boldsymbol{\theta})$ as an interior solution. This implies that the solution to the maximization problem in the square bracket of (FS') and (SS') is given by the following $(L-1)$-dimensional system of equations:

$$\begin{bmatrix} P_{q^1}(s, \mathbf{q}(s, \boldsymbol{\theta})) \\ \vdots \\ P_{q^{L-1}}(s, \mathbf{q}(s, \boldsymbol{\theta})) \end{bmatrix} = \begin{bmatrix} C_{q^1}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \\ \vdots \\ C_{q^{L-1}}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \end{bmatrix}. \tag{3}$$

Given the result, the bidder's *induced utility*, discussed in Hanazono et al. (2013), is well defined and given as

$$u(s, \boldsymbol{\theta}) = P(s, \mathbf{q}(s, \boldsymbol{\theta})) - C(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}).$$

The induced utility represents the conditional payoff upon winning given that $s = s_i (= s_{(1)})$ in the FS auction and $s = s_{(2)}$ in the SS auction. Then, replacing the square brackets in (FS') and (SS') with $u(\cdot)$, we can replicate bidder i's problems in the FS and SS auctions, (FS) and (SS), with the following one-dimensional optimization problems in the score-bidding games:

$$\max_{s_i \in \mathcal{S}} u(s_i, \boldsymbol{\theta}_i) \Pr\{\text{win}|s_i\}, \tag{FS''}$$

$$\max_{s_i \in \mathcal{S}} u(s_{(2)}, \boldsymbol{\theta}_i) \Pr\{\text{win}|s_i\}. \tag{SS''}$$

We finally discuss the smoothness of $u(s, \boldsymbol{\theta})$; let $u_s(s, \boldsymbol{\theta})$ and $u_{\theta^k}(s, \boldsymbol{\theta})$ denote the partial derivatives of $u(\cdot)$ with respect to s and θ^k with $k = 0, \dots, K-1$, respectively. Because both $P(s, \mathbf{q})$ and $C(\mathbf{q}, \boldsymbol{\theta})$ are smooth, we have $P_s(\cdot) = 1/S_p(\cdot) > 0$ and $C_{\theta^k} > 0$. Therefore, the derivatives of $u(s, \boldsymbol{\theta})$ exist as:

$$\begin{aligned} u_s(s, \boldsymbol{\theta}) &= P_s(s, \mathbf{q}(s, \boldsymbol{\theta})) > 0, \\ u_{\theta^k}(s, \boldsymbol{\theta}) &= -C_{\theta^k}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) < 0. \end{aligned} \tag{4}$$

Moreover, it is easy to see that $u_s(\cdot)$ and $u_{\theta^k}(\cdot)$ are differentiable with respect to s and that u_s is differentiable with respect to $\boldsymbol{\theta}$; applying the implicit function theorem to (3), we see that $\mathbf{q}(s, \boldsymbol{\theta})$ is differentiable with respect to both s and $\boldsymbol{\theta}$. Let q_s and q_{θ^k} denote the derivatives with respect to s and θ^k. Then, for all $k = 0, \ldots, K-1$, we have $u_{s\theta^k} = -\sum_{\ell=1}^{L-1} C_{q^\ell \theta^k} q_s^\ell$ and $u_{ss} = P_{ss} + \sum_{\ell=1}^{L-1} P_{sq^\ell} q_s^\ell$. Note that $P_{ss}(\cdot)$ and $P_{sq^\ell}(\cdot)$ are bounded because $S(\cdot)$ is assumed to be sufficiently smooth.

3.2 Equilibrium in the SS auction

Hanazono et al. (2013) characterize the equilibrium of the SS auction under the general multidimensional type-space environment. In the rest of this subsection, we discuss several features of Hanazono et al. (2013) that are relevant to our analysis.

In the SS bidding game, there exists a dominant strategy equilibrium, $\sigma_{\text{II}} : \boldsymbol{\Theta} \to \mathcal{S}$, such that bidder i with $i = 1, \ldots, n$ chooses $\sigma_{\text{II}}(\boldsymbol{\theta}_i) = z(\boldsymbol{\theta}_i) \in \mathcal{S}$, where

$$z(\boldsymbol{\theta}) = \{s | u(s, \boldsymbol{\theta}) = 0\}$$

is named as the break-even score in Hanazono et al. (2013), representing the minimum score subject to non-negative payoffs. If the scoring rule is QL, $z(\boldsymbol{\theta})$ is the bidder's *pseudo-type* discussed in Asker and Cantillon (2008). Hanazono et al. (2013) show that $z(\boldsymbol{\theta})$ is well-defined and strictly monotone in $\boldsymbol{\theta}$ under the multidimensional type-space environment.

By (3), the optimal quality choice in the SS bidding game is given by

$$\mathbf{q}^{FB}(\boldsymbol{\theta}) := \{\mathbf{q} | P(z(\boldsymbol{\theta}), \mathbf{q}) - C(\mathbf{q}, \boldsymbol{\theta}) = 0\}.$$

Assumption 2 ensures the well-definedness and uniqueness of $\mathbf{q}^{FB}(\boldsymbol{\theta})$. Therefore, the equilibrium in the SS auction is summarized as follows:

Proposition 1. *In the SS auction, there exists a dominant strategy equilibrium in which bidder i submits $(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i))$ such that*

$$(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i)) = (P(z(\boldsymbol{\theta}_i), \mathbf{q}^{FB}(\boldsymbol{\theta}_i)), \mathbf{q}^{FB}(\boldsymbol{\theta}_i)), \tag{5}$$

where $z(\boldsymbol{\theta}_i)$ and $\mathbf{q}^{FB}(\boldsymbol{\theta}_i)$ are given by

$$\begin{aligned}
u(z(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i) &= 0, \\
P_{q^1}(z(\boldsymbol{\theta}_i), \mathbf{q}^{FB}(\boldsymbol{\theta}_i)) &= C_{q^1}(\mathbf{q}^{FB}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i), \\
&\vdots \\
P_{q^{L-1}}(z(\boldsymbol{\theta}_i), \mathbf{q}^{FB}(\boldsymbol{\theta}_i)) &= C_{q^{L-1}}(\mathbf{q}^{FB}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i).
\end{aligned}$$

The smoothness of $z(\boldsymbol{\theta})$ in $\boldsymbol{\theta}$ is easily shown (while it is unnecessary for the equilibrium analysis of the SS auction but will, in turn, be required for the analysis of the FS auction.). Because $z(\cdot)$ satisfies $u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, the implicit function theorem ensures that for all $\boldsymbol{\theta}$, the partial derivative of $z(\boldsymbol{\theta})$ with respect to θ^k exists locally for all $k = 0, \ldots, K-1$. It is also easily seen that $z(\boldsymbol{\theta})$ is strictly increasing in θ^k. Let $z_{\theta^k}(\boldsymbol{\theta})$ denote the partial derivative of $z(\boldsymbol{\theta})$ with respect to θ^k. Then, taking the derivative on both sides of $u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0$ with respect to θ^k gives

$$u_s(z(\boldsymbol{\theta}), \boldsymbol{\theta}) z_{\theta^k}(\boldsymbol{\theta}) + u_{\theta^k}(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0.$$

Given that $u_{\theta^k} < 0$ and that $u_s > 0$, we have $z_{\theta^k}(\boldsymbol{\theta}) > 0$.

3.3 Equilibrium in the FS auction

We next consider the FS auction with a reserve score $s^r \in (\min_{\boldsymbol{\theta}} z(\boldsymbol{\theta}), \max_{\boldsymbol{\theta}} z(\boldsymbol{\theta}))$. We introduce a reserve score to ensure the uniqueness of the pure monotone equilibrium, following Maskin and Riley (1984).[17] Note that the argument below works even if the reserve

[17] The reserve score guarantees the smoothness of the bidder's first-order condition at the boundary, as shown in Maskin and Riley (1984). In the analysis on scoring auctions, Hanazono et al. (2013) employ a

score is equal to the break-even score of the least efficient supplier.

Then, there are three sets of supplier types:

$$
\begin{aligned}
\text{i) } \boldsymbol{\Theta}_{r+} &:= \{\boldsymbol{\theta} | z(\boldsymbol{\theta}) > s^r\}, \\
\text{ii) } \boldsymbol{\Theta}_{r} \;\; &:= \{\boldsymbol{\theta} | z(\boldsymbol{\theta}) = s^r\}, \\
\text{iii) } \boldsymbol{\Theta}_{r-} &:= \{\boldsymbol{\theta} | z(\boldsymbol{\theta}) < s^r\}.
\end{aligned}
$$

Each of these sets corresponds to i) a set of supplier types that earn negative profits from bidding s^r; ii) those that earn zero profits from bidding s^r; and iii) those that earn positive profits from bidding s^r. Because $z(\boldsymbol{\theta})$ is strictly increasing and smooth, $\boldsymbol{\Theta}_{r-}$, $\boldsymbol{\Theta}_{r}$, and $\boldsymbol{\Theta}_{r+}$ are mutually exclusive with $\boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_{r} \cup \boldsymbol{\Theta}_{r+} = \boldsymbol{\Theta}$.[18]

Let $\bar{x}$ denote the probability that the supplier type is either in $\boldsymbol{\Theta}_{r-}$ or $\boldsymbol{\Theta}_{r}$, which is given by

$$
\bar{x} = \Pr\{s^r \geqq z(\boldsymbol{\theta})\} = \int_{\{\widetilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_{r}\}} f(\widetilde{\boldsymbol{\theta}}) d\widetilde{\boldsymbol{\theta}}.
$$

Because nether $\boldsymbol{\Theta}_{r+}$ nor $\boldsymbol{\Theta}_{r} \cup \boldsymbol{\Theta}_{r-}$ is empty, we have $\bar{x} \in (0, 1)$.

Then, we demonstrate the existence and the uniqueness of the symmetric pure monotone equilibrium strategy in the FS bidding game, $\sigma_{\text{I}} : \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_{r} \to \mathcal{S}$. Given σ_{I}, bidder i's equilibrium multidimensional bid in the original FS auction is given by

$$
\left(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i)\right) = \left(P(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \mathbf{q}(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)), \mathbf{q}(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)\right), \tag{6}
$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_{r}$ and staying out for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r+}$.

Suppose that all bidders except for bidder i follow σ_{I}. Let $G(s)$ and $g(s)$ denote the distribution and density of the score by bidder i's rival. Then, problem (FS") for bidder i is

binding reserve score to show the existence of a unique equilibrium.

[18] Suppose, by contradiction, that $\exists \boldsymbol{\theta}$ such that $\boldsymbol{\theta} \in \{\boldsymbol{\Theta}_{r+} \cap \boldsymbol{\Theta}_{r}\}$. Because $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r}$, we have $z(\boldsymbol{\theta}) = s^r$. On the other hand, we have $z(\boldsymbol{\theta}) > s^r$ because $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r+}$. We thus have a contradiction. A similar argument holds for the remaining cases. It is obvious that the smoothness of $z(\boldsymbol{\theta})$ implies $\boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_{r} \cup \boldsymbol{\Theta}_{r+} = \boldsymbol{\Theta}$

given by

$$\max_{s_i \in \mathcal{S}} \pi(s_i, \boldsymbol{\theta}_i) = u(s_i, \boldsymbol{\theta}_i)[1 - G(s_i)]^{n-1}, \tag{7}$$

and the first-order condition is given by

$$\frac{1 - G(s_i)}{(n-1)g(s_i)} = \frac{u(s_i, \boldsymbol{\theta}_i)}{u_s(s_i, \boldsymbol{\theta}_i)} \tag{8}$$

if $g(\cdot) \neq 0$.

McAdams (2003) has shown that a pure monotone equilibrium exists if $\pi(s, \boldsymbol{\theta})$ satisfies the *single-crossing property* – i.e., $\partial^2 \pi(s, \boldsymbol{\theta})/\partial s \partial \theta^k \geqq 0$ for all $\ell = 0, \ldots, L-1$. Here, we demonstrate that the sorting condition defined below, which is a bit stricter condition than the single-crossing property, also guarantees the uniqueness of the symmetric pure monotone equilibrium.

Definition 1 (Sorting condition)**.** *For any $s \in \mathcal{S}$, $u(s, \boldsymbol{\theta})$ satisfies the sorting condition if*

$$\frac{\partial}{\partial \theta^k} \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} < 0,$$

for all $k = 0, \ldots, K-1$.

The following lemma illustrates that Assumption 3 is a set of sufficient primitive conditions for the sorting condition. Note that the lemma is an extension of Proposition 1 in Hanazono et al. (2013) to settings with multidimensional types.[19]

Lemma 1 (Sorting condition)**.** *If Assumption 3 holds, then $u(s, \boldsymbol{\theta})$ satisfies the sorting condition.*

Proof. See Appendix A. □

Then, we have a proposition regarding the characteristics and the uniqueness of a symmetric pure monotone equilibrium in the FS auction.

[19]In fact, the sorting condition in Hanazono et al. (2013) does not require the smoothness of $u(s, \theta)/u_s(s, \theta)$ in order to accommodate the case of a binding reserve price.

Proposition 2. *The symmetric pure monotone equilibrium strategy, $\sigma_{\mathrm{I}}(\cdot)$, is unique. Moreover, $\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)$ and $\mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)$ satisfy the L-dimensional system of best-response functions:*

$$\frac{1 - G(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i))}{(n-1)g(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i))} = \frac{u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}{u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}, \tag{9}$$

$$\begin{aligned} P_{q^1}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)) &= C_{q^1}(\mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i), \boldsymbol{\theta}_i) \\ &\vdots \\ P_{q^{L-1}}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)) &= C_{q^{L-1}}(\mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i), \boldsymbol{\theta}_i). \end{aligned} \tag{10}$$

It is easy to see that expression (10) is given by expression (3) with $s = \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)$. Hence, in what follows, we show that there uniquely exists an equilibrium strategy, $\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)$, that satisfies (9) and is distributed according to $G(\cdot)$.

We first show that the sorting condition guarantees a unique solution to (7). We guess that both $G(s)$ and $g(s)$ satisfy the following properties:

$$\begin{cases} G(s) \in [0, \bar{x}) \text{ for all } s < s^r \text{ and } G(s^r) = \bar{x}; \\ G(s) \text{ is strictly increasing and differentiable for all } s < s^r; \\ \lim_{s \to s^r} g(s) = \infty. \end{cases} \tag{Guess}$$

In our situation, the cross partial derivative of $u(s, \boldsymbol{\theta})$ with respect to s and $\boldsymbol{\theta}$ exists. Therefore, given (Guess), the sorting condition is equivalent to *strict increasing differences* in the bidder's interim expected profit, $\pi(s, \boldsymbol{\theta})$. It is well known that this property ensures the pseudoconcavity of the bidder's objective function with respect to s_i.[20] It follows that maximization problem (7) has a unique global maximizer if $G(\cdot)$ and $g(\cdot)$ satisfy (Guess). Moreover, the sorting condition implies that the solution is strictly increasing in $\boldsymbol{\theta}$.[21]

We then conclude the proof by showing that there uniquely exists $G(\cdot)$ that satisfies (9). Let $\bar{t}(s) := \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ and $\underline{t}(s) := \inf_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ denote the upper and lower bounds of $u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ for some $s \in \mathcal{S}$, and let $T(s) := [\underline{t}(s), \bar{t}(s)]$. Note that $u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ is a random variable given s. Thus, we define a survival function of

[20] See, e.g., Matthews (1995); Hanazono et al. (2013).

[21] In Online Appendix I, we verify these points.

$u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$ as:

$$\xi(t;s) := \Pr\left\{ \frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})} \geqq t \,\middle|\, s \right\} = \int_t^{\bar{t}(s)} \int_{\{\tilde{\boldsymbol{\theta}} | u(s,\tilde{\boldsymbol{\theta}})/u_s(s,\tilde{\boldsymbol{\theta}})=\hat{t}\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} d\hat{t},$$

for any $s \in \mathcal{S}$ and $t \in T(s)$.

Note that $\xi(t;s)$ is smooth and strictly decreasing in t.[22] Moreover, $\xi(\bar{t}(s);s) = 0$ and $\xi(\underline{t}(s);s) = 1$ for any $s \in \mathcal{S}$. Therefore, $\xi : T(s) \to [0,1]$ is bijective given s. It follows that $\xi(t;s)$ is invertible with respect to t for all $s \in \mathcal{S}$.

Let $\xi^{-1}(x;s)$ denote the inverse of $\xi(t;s)$ with respect to t. Because $\xi^{-1}(x;s) \in T(s)$ by definition, $\xi^{-1}(\cdot)$ is bounded. In addition, given x, $\xi^{-1}(x;s)$ is smooth in s, as shown in the next lemma.

Lemma 2 (Differentiability of $\xi^{-1}(x;s)$)**.**
For any $x \in [0,1]$, $\xi^{-1}(x;s)$ is differentiable with respect to s in the interior of S.

Proof. See Appendix B. □

Finally, we have a lemma, which addresses that $\xi(\cdot)$ is equivalent to $G(\cdot)$ if bidder i with $\boldsymbol{\theta}_i \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_r$ also plays $\sigma_{\mathrm{I}}(\cdot)$.

Lemma 3 (Distribution of $u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$)**.**
For all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_r$,

$$G(s) \equiv \xi\left(\frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})}; s \right) \tag{11}$$

if $s = \sigma_{\mathrm{I}}(\boldsymbol{\theta})$.

Proof. See Appendix C. □

[22] Taking the derivative with respect to t, we have

$$\frac{\partial}{\partial t}\xi(t;s) = -\int_{\{\tilde{\boldsymbol{\theta}} | u(s,\tilde{\boldsymbol{\theta}})/u_s(s,\tilde{\boldsymbol{\theta}})=t\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} < 0.$$

This is bounded for any $s \in \mathcal{S}$ and $t \in T(s)$, because $f(\boldsymbol{\theta})$ is bounded for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Using $\xi(\cdot)$ and its inverse, we investigate the symmetric equilibrium in the FS auction. By transforming (8) with $\xi(\cdot; s_i)$, we obtain:

$$\xi\left(\frac{1-G(s_i)}{(n-1)g(s_i)}; s_i\right) = \xi\left(\frac{u(s_i, \boldsymbol{\theta}_i)}{u_s(s_i, \boldsymbol{\theta}_i)}; s_i\right),$$

for all $\boldsymbol{\theta}_i \in \boldsymbol{\Theta}_{r-}$. By Lemma 3, the right-hand side is equal to $G(s_i)$ if $s_i = \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)$. This implies that, if bidder i also plays $\sigma_{\mathrm{I}}(\cdot)$, then $G(s)$ and $g(s)$ must satisfy:

$$\xi\left(\frac{1-G(s)}{(n-1)g(s)}; s\right) = G(s) \tag{12}$$

subject to $s = \sigma_{\mathrm{I}}(\boldsymbol{\theta})$.

In Appendix D, we show that this above differential equation has a unique solution with the boundary condition, $G(s^r) = \bar{x}$, if $s^r < \max_{\boldsymbol{\theta}} z(\boldsymbol{\theta})$. In particular, the smoothness of $\xi^{-1}(x; s)$, shown in Lemma 2, ensures the existence and the uniqueness of $G(s)$ (and $g(s)$) that satisfy (12). In the appendix, we also demonstrate that $G(s)$ (and $g(s)$) satisfy all the items in (Guess). Recall that $\sigma_{\mathrm{I}}(\boldsymbol{\theta})$ is the unique solution to (9) given $G(\cdot)$. Hence, we conclude that $\sigma_{\mathrm{I}}(\cdot)$ is the unique symmetric pure monotone equilibrium.

4 Structural estimation of the scoring auction model

4.1 Outline

In this section, we demonstrate that the K-dimensional i.i.d. signal is identified from L-dimensional bids under Assumptions 1 through 4 if $K \leqq L$. The key to our identification strategy is the invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$ with respect to $\boldsymbol{\theta}$. We show this by applying the global inverse function theorem. In the FS auction, we assume, for simplicity, that the auctioneer sets the reserve score as $s^r = z(\bar{\boldsymbol{\theta}})$, i.e., the least efficient type is indifferent between bidding and staying out in the FS auction.

4.2 Identification of the multidimensional signal in FS and SS auctions

We first examine the FS auction. Let $(p^*, \mathbf{q}^*)$ denote an observed multidimensional bid, and let s^* denote the associated score, given by $s^* = S(p^*, \mathbf{q}^*)$. Suppose that p^* and $\mathbf{q}^*$ are generated by equilibrium strategy $\sigma_{\mathrm{I}}(\cdot)$, as discussed in (6). Then, s^* satisfies the first-order condition, (9), as

$$\frac{1 - G(s^*)}{(n-1)g(s^*)} = \frac{u(s^*, \boldsymbol{\theta})}{u_s(s^*, \boldsymbol{\theta})}.$$

Given that the observed quality bid satisfies $\mathbf{q}^* = \mathbf{q}(s^*, \boldsymbol{\theta})$, we have $u(s^*, \boldsymbol{\theta}) = P(s^*, \mathbf{q}^*) - C(\mathbf{q}^*, \boldsymbol{\theta})$ and $u_s(s^*, \boldsymbol{\theta}) = P_s(s^*, \mathbf{q}^*)$. Then, we rearrange the first-order condition as:

$$C(\mathbf{q}^*, \boldsymbol{\theta}) = p^* - P_s(s^*, \mathbf{q}^*)\frac{1 - G(s^*)}{(n-1)g(s^*)}. \tag{13}$$

Moreover, $\mathbf{q}^* = \mathbf{q}(s^*, \boldsymbol{\theta})$ satisfies (3) such that

$$C_{q^\ell}(\mathbf{q}^*, \boldsymbol{\theta}) = P_{q^\ell}(s^*, \mathbf{q}^*) \tag{14}$$

for all $\ell = 1, \ldots, L-1$. Then, from equations (13) and (14), we have the following system of nonlinear equations:

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*, \tag{15}$$

where

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \begin{bmatrix} C(\mathbf{q}^*, \boldsymbol{\theta}) \\ C_{q^1}(\mathbf{q}^*, \boldsymbol{\theta}) \\ \vdots \\ C_{q^{L-1}}(\mathbf{q}^*, \boldsymbol{\theta}) \end{bmatrix}; \quad \mathbf{b}^* = \begin{bmatrix} p^* - P_s(s^*, \mathbf{q}^*)(1 - G(s^*))/(n-1)g(s^*) \\ P_{q^1}(s^*, \mathbf{q}^*) \\ \vdots \\ P_{q^{L-1}}(s^*, \mathbf{q}^*) \end{bmatrix}. \tag{16}$$

Given that bidders follow a strictly increasing strategy $\sigma_{\mathrm{I}}(\cdot)$, $b^0 \equiv p - P_s(s, \mathbf{q})(1 - G(s))/(n-1)g(s)$ is monotone in s given $\mathbf{q}$. This implies that $\mathbf{b}$ and s are one-to-one with each other. In other words, for any observables $(p^*, \mathbf{q}^*)$, $\mathbf{b}^*$ is uniquely given. Therefore, the monotonicity of b^0 gives a refutable restriction on the distribution of s in the FS scoring auction model.

The case of the SS auction is analogous; suppose that each bidder's multidimensional bid, $(p^*, \mathbf{q}^*)$, is generated by the equilibrium strategy $\sigma_{\text{II}}(\cdot)$.[23] Then, the associated score, $s^* = S(p^*, \mathbf{q}^*)$, is given by

$$s^* = \sigma_{\text{II}}(\boldsymbol{\theta}) = z(\boldsymbol{\theta}),$$

where p^* and $\mathbf{q}^*$ are given by

$$p^* = P(s^*, \mathbf{q}^*), \text{ and} \tag{17}$$

$$C_{q^\ell}(\mathbf{q}^*, \boldsymbol{\theta}) = P_{q^\ell}(s^*, \mathbf{q}^*) \tag{18}$$

for all $\ell = 1, \ldots, L-1$.

Then, the system of nonlinear equations is given as $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$ with

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \begin{bmatrix} C(\mathbf{q}^*, \boldsymbol{\theta}) \\ C_{q^1}(\mathbf{q}^*, \boldsymbol{\theta}) \\ \vdots \\ C_{q^{L-1}}(\mathbf{q}^*, \boldsymbol{\theta}) \end{bmatrix}; \quad \mathbf{b}^* = \begin{bmatrix} p^* \\ P_{q^1}(s^*, \mathbf{q}^*) \\ \vdots \\ P_{q^{L-1}}(s^*, \mathbf{q}^*) \end{bmatrix}. \tag{19}$$

Now, we discuss our approach to identification. First, $P(\cdot)$ is a known function. For the FS auction, $g(\cdot)$ and $G(\cdot)$ can be obtained from observations on $s^* = S(p^*, \mathbf{q}^*)$. Hence, all elements of $\mathbf{b}^*$ can be evaluated from the observed multidimensional bid, $(p^*, \mathbf{q}^*)$, in both FS and SS cases. Furthermore, $C(\mathbf{q}, \boldsymbol{\theta})$ is known except for $\boldsymbol{\theta}$. In other words, only $\boldsymbol{\theta}$ is the unknown element in the nonlinear system, $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$. Therefore, $\boldsymbol{\theta}$ is identified from observations if function $A(\boldsymbol{\theta}; \mathbf{q})$ is invertible with respect to $\boldsymbol{\theta}$. More specifically, let $A^{-1}(\mathbf{b}; \mathbf{q})$ denote the inverse function of $A(\boldsymbol{\theta}; \mathbf{q})$ with respect to $\boldsymbol{\theta}$. Then, if $A(\boldsymbol{\theta}; \mathbf{q})$ is invertible with respect to $\boldsymbol{\theta}$, we can recover $\boldsymbol{\theta}$ as the unique solution to the nonlinear system. In what follows, we give a formal argument for this by demonstrating that the nonlinear system has a unique solution.

[23] In the SS auction, the winner also chooses $(p^{post}, \mathbf{q}^{post})$, which is also observable. In this analysis, we ignore the effect of this additional observation on identification.

The unique solution to the nonlinear system is shown by the global inverse function theorem.[24] In our situation, we have to show the following conditions: (i) $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$; (ii) $A(\boldsymbol{\theta}; \mathbf{q})$ is a proper mapping for any $\boldsymbol{\theta}$ and $\mathbf{q}$; and (iii) $\boldsymbol{\Theta}$ is arcwise connected, and the image of $A(\boldsymbol{\theta}; \mathbf{q})$ is simply connected for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$.

The following lemma demonstrates that Assumption 4 is equivalent to the local invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$.

Lemma 4. *Suppose that $K \leqq L$. Then, the cost function, $C(\mathbf{q}, \boldsymbol{\theta})$, satisfies Assumption 4 if and only if the Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q}) = (C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \ldots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^{\mathrm{T}}$ with respect to $\boldsymbol{\theta}$ is full column rank for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$.*

Proof. See Appendix E. □

By the local inverse function theorem, Lemma 4 guarantees the local invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$ if $K \leqq L$.

Then, we have a proposition regarding the global invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$. Given the local invertibility of $A(\cdot)$, our proof focuses on demonstrating the remaining conditions, (ii) and (iii), for the global inverse function theorem.

Proposition 3. *Suppose that $K \leqq L$. Then, under Assumptions 1 through 4, vector-valued function $A(\boldsymbol{\theta}; \mathbf{q})$ is globally invertible with respect to $\boldsymbol{\theta}$ for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$.*

Proof. See Appendix G. □

The following corollary is an immediate consequence of Proposition 3.

Corollary 1 (Identification)**.** *Under Assumptions 1 through 4, the bidder's K-dimensional signal is identified from L-dimensional bid samples if $K \leqq L$.*

Several remarks are in order. First, we briefly discuss the case in which Assumption 4 is not satisfied. As illustrated in the example in Section 2, if the cost function exhibits the rank-deficient Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q})$ at $\mathbf{q}^*$, then the impact of a dimension of $\boldsymbol{\theta}$ on the total and marginal costs is identical to that of another dimension or a linear combination

[24] See Ambrosetti and Prodi (1995) for more details.

of a set of other dimensions of $\boldsymbol{\theta}$ at $\mathbf{q}^*$. If K-dimensional $\boldsymbol{\theta}$ exhibits such dependence in the cost function, then two different bidder types $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$ have the same total and marginal costs at $\mathbf{q}^*$ – i.e., $A(\boldsymbol{\theta}; \mathbf{q}^*) = A(\boldsymbol{\theta}'; \mathbf{q}^*)$. Given that the optimal choice in s depends solely on $\boldsymbol{\theta}$ in the scoring auction, these two bidders are observationally equivalent, as their score and quality are identical to s^* and $\mathbf{q}^*$, respectively. Therefore, the multidimensional signal is not identified.

Second, we make a note on identification when the solution to $\mathbf{q}$ is corner (i.e., cases arising if Assumption 2 is not satisfied). For instance, expression (14) becomes inequality if a quality upper bound binds:

$$C_{q^\ell}(\mathbf{q}^*, \boldsymbol{\theta}) \geqq P_{q^\ell}(s^*, \mathbf{q}^*), \tag{14$'$}$$

for all $\ell = 1, \ldots, L-1$. That is, one obtains $A(\boldsymbol{\theta}; \mathbf{q}^*) \geqq \mathbf{b}^*$, suggesting that $\boldsymbol{\theta}$ is not identified. In this case, one may need to exploit additional observations or constraints on primitives for identification or to use partial identification.

Finally, we explore the specification test for the cost function. Suppose that the researcher uses a cost function, $\widehat{C}(\mathbf{q}, \boldsymbol{\theta})$, that may not be the true cost function, $C(\mathbf{q}, \boldsymbol{\theta})$. Given that the observation of the scoring auction data is L-dimensional, one needs additional variations in data to identify signals of $(L+1)$ or higher dimensions. This, in turn, implies that there is no way to test the cost function with L-dimensioning bid data only.

Several ways have been proposed to obtain additional dimensions of information, such as exogenous variations in the scoring rules and in the number of bidders.[25] However, in Appendix H, we show that at least the exogenous variation in the number of bidders does not help to test the cost function if the scoring rule is QL.

Note that, while the cost function may be testable with non-QL scoring rule, it generally has some limitations, as discussed in Athey and Haile (2007). For instance, the alternative hypothesis is that some component of the specification is incorrect. A failure of the test may indicate the presence of unobserved heterogeneity, risk aversion, non-equilibrium bidding behavior, etc.

[25]The idea to exploit a variation in the scoring rule is seen in Asker and Cantillon (2008). For more detailed arguments on the use of a variation in the number of bidders, see Athey and Haile (2002, 2007).

4.3 Estimation for the distribution of $\boldsymbol{\theta}$

Let T be the number of scoring auction samples, each indexed by $t = 1, \ldots, T$. Let $\hat{\boldsymbol{\theta}}_{i,t} = (\hat{\theta}^0_{i,t}, \ldots, \hat{\theta}^{K-1}_{i,t})$ with $K \leqq L$ be the solution to $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$, where $\mathbf{b}^*$ is given by (16) and (19) for the FS and SS auctions, respectively.

In the FS auction, both $G(s)$ and $g(s)$ are estimated by the standard kernel estimator. Auction-specific heterogeneities, such as the number of bidders, properties of the item to be purchased, etc., are controlled; let n_t and $\mathbf{x}_t = (x^1_t, \ldots, x^d_t)$ denote the number of bidders and the covariates of auction t, respectively. Let $g(s, n, \mathbf{x})$ denote the joint density function of s, n, and $\mathbf{x}$. Then, the kernel estimator for $G(s, n, \mathbf{x}) := \int_{-\infty}^{s} g(v, n, \mathbf{x})dv$ is provided by

$$\hat{G}(s, n, \mathbf{x}) = \frac{1}{T h_{G_n} h^d_{G_x}} \sum_{t=1}^{T} \frac{1}{n_t} \sum_{i=1}^{n_t} \mathbf{1}(s_{i,t} \leqq s) K_G \left(\frac{n - n_t}{h_{G_n}}, \frac{x_1 - x_{1,t}}{h_{G_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{G_x}} \right), \tag{20}$$

where $\mathbf{1}(\cdot)$ is an indicator function, K_G is a kernel with a bounded support, and h_{G_n} and h_{G_x} are bandwidths. Similarly, the kernel density estimator for $g(s, n, \mathbf{x})$ is given by

$$\hat{g}(s, n, \mathbf{x}) = \frac{1}{T h_s h_{g_n} h^d_{g_x}} \sum_{t=1}^{T} \frac{1}{n_t} \sum_{i=1}^{n_t} K_g \left(\frac{s - s_{i,t}}{h_s}, \frac{n - n_t}{h_{g_n}}, \frac{x_1 - x_{1,t}}{h_{g_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{g_x}} \right), \tag{21}$$

where K_g is a kernel with a bounded support and h_s, h_{g_n}, and h_{g_x} are bandwidths. In practice, the discrete variables, such as the number of bidders and the maximum quality level, are smoothed out in the way that Li and Racine (2006) discuss.

Corollary 1 suggests that $\hat{\boldsymbol{\theta}}_{i,t}$ is recovered in both FS and SS auctions. The estimation for $F(\boldsymbol{\theta}, \mathbf{x}) := \int_{-\infty}^{\theta^0} \cdots \int_{-\infty}^{\theta^{K-1}} f(\boldsymbol{\tau}, \mathbf{x}) d\tau^0 \cdots d\tau^{K-1}$ is given by the standard kernel method:

$$\hat{F}(\boldsymbol{\theta}, \mathbf{x}) = \frac{1}{T h^d_{F_x}} \sum_{t=1}^{T} \sum_{i=1}^{n_t} \mathbf{1}(\boldsymbol{\theta} \leq \boldsymbol{\theta}_{i,t}) K_F \left(\frac{x_1 - x_{1,t}}{h_{F_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{F_x}} \right),$$

where K_F is a kernel with a bounded support, and h_{F_x} is a bandwidth. Similarly, the kernel density estimator for the joint density function of $\boldsymbol{\theta}$ and the covariate vector $\mathbf{x}$ is given by

$$\hat{f}(\boldsymbol{\theta}, \mathbf{x}) = \frac{1}{T h_{f_0} \cdots h_{f_{K-1}} h^d_{f_x}} \sum_{t=1}^{T} K_f \left(\frac{\theta^0 - \theta^0_{i,t}}{h_{f_0}}, \ldots, \frac{\theta^{K-1} - \theta^{K-1}_{i,t}}{h_{f_{K-1}}}, \frac{x_1 - x_{1,t}}{h_{f_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{f_x}} \right),$$

where K_f is a kernel with bounded support, and $h_{f_0}, \ldots, h_{f_{K-1}}$, and h_{f_x} are bandwidths.

5 An Empirical experiment

5.1 Data and Institution

The data used in our analysis contain the bid results of the procurement auctions for civil engineering projects conducted from January 2010 through August 2014 by the Ministry of Land, Infrastructure, and Transportation (MLIT) in Japan. The data include project names, dates of auctions, engineers' estimates, scoring auctions or not, and submitted bids with the bidder's identity. The MLIT procures 21 types of construction work, including civil engineering (or heavy and general construction work), buildings, bridges, paving, dredging, and painting. The civil engineering projects cost approximately 750 billion yen a year, which accounts for approximately 54 percent of the entire expenditure of the ministry, as well as for approximately three to four percent of the public construction investment in Japan. The number of civil engineering projects let by the ministry during the study period was 18,183.

Among these, 6,610 projects were allocated through scoring auctions in which bidders were asked to submit a technical proposal.[26] After removing samples with only one bidder and possibly misrecorded auctions, we are left 5,142 scoring auction samples.[27]

Table 1 reports the sample statistics. The mean of winners' bids and engineers' estimated prices were approximately 423 or 477 million yen, respectively. The quality bids ranged from approximately 130 through 200. The score is calculated as the quality bid divided by the price bid. The bidder with the highest score wins the project. Note that, taking into account the project-size heterogeneity, we report as the *score* the observed score

[26]There are three types of scoring auctions: Technical Proposal Type (*Kodo Gijutsu Teian Gata*); Regular Type (*Hyojun Gata*); and Simple Type (*Kan-i Gata*). We use Technical Proposal Type and Regular Type. In the Simple Type, bidders are not asked to turn in any proposal; instead, the buyer evaluates the bidder's past experience and the technology levels as non-price attributes. Hence, we removed these auctions from our samples. The Simple Type is used for relatively smaller projects.

[27]Misrecorded auctions include those in which quality or price bids are too low or too high (outside of the feasible level for the quality bid or less than 10% or greater than 200% of the engineer's estimate for the price bid).

multiplied by the engineer's estimate for each auction.[28] In each auction, approximately ten firms participated, on average.

[Table 1 about here.]

In the scoring auctions of the MLIT, the quality-bid point is given by a weighted sum of all non-price attributes, which include noise level, completion time, and bidder experience. The method of converting a technical proposal into a quality bid differs for each project. For instance, each one-decibel reduction in noise accounts for five additional quality-bid points. Our data records the quality-bid point, but not each of itemized points. The lower bound of the quality bid is 100 for all auctions, and the upper bound is 150 to 200, depending on the auction. The bidder proposing nothing has a quality bid equal to 100. Table 2 reports the sample statistics by upper bound in quality.

[Table 2 about here.]

5.2 Specifications

5.2.1 Percentage bids

Let $B_{i,t}$ and $q_{i,t}$ denote the raw values of bidder i's price and quality bids, respectively, in scoring auction $t \in T$. Under the inverse PQR scoring rule that the MLIT uses, the actual score is given by $q_{i,t}/B_{i,t}$. In our analysis, we normalize the raw price bid by dividing by the engineer's estimate to control for the project-size heterogeneity. Let $\bar{B}_t$ denote the engineer's estimated price in auction t. Then, the normalized price bid of bidder i in auction t is defined as $p_{i,t} = B_{i,t}/\bar{B}_t$. Let q^{post} and p^{post} denote the winning bidder's contracted quality and normalized price bid. Then, we assume that the procurement buyer's utility from auction t is defined as:

$$w_t = \frac{q_t^{post}}{p_t^{post}}, \tag{22}$$

[28] In other words, we compute the *score* by dividing the quality-bid points by the percentage price bid in terms of the engineer's estimate. We will use the *score* as the buyer's utility in the counterfactual analysis.

For estimation purposes, we use the inverse of the buyer's utility as bidder i's *score bid*:[29]

$$s_{i,t} = \frac{p_{i,t}}{q_{i,t}}. \tag{23}$$

The associated scoring rule is given by $S(p_{i,t}, q_{i,t}) = p_{i,t}/q_{i,t}$. Figure 1 shows the histogram of $s_{i,t}$ for the auction samples with the number of bidders being equal to ten.

[Figure 1 about here.]

Finally, while we control the heterogeneity in project size by normalizing the price bid with the engineer's estimates, our sample still involves heterogeneity in the number of bidders and the quality upper bound. Thus, the covariate x that we use is the quality upper bound.

5.2.2 Cost function

Given the data, we assume that $L = K = 2$. We use (2) as the cost function with $\alpha^0 = \alpha^1 = 0$:

$$C(q, \boldsymbol{\theta}) = \begin{cases} (q + \theta^1)^\beta + \theta^0 & \text{if } q > -\theta^1 \\ \theta^0 & \text{otherwise,} \end{cases} \tag{24}$$

where β is 2, 3, or 4. This cost function satisfies Assumptions 1 through 4.

5.3 Estimation of $\boldsymbol{\theta}$

Let $s_{i,t} = S(p_{i,t}, q_{i,t})$. Then, the signal is estimated from $p_{i,t}$, $q_{i,t}$, $\hat{g}(\cdot)$, and $\hat{G}(\cdot)$ as follows:

Given that $P = p_{i,t}$ and $P_s = q_{i,t}$ under the PQR scoring rule, we have $b^0 = p_{i,t} -$

[29] Recall that the outcome of the scoring auction is invariant to any monotone transformation of the scoring rule.

$q_{i,t}(1-\hat{G}(s_{i,t}))/[(n-1)\hat{g}(s_{i,t})]$ and $b^1 = s_{i,t}$. Hence, we have

$$\hat{\theta}^0 = p_{i,t} - q_{i,t}\frac{1}{n-1}\frac{1-\hat{G}(s_{i,t})}{\hat{g}(s_{i,t})} - \left(\frac{s_{i,t}}{\beta}\right)^{\frac{\beta}{\beta-1}},$$
$$\hat{\theta}^1 = \left(\frac{s_{i,t}}{\beta}\right)^{\frac{1}{\beta-1}} - q_{i,t}.$$

For estimating $\hat{G}$ and $\hat{g}$, we use the triweight kernel:

$$K(u) = \frac{35}{32}(1-u^2)^3\mathbf{1}(|u|<1).$$

As usual, the bandwidths h_s and h_x are given by the so-called rule of thumb; $h_s = \eta_s(\sum_{k=1}^{T} n_k)^{-1/5}$ and $h_x = \eta_x(\sum_{k=1}^{T} n_k)^{-1/5}$, where $\eta_s = 1.06\hat{\rho}_s$ and $\eta_x = 1.06\hat{\rho}_x$, respectively. Both $\hat{\rho}_s$ and $\hat{\rho}_x$ are sample standard deviations of the normalized scoring bids and the observed covariate, respectively. The following figures are the estimated joint density functions assuming that the cost function is the quadratic polynomial ($\beta = 2$). Axes x (horizontal) and y (depth) represent θ^0 and θ^1, respectively.

[Figure 2 about here.]

5.4 Counterfactual analyses

5.4.1 Second-price vs. FS auctions

One of the appeals of multidimensional auctions is that both the auctioneer and the bidders increase welfare from a more complete comparison of suppliers' attributes.(See, e.g., Milgrom (2004).) Our first empirical examination, thus, measures the gains from the use of scoring auctions for both buyer and suppliers.

We create a series of counterfactual second-price auctions, in each of which the quality level is fixed at $\bar{q} = 130$, 140, 150, 160, and 170. Using the estimated cost functions, we can point-estimate bidders' costs at any level of $\bar{q} = 130, \ldots, 170$. We then select the second-lowest cost as $\hat{p}_t^{post}$, the contract price of the counterfactual second-price auctions. The buyer's utility in the price-only auction is given according to (22) as $w_t = \bar{q}_t/\hat{p}_t^{post}$.

Because the bidder's cost functions are differentiated by $\beta = 2$, 3, and 4, fifteen types of counterfactual second-price auctions are generated.

Table 3 compares the procurement buyer's utilities in the observed FS auction versus a series of counterfactual price-only auctions. The extent of the government's expected gain from the scoring auction depends crucially on the fixed quality standard in the price-only auction. While the government utilities would drop by more than seven percent if the quality standard in the price-only auction were $\bar{q} = 130$, the drop would be not very large (merely .96 percent) if, for instance, $\bar{q} = 160$ with the Quadratic cost function. Considering the fact that the buyer must incur substantial costs to evaluate the quality bids in the scoring auction, our results suggest that a simple low-price auction still works well, as long as the buyer can appropriately design the quality standard of the price-only auction.

Table 4 reports the winning bidders' (normalized) expected payoffs, which is computed by taking the average of the estimated (nominal) payoff divided by the engineer's estimate for each auction. The results show that the bidder's payoff also varies, depending on the quality standard in the price-only auction. Note that the positive relationship between payoffs and quality standards is due to greater information rents left over to bidders with a higher quality standard, as suggested by Che (1993).

[Table 3 about here.]

[Table 4 about here.]

The results do not take into account the bidder's participation decision and the bidder's cost for preparing multidimensional bidding. Given that bidders earn more in the scoring auction, scoring auctions can encourage bidders' participation. On the other hand, if the bid preparation costs are significantly greater in a scoring auction than in a price-only auction, then participation is discouraged. Given the buyer's small gain from the scoring auction in comparison to the price-only auction with $\bar{q} = 160$, our results suggest that a price-only auction with an appropriate $\bar{q}$ is still a good mechanism to allocate the government contract.

5.4.2 SS vs. FS auctions

We next estimate the extent to which the buyer's expected utility would be changed by introducing SS auctions. In our setting, the second derivative of $u(s, \boldsymbol{\theta})$ is strictly positive.[30] Therefore, theory predicts that the expected s (i.e., the inverse of the buyer's utility) will be lower in SS than in FS auctions (Theorem 3 in Hanazono et al. (2013)). We conduct a counterfactual analysis to measure the difference between FS and SS auctions regarding the buyer's and bidders' utilities and the contract quality level. In Online Appendix II, we show the way to generate counterfactual SS auction samples from the estimated parameters, $\hat{\boldsymbol{\theta}}_{i,t}$.

Table 5 shows the buyer's utilities (the inverse of s). The means are lower by .71 to .72 percent in the SS than in the FS auction. While the result is in line with the theoretical prediction, the advantage may be offset by the greater variance in the SS auction.

Table 6 reports the quality level finalized in the contract. The contract quality level declines, on average, by approximately .05 to .06 percents if SS auctions are used. This suggests that the higher expected s is due to excessive quality proposal in the FS auction. In fact, bidders earn larger payoffs in the FS auction, on average (about 2.3 percent), which Table 7 shows. This suggests that, while the FS auction would result in a higher expected score (or, equivalently, lower buyer utilities), the drawback can be remedied by more intensified competition as the FS auction is more profitable for bidders under the price-over-quality ratio (PQR) scoring rule.

[Table 5 about here.]

[Table 6 about here.]

[Table 7 about here.]

5.4.3 QL vs. PQR rules

Finally, we examine the impact of the change in the scoring rule; we generate a QL scoring rule that dominates the current PQR scoring rule from the viewpoint of the buyer's welfare.

[30] Under the PQR scoring rule, $u_s(s, \boldsymbol{\theta}) = P_s(s, q(s, \boldsymbol{\theta})) = q(s, \boldsymbol{\theta})$. Then, $u_{ss}(s, \boldsymbol{\theta}) = q_s(s, \boldsymbol{\theta}) = P_{sq}(s, q)/(C_{qq}(q(s, \boldsymbol{\theta}), \boldsymbol{\theta}) - P_{qq}(s, q(s, \boldsymbol{\theta}))$. Note that $P_{qq} = 0$ and $P_{sq} = 1$ in our setting. Thus, $u_{ss} = 1/C_{qq} > 0$.

The QL rules that we consider are given as

$$S(p, q) = p - \phi(\beta)q, \tag{25}$$

for some $\phi > 0$. Given the theoretical prediction that the lower performance of the FS auction under the PQR rule is due to over-provision in quality, we seek appropriate ϕs that induce bidders to bid lower q. We choose $\phi(\beta) \approx .0058$ (the slope of the scoring function, $P_q(\cdot)$), for example, for the auctions with the quality upper bound is equal to 160. Using $\phi(\beta)$, we predict the expected winning score in the QL scoring auction, which is given by the mean of the second-lowest pseudotype due to the expected score equivalence. In Online Appendix III, we show the way to generate the counterfactual SS auction samples with the QL scoring rule.

Table 8 reports the buyer's utility from the counterfactual QL scoring auctions. In all cases, utilities rise by .7 percent, on average. Note that standard deviations are larger in our counterfactual QL scoring auctions because we use the SS auction to generate the QL scoring auction samples.

Table 9 shows the winning bidder's profits. The profits drop by about 3.8 to 4.0 percent. This suggests that, using an appropriate QL scoring rule, the buyer can extract more rents from bidders.

[Table 8 about here.]

[Table 9 about here.]

Table 10 compares the contracted quality levels in the observed FS auction and in simulated QL scoring auctions. The quality bids rise by approximately .04 to .06 percent under the well-designed QL scoring rule. This suggests that the QL scoring rule can limit the winner's informational rent while promoting higher quality proposals.

[Table 10 about here.]

6 Conclusion

In this research, we provide a method to analyze the scoring auction theoretically and econometrically. Allowing a broad class of scoring rules, we demonstrate the existence and the characterization of a symmetric monotone equilibrium of the scoring auction. Based on our theoretical model, we then examine identification of the scoring auction model. Furthermore, we take our framework to the scoring auction data to quantify the impact of the change in design of scoring auctions and the adoption of the scoring auction instead of price-only auctions.

We restrict attention to the independent scoring rule, in which the bidder's score depends only on his or her price and quality bids. Especially in the real-world procurement auctions, scoring rules are used in which the bidder's score depends also on the other bidders' price and quality bids (an interdependent scoring rule). Albano et al. (2009) suggest that the interdependent scoring rule causes the efficiency loss, estimating the auctioneer's loss to be approximately 11 percent. Interesting future research may lie in doing an analysis similar to that of this paper, but on the scoring auction with an interdependent scoring rule. A counterfactual analysis would quantify the expected score difference between the FS and SS auctions with an interdependent scoring rule.

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Appendix A Proof of Lemma 1

Proof. Let $q^{\ell}_{\theta^k}(s,\boldsymbol{\theta}) = \partial q^{\ell}(s,\boldsymbol{\theta})/\partial\theta^k$. Then, it is sufficient to show that

$$\left[\sum_{\ell=1}^{L-1} \frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})} P_{sq^{\ell}}(s,\mathbf{q}(s,\boldsymbol{\theta}))q^{\ell}_{\theta^k}(s,\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q},\boldsymbol{\theta}) \tag{A-1}$$

is strictly positive and bounded.

We first demonstrate that Assumptions 1-(ii), 2, and 3-(i) imply that

$$\left[\sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) q^\ell_{\theta^k}(s,\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) \geqq 0. \tag{A-2}$$

By expression (3), $\mathbf{q}(s,\boldsymbol{\theta})$ satisfies

$$P_{q^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta})) = C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}).$$

for all $s \in \mathcal{S}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and for all $\ell = 1,\ldots,L-1$. Taking the derivative on both sides of this expression with respect to θ^k with $k = 0,\ldots,K-1$, we obtain

$$\sum_{m=1}^{L-1} [P_{q^\ell q^m}(s,\mathbf{q}(s,\boldsymbol{\theta})) - C_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})] q^\ell_{\theta^k}(s,\boldsymbol{\theta}) = C_{q^\ell \theta^k}(\mathbf{q},\boldsymbol{\theta}) \geqq 0, \tag{A-3}$$

where the inequality is due to Assumption 1-(ii). Then, by Assumption 2, we have $q^\ell_{\theta^k}(s,\boldsymbol{\theta}) \leqq 0$ for all $s \in \mathcal{S}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Recall that $P_{q^\ell q^m} - C_{q^\ell q^m}$ is strictly negative at $\mathbf{q} = \mathbf{q}(s,\boldsymbol{\theta})$ by Assumption 2. It follows that $q^\ell_{\theta^k}(s,\boldsymbol{\theta}) = 0$ if $C_{q^\ell \theta^k} = 0$. Therefore, if $C_{q^\ell \theta^k} = 0$, expression (A-2) holds (because $C_{\theta^k} > 0$, and the first term in (A-2) vanishes).

To show that (A-2) holds under $C_{q^\ell \theta^k} > 0$, we impose Assumption 3-(i): $\frac{\partial}{\partial q^m} \frac{C_{q^\ell}(\mathbf{q},\boldsymbol{\theta})}{C_{\theta^k}(\mathbf{q},\boldsymbol{\theta})} \geqq 0$ for all $\ell, m = 1,\ldots,L-1$ and $k = 0,\ldots,K-1$. Assumption 3-(i) implies that

$$C_{q^\ell q^m}(\mathbf{q},\boldsymbol{\theta}) C_{\theta^k}(\mathbf{q},\boldsymbol{\theta}) - C_{q^\ell}(\mathbf{q},\boldsymbol{\theta}) C_{q^m \theta^k}(\mathbf{q},\boldsymbol{\theta}) \geqq 0.$$

for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Given the assumption that $P_{q^\ell q^m} \leqq 0$ for all s, $\mathbf{q}$ and for all $\ell, m = 1,\ldots,L-1$, the inequality implies that

$$[C_{q^\ell q^m}(\mathbf{q},\boldsymbol{\theta}) - P_{q^\ell q^m}(s,\mathbf{q})] C_{\theta^k} - C_{q^\ell}(\mathbf{q},\boldsymbol{\theta}) C_{q^m \theta^k}(\mathbf{q},\boldsymbol{\theta}) \geqq 0. \tag{A-4}$$

Evaluating this at $\mathbf{q} = \mathbf{q}(s, \boldsymbol{\theta})$ and multiplying by $q^{\ell}_{\theta^k}/C_{q^m\theta^k} < 0$ gives

$$C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})q^{\ell}_{\theta^k}(s,\boldsymbol{\theta}) \geqq \frac{C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}{C_{q^m\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}[C_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) - P_{q^\ell q^m}(s,\mathbf{q}(s,\boldsymbol{\theta}))]q^{\ell}_{\theta^k}(s,\boldsymbol{\theta}).$$

Using this expression and expression (A-3), we rewrite (A-2) as

$$\begin{aligned}
&\left[\sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})q^{\ell}_{\theta^k}(s,\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}),\\
\geqq&\frac{C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}{C_{q^m\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}\left[\sum_{\ell=1}^{L-1}[C_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) - P_{q^\ell q^m}(s,\mathbf{q}(s,\boldsymbol{\theta}))]q^{\ell}_{\theta^k}(s,\boldsymbol{\theta})\right]\\
&\qquad + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}),\\
=&\frac{C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}{C_{q^m\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}\left[-C_{q^\ell\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) = 0, \qquad \text{(A-5)}
\end{aligned}$$

where the inequality is due to the above expression and the first-equality is given by (A-3). Therefore, (A-2) holds if $C_{q^\ell\theta^k} > 0$.

Finally, we show that (A-2) implies (A-1). By Assumption 3-(ii), we have

$$\frac{d}{dq^\ell}\frac{P(s,\mathbf{q})}{P_s(s,\mathbf{q})} = \frac{1}{(P_s(s,\mathbf{q}))^2}\left[P_{q^\ell}(s,\mathbf{q})P_s(s,\mathbf{q}) - P(s,\mathbf{q})P_{sq^\ell}(s,\mathbf{q})\right] \geqq 0.$$

Given that $C(\mathbf{q},\boldsymbol{\theta}) > 0$ and that $P_s(s,\mathbf{q}) > 0$, this inequality implies that

$$\frac{P(s,\mathbf{q}) - C(\mathbf{q},\boldsymbol{\theta})}{P_s(s,\mathbf{q})}P_{sq^\ell}(s,\mathbf{q}) < P_{q^\ell}(s,\mathbf{q})$$

for all $s \in \mathcal{S}$ and $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$ with $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Let us evaluate this inequality at $\mathbf{q} = \mathbf{q}(s,\boldsymbol{\theta})$. Then, we have $P_{q^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta})) = C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})$. Thus, by multiplying $q^{\ell}_{\theta^k}(s,\boldsymbol{\theta}) < 0$ on both sides, we obtain

$$\frac{P(s,\mathbf{q}(s,\boldsymbol{\theta})) - C(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})}{P_s(s,\mathbf{q}(s,\boldsymbol{\theta}))}P_{sq^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta}))q^{\ell}_{\theta^k}(s,\boldsymbol{\theta}) > C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})q^{\ell}_{\theta^k}(s,\boldsymbol{\theta})$$

for all s and $\boldsymbol{\theta}$. Using this inequality and (A-2), we bound expression (A-1) from below as

$$\begin{aligned}
&\left[\sum_{\ell=1}^{L-1} \frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})} P_{sq^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta})) q^\ell_{\theta^k}(s,\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}),\\
>&\left[\sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) q^\ell_{\theta^k}(s,\boldsymbol{\theta})\right] + C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}),\\
\geqq& 0,
\end{aligned}$$

for all $s \in \mathcal{S}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. □

Appendix B Proof of Lemma 2

Proof. For notational convenience, we let

$$t(s,\boldsymbol{\theta}) := \frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})}.$$

We then consider $\boldsymbol{\Theta}(s,x) = \{\boldsymbol{\theta} | \xi(t(s,\boldsymbol{\theta}); s) = x\}$, denoting the set of $\boldsymbol{\theta}$ such that $\xi(t(s,\boldsymbol{\theta}); s)$ is constant given s. Recall that, given s, $u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$ is a surjective mapping such that $\boldsymbol{\Theta} \to T(s)$, and $\xi(t; s)$ is a bijective mapping such that $T(s) \to [0,1]$. Therefore, for any $x \in [0,1]$ and $s \in \mathcal{S}$, $\boldsymbol{\Theta}(s,x)$ is nonempty.

Now, we first show that for all $x \in [0,\bar{x}]$ and for all $s_1, s_2 \in \mathcal{S}$,

$$\boldsymbol{\Theta}(s_1,x) \cap \boldsymbol{\Theta}(s_2,x)$$

is nonempty.

If $s_1 = s_2$, the statement is true trivially. To see the case that $s_1 \neq s_2$, suppose, by contradiction, that $\boldsymbol{\Theta}(s_1,x) \cap \boldsymbol{\Theta}(s_1,x)$ is empty if $s_1 \neq s_2$. Then, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s_1,x)$, we have either

Case 1: $\xi(t(s_2,\boldsymbol{\theta}); s_2) < x,$ or

Case 2: $\xi(t(s_2,\boldsymbol{\theta}); s_2) > x.$

Note that the sorting condition implies that $t(s, \boldsymbol{\theta})$ is strictly decreasing in $\boldsymbol{\theta}$. It follows that, for any $\boldsymbol{\theta} \in \Theta(s, x)$, $\xi(t(s, \boldsymbol{\theta}'); s) > x$ if $\boldsymbol{\theta}' \geq \boldsymbol{\theta}$ (i.e., $\boldsymbol{\theta}' \geqq \boldsymbol{\theta}$ and $\boldsymbol{\theta}' \neq \boldsymbol{\theta}$), and $\xi(t(s, \boldsymbol{\theta}'); s) < x$ if $\boldsymbol{\theta}' \leq \boldsymbol{\theta}$.

Then, define $\boldsymbol{\Theta}^L(s, x) = \{\boldsymbol{\theta} | \xi(t(s, \boldsymbol{\theta}); s) \leqq x\}$. That is, $\boldsymbol{\Theta}^L(s, x)$ denotes the lower contour set of $\boldsymbol{\theta}$ such that, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}^L(s, x)$, $\xi(t(s, \boldsymbol{\theta}); s)$ is less than or equal to x given s. Note that the sorting condition implies that if $\boldsymbol{\theta}$ is in $\boldsymbol{\Theta}^L(s, x)$, so is any $\boldsymbol{\theta}' \leq \boldsymbol{\theta}$. More formally, ($\boldsymbol{\theta} \in \boldsymbol{\Theta}^L(s, x)$ and $\boldsymbol{\theta}' \leq \boldsymbol{\theta}$) $\Rightarrow$ ($\boldsymbol{\theta}' \in \boldsymbol{\Theta}^L(s, x)$) for all s and x.[31]

Using the feature of $\boldsymbol{\Theta}^L(s, x)$, we consider Case 1 first. In this case, $\boldsymbol{\Theta}(s_1, x)^L$ is a strict subset of $\boldsymbol{\Theta}^L(s_2, x)$, because, for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}^L(s_1, x)$, there exists $\boldsymbol{\theta}' \in \boldsymbol{\Theta}(s_2, x)$ with $\boldsymbol{\theta}' \geq \boldsymbol{\theta}$. Then, set $\boldsymbol{\Theta}^L(s_2, x) \backslash \boldsymbol{\Theta}^L(s_1, x)$ is nonempty. Then, we have

$$\begin{aligned}
x &= \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_2, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_1, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} + \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_2, x) \backslash \boldsymbol{\Theta}^L(s_1, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= x + \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_2, x) \backslash \boldsymbol{\Theta}^L(s_1, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&> x.
\end{aligned}$$

The last inequality holds because $f(\boldsymbol{\theta})$ is strictly positive for all $\boldsymbol{\theta}$. Therefore, we have a contradiction.

Obtaining a contradiction in Case 2 is analogous. In Case 2, $\boldsymbol{\Theta}(s_2, x)$ is a strict subset of $\boldsymbol{\Theta}^L(s_1, x)$. Then, we have a contradiction:

$$\begin{aligned}
x &= \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_1, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_2, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} + \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_1, x) \backslash \boldsymbol{\Theta}^L(s_2, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= x + \int_{\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}^L(s_1, x) \backslash \boldsymbol{\Theta}^L(s_2, x)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&> x.
\end{aligned}$$

[31] This means that $\boldsymbol{\Theta}(s, x)$ is the frontier of $\boldsymbol{\Theta}^L(s, x)$ as being similar in spirit to the production possibility frontier of the production set in the firm theory. The sorting condition plays the same role as the free-disposal assumption.

Therefore, for all $x \in [0,1]$ and for all $s_1, s_2 \in \mathcal{S}$, there exists $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s_1, x) \cap \boldsymbol{\Theta}(s_2, x)$.

Using the result, we next show the differentiability of $\xi(\cdot)$. By definition, we have

$$x \equiv \xi(t(s, \boldsymbol{\theta}); s).$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s, x)$ with $x \in [0,1]$ and $s \in \mathcal{S}$. Given that $\xi^{-1}(x; s)$ is a strictly decreasing function of x, we transform both sides of this identity with $\xi^{-1}(\cdot; s)$ to obtain

$$\xi^{-1}(x; s) \equiv t(s, \boldsymbol{\theta}).$$

From the above result, there exists $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s_1, x) \cap \boldsymbol{\Theta}(s_2, x)$ such that

$$\frac{\xi^{-1}(x; s_1) - \xi^{-1}(x; s_2)}{s_1 - s_2} = \frac{t(s_1, \boldsymbol{\theta}) - t(s_2, \boldsymbol{\theta})}{s_1 - s_2}.$$

Without loss, we assume $s_1 < s_2$. Note that $t(s, \boldsymbol{\theta})$ is differentiable with respect to s. Therefore, by the mean-value theorem, there exists $s' \in [s_1, s_2]$ such that

$$\frac{t(s_1, \boldsymbol{\theta}) - t(s_2, \boldsymbol{\theta})}{s_1 - s_2} = \frac{\partial}{\partial s} t(s', \boldsymbol{\theta}).$$

This holds in the limit where $s_2 \to s_1$. That is, there exists $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s_1, x)$ such that

$$\frac{d}{ds} \xi^{-1}(x; s_1) = \lim_{s_2 \to s_1} \frac{t(s_1, \boldsymbol{\theta}) - t(s_2, \boldsymbol{\theta})}{s_1 - s_2} = \frac{\partial}{\partial s} t(s_1, \boldsymbol{\theta}).$$

It follows that, for all s in the interior of S, $x \in [0, \bar{x}]$, there exists $\boldsymbol{\theta} \in \boldsymbol{\Theta}(s, x)$ such that

$$\frac{d}{ds} \xi^{-1}(x; s) := \frac{\partial}{\partial s} t(s, \boldsymbol{\theta}).$$

$\square$

Appendix C Proof of Lemma 3

Proof. Suppose that bidder i with $\boldsymbol{\theta}_i \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_r$ also plays $\sigma_{\mathrm{I}}(\cdot)$ as an optimal bidding strategy. Then, the first-order condition gives

$$\frac{1 - G(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'))}{(n-1)g(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'))} - \frac{u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)}{u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)} \gtreqqless 0,$$

for any $\boldsymbol{\theta}' \in \boldsymbol{\Theta}_{r-} \cup \boldsymbol{\Theta}_r$ with $u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}')/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}') \lesseqqgtr u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)$. This suggests that choosing $\sigma_{\mathrm{I}}(\boldsymbol{\theta}')$ is suboptimal for bidder i – i.e., too low (or too high) – if $\boldsymbol{\theta}'$ is such that $u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}')/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}') < u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)$ (or $u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}')/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}') > u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)$). Hence, $\sigma_{\mathrm{I}}(\boldsymbol{\theta}') \lesseqqgtr \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)$ if and only if $u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}')/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}') \lesseqqgtr u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)/u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}'), \boldsymbol{\theta}_i)$.

This implies that if i also plays $\sigma_{\mathrm{I}}(\cdot)$ as an optimal strategy, then $G(\cdot)$ satisfies

$$\begin{aligned}
G(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)) &= \Pr\{\sigma_{\mathrm{I}}(\boldsymbol{\theta}) < \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i)\}, \\
&= \Pr\left\{ \frac{u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta})}{u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta})} > \frac{u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}{u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)} \middle| \, \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i) \right\}, \\
&= \xi\left(\frac{u(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}{u_s(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}; \sigma_{\mathrm{I}}(\boldsymbol{\theta}_i) \right).
\end{aligned}$$

□

Appendix D The existence and uniqueness of the solution to (12)

Proof. Transforming back both sides of expression (12) with $\xi^{-1}(\cdot; s)$, we obtain:

$$\frac{1 - G(s)}{(n-1)g(s)} = \xi^{-1}(G(s); s). \tag{A-6}$$

Now, let $x = G(s)$. Given that $G(s)$ is strictly increasing, $G(s)$ has its inverse. Then, let $y(\cdot)$ denote the inverse of $G(\cdot)$. Then, from (Guess) and (A-6), we obtain an ordinary differential equation:

$$\begin{cases} y'(x) = \dfrac{n-1}{1-x}\xi^{-1}(x; y(x)), \\ y(\bar{x}) = s^r. \end{cases} \tag{A-7}$$

The reason that the ordinary differential equation has a unique solution is given as follows. First, the right-hand side is differentiable with respect to s for all $x \in [0, \bar{x}]$. This is because, by Lemma 2, $\xi^{-1}(x; s)$ is differentiable with respect to s for all $x \in [0, \bar{x}]$. Second, $\xi^{-1}(x; s) \in T(s)$ is bounded for all $x \in [0, \bar{x}]$ and $s \in \mathcal{S}$. These ensure that the right-hand side of (A-7) is Lipschitz continuous with respect to $s \in \mathcal{S}$ for any $x \in [0, \bar{x}]$. Then, the standard argument of the ordinary differential equation applies to see that $y(\cdot)$ is a unique solution to (A-7).

It is easy to see that $y'(\bar{x}) = 0$ because $\xi^{-1}(\bar{x}, s^r) = \underline{t}(s^r) = 0$. In addition, $y'(\cdot)$ is strictly positive and bounded for all $x \in [0, \bar{x}]$. It follows that $G(\cdot) = y^{-1}(\cdot)$ satisfies (Guess). Thus, there uniquely exists $G(\cdot)$ that satisfies (Guess) and (A-6).

Note that this argument holds even if the reserve price is equal to $z(\bar{\boldsymbol{\theta}})$, i.e., the least efficient supplier, $\bar{\boldsymbol{\theta}}$ is indifferent between bidding and staying out. In this case, the right-hand side of (A-7) goes to infinity as $x \to 1$ for some s and fails to meet the Lipschitz condition.[32] The differential equation has multiple solutions for three initial values of $y(1)$: $-\infty$, s^r, and ∞. If $y(1)$ is negative infinite, then $y(x)$ is decreasing at some x close to 1. Hence, it is not a monotone equilibrium. If $y(1)$ is positive infinite, $y(x)$ is not an equilibrium given that s^r is finite. □

Appendix E Proof of Lemma 4

Proof. First, we show that Assumption 4 implies that $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$.

Fix $\mathbf{q}$ in $\mathcal{Q}$. Let $\boldsymbol{\Gamma}$ denote a nonsingular matrix, and let $\tilde{\boldsymbol{\theta}} := \boldsymbol{\Gamma}\boldsymbol{\theta}$. Let $\mathrm{int}\boldsymbol{\Theta}$ denote the interior of $\boldsymbol{\Theta}$. Then, there exists $\epsilon > 0$ such that $B_\epsilon(\theta) := \{\boldsymbol{\theta}' | d(\boldsymbol{\theta}', \boldsymbol{\theta}) < \epsilon\}$ is in $\mathrm{int}\boldsymbol{\Theta}$. Then, $\boldsymbol{\Gamma} B_\epsilon(\boldsymbol{\theta})$ is also open in $\boldsymbol{\Theta}$. Then, for any $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$, let $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ denote the Jacobian

[32] The argument follows Matthews (1995).

matrix of $A(\boldsymbol{\theta};\mathbf{q})$ with respect to $\boldsymbol{\theta}$; namely:

$$\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q}) = \begin{bmatrix} C_{\theta^0}(\mathbf{q},\boldsymbol{\theta}) & C_{\theta^1}(\mathbf{q},\boldsymbol{\theta}) & \cdots & C_{\theta^{K-1}}(\mathbf{q},\boldsymbol{\theta}) \\ C_{q^1\theta^0}(\mathbf{q},\boldsymbol{\theta}) & C_{q^1\theta^1}(\mathbf{q},\boldsymbol{\theta}) & \cdots & C_{q^1\theta^{K-1}}(\mathbf{q},\boldsymbol{\theta}) \\ \vdots & \vdots & \ddots & \vdots \\ C_{q^{L-1}\theta^0}(\mathbf{q},\boldsymbol{\theta}) & C_{q^{L-1}\theta^1}(\mathbf{q},\boldsymbol{\theta}) & \cdots & C_{q^{L-1}\theta^{K-1}}(\mathbf{q},\boldsymbol{\theta}) \end{bmatrix}.$$

Then, $\tilde{C}(\mathbf{q},\tilde{\boldsymbol{\theta}}) := C(\mathbf{q},\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}})$ is well defined in the neighborhood of $\tilde{\boldsymbol{\theta}}$.

Now, let $\tilde{A}(\tilde{\boldsymbol{\theta}};\mathbf{q}) := (\tilde{C}(\mathbf{q},\tilde{\boldsymbol{\theta}}), C_{q^1}(\mathbf{q},\boldsymbol{\theta}),\dots,C_{q^{L-1}}(\mathbf{q},\boldsymbol{\theta}))^{\mathrm{T}}$. Then, the Jacobian matrix of $\tilde{A}(\tilde{\boldsymbol{\theta}};\mathbf{q})$ at $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\tilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}};\mathbf{q}) &= \mathbf{J}_{\boldsymbol{\theta}}\boldsymbol{\Gamma}^{-1}, \\ &= \begin{bmatrix} \tilde{C}_{\tilde{\theta}^0}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \tilde{C}_{\tilde{\theta}^1}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \cdots & \tilde{C}_{\tilde{\theta}^{K-1}}(\mathbf{q},\tilde{\boldsymbol{\theta}}) \\ \tilde{C}_{q^1\tilde{\theta}^0}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \tilde{C}_{q^1\tilde{\theta}^1}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \cdots & \tilde{C}_{q^1\tilde{\theta}^{K-1}}(\mathbf{q},\tilde{\boldsymbol{\theta}}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_{q^{L-1}\tilde{\theta}^0}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \tilde{C}_{q^{L-1}\tilde{\theta}^1}(\mathbf{q},\tilde{\boldsymbol{\theta}}) & \cdots & \tilde{C}_{q^{L-1}\tilde{\theta}^{K-1}}(\mathbf{q},\tilde{\boldsymbol{\theta}}) \end{bmatrix}.\end{aligned}$$

Then, $\tilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}};\mathbf{q})$ is a strictly diagonally dominant matrix by Assumption 4. Therefore, $\tilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}};\mathbf{q})$ is nonsingular by the Levy-Desplanques theorem. Note that $\boldsymbol{\Gamma}^{-1}$ is nonsingular by definition. Thus, $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q})$ is also nonsingular.

The above argument holds for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$ and for all $\mathbf{q} \in \mathcal{Q}$. Therefore, $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q})$ is nonsingular for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$ and for all $\mathbf{q} \in \mathcal{Q}$.

Second, we show that the nonsingularity of $\mathbf{J}_{\boldsymbol{\theta}}$ implies Assumption 4. Suppose that $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q})$ is nonsingular for all $\boldsymbol{\theta}$ in the interior of $\boldsymbol{\Theta}$ and $\mathbf{q} \in \mathcal{Q}$. Then, set $\boldsymbol{\Gamma} := \mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta};\mathbf{q})$ for all $\mathbf{q}$ and $\boldsymbol{\theta}$ so that we have $\mathbf{J}_{\boldsymbol{\theta}}\boldsymbol{\Gamma}^{-1} = I_L$. Then, Assumption 4 is satisfied. $\square$

Appendix F Local invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$ under $K := \dim(\boldsymbol{\theta}) < L$

We first show that Assumption 4 implies the local invertibility of $A(\cdot)$. Let $\mathbf{D}$ denote an $L \times (L-K)$ matrix such that its ℓth column with $\ell = \{K, \ldots, L-1\}$ is given by

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \sum_{k=0}^{K-1} C_{q^K \theta^k} + \epsilon & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{K-1} C_{q^{K+1} \theta^k} + \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \sum_{k=0}^{K-1} C_{q^{L-1} \theta^k} + \epsilon \end{bmatrix},$$

with some $\epsilon > 0$. Then, for some $\mathbf{z} \in \mathbb{R}^{L-K}$, define $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q}) := A(\boldsymbol{\theta}; \mathbf{q}) + \mathbf{Dz}$. Now, let $\widehat{\mathbf{J}}_{(\boldsymbol{\theta},\mathbf{z})}$ denote the Jacobian matrix of $\widehat{A}$. Then, for any $(\boldsymbol{\theta}, \mathbf{0}) \in \mathrm{int}\boldsymbol{\Theta} \times \mathbb{R}^{L-K}$, $\widehat{\mathbf{J}}_{(\boldsymbol{\theta},\mathbf{z})}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q})$ is a full-rank matrix. Hence, applying the local inverse function theorem, we have the inverse of $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q})$. Let $\widehat{A}^{-1}(\cdot)$ denote the inverse and T denote an operator that trims $L-K$ elements of an L-dimensional vector from the end. Then, because $\widehat{A}^{-1} := \left(\widehat{A}_0^{-1}, \cdots, \widehat{A}_{K-1}^{-1}, \widehat{A}_K^{-1}, \cdots, \widehat{A}_{L-1}^{-1}\right)$, we have $A^{-1} \equiv (\widehat{A}_0^{-1}, \cdots, \widehat{A}_{K-1}^{-1}) = T\widehat{A}^{-1}(\cdot)$. Note that we have $(\boldsymbol{\theta}, \mathbf{0}) = \widehat{A}^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q})$. Therefore,

$$\begin{aligned} \boldsymbol{\theta} &= T(\boldsymbol{\theta}, \mathbf{0}) \\ &= T(\widehat{A}^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q})) \\ &= A^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q}). \end{aligned}$$

Hence, $A(\cdot)$ is locally invertible for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and for all $\mathbf{q} \in \mathcal{Q}$.

Next, we show the converse. Suppose that $A(\cdot)$ is locally invertible for any $\boldsymbol{\theta}$ and $\mathbf{q}$.

Then, from the above argument, $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q}) := A(\boldsymbol{\theta}; \mathbf{q}) + \mathbf{Dz}$ has the nonsingular Jacobian matrix for any $(\boldsymbol{\theta}, \mathbf{0})$ and for any $\mathbf{q}$. Let $\widehat{\mathbf{J}}_{(\boldsymbol{\theta},\mathbf{0})}(\boldsymbol{\theta}, \mathbf{0}; \mathbf{q})$ denote the Jacobian matrix. Then, set $\boldsymbol{\Gamma} = \widehat{\mathbf{J}}$ so that we have $\widehat{\mathbf{J}}\boldsymbol{\Gamma}^{-1} = I_L$. Then, Assumption 4 is satisfied.

Appendix G Proof of Proposition 3

Proof. By the global inverse function, $A(\boldsymbol{\theta}; \mathbf{q})$ is globally invertible if, for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$,

1. $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible and its inverse function is continuous;
2. $A(\boldsymbol{\theta}; \mathbf{q})$ is proper;
3. $\boldsymbol{\Theta}$ is arcwise connected, and the image of $A(\boldsymbol{\theta}, \mathbf{q})$ is simply connected.

First, we already have known that $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$ (where $\mathrm{int}\boldsymbol{\Theta}$ denotes the interior of $\boldsymbol{\Theta}$) and $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$. We show that $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for any $\boldsymbol{\theta} \in \partial\boldsymbol{\Theta}$. (where $\partial\boldsymbol{\Theta} := \boldsymbol{\Theta} \setminus \mathrm{int}\boldsymbol{\Theta}$, the boundary of $\boldsymbol{\Theta}$.) Let $B_\epsilon(\tilde{\boldsymbol{\theta}}) := \{\boldsymbol{\theta} | d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) < \epsilon\}$ denote the ϵ-neighborhood of $\tilde{\boldsymbol{\theta}}$. Suppose that $A(\tilde{\boldsymbol{\theta}}; \mathbf{q})$ is not locally injective at $\tilde{\boldsymbol{\theta}} \in \partial\boldsymbol{\Theta}$. Then, for a fixed $\epsilon > 0$, there exists $\boldsymbol{\theta} \in B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}$ such that $A(\tilde{\boldsymbol{\theta}}; \mathbf{q}) = A(\boldsymbol{\theta}; \mathbf{q})$. Let $\delta := d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta})$. Then, for a fixed $\epsilon' \in (0, \delta)$, there exists $\boldsymbol{\theta}' \in B_{\epsilon'}(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}$ such that $A(\tilde{\boldsymbol{\theta}}; \mathbf{q}) = A(\boldsymbol{\theta}'; \mathbf{q})$. Since $\boldsymbol{\theta} \notin B_{\epsilon'}(\tilde{\boldsymbol{\theta}})$, we have $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$ and $A(\boldsymbol{\theta}; \mathbf{q}) = A(\boldsymbol{\theta}'; \mathbf{q})$. However, by Assumption 4, $A(\boldsymbol{\theta}; \mathbf{q})$ is injective for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$. Thus, there exists $\epsilon > 0$, such that $A(\tilde{\boldsymbol{\theta}}; \mathbf{q}) \neq A(\boldsymbol{\theta}; \mathbf{q})$ for all $\boldsymbol{\theta} \in B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}$. Then, $A(\cdot; \cdot)$ is locally invertible with respect to $\boldsymbol{\theta}$ at $\tilde{\boldsymbol{\theta}} \in \partial\boldsymbol{\Theta}$ with the inverse $A^{-1}(\cdot; \cdot) : A(B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}; \mathbf{q}) \to B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}$, where $A(B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}; \mathbf{q}) := \{A(\boldsymbol{\theta}; \mathbf{q}) | \boldsymbol{\theta} \in B_\epsilon(\tilde{\boldsymbol{\theta}}) \cap \boldsymbol{\Theta}\}$.

Next, we show that $A^{-1}(\cdot)$ is continuous at any boundary points $\tilde{\boldsymbol{\theta}} \in \partial\boldsymbol{\Theta}$. Take a sufficiently small $\epsilon > 0$. Let $\bar{\delta} := \sup_{d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) < \epsilon} d(A(\tilde{\boldsymbol{\theta}}; \mathbf{q}), A(\boldsymbol{\theta}; \mathbf{q}))$. Then, since $A(\cdot)$ is continuous at $\tilde{\boldsymbol{\theta}} \in \partial\boldsymbol{\Theta}$, for any $\delta \in (0, \bar{\delta})$, there is an $\epsilon' \in (0, \epsilon)$ such that if $d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) < \epsilon'$, we have $d(A(\tilde{\boldsymbol{\theta}}; \mathbf{q}), A(\boldsymbol{\theta}; \mathbf{q})) < \delta$. Furthermore, by the definition of δ, if $d(A(\tilde{\boldsymbol{\theta}}; \mathbf{q}), A(\boldsymbol{\theta}; \mathbf{q})) < \delta$, then $d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) < \epsilon$ holds. Therefore, if $d(A(\tilde{\boldsymbol{\theta}}; \mathbf{q}), A(\boldsymbol{\theta}; \mathbf{q})) < \delta$, we have

$$\begin{aligned} d(A^{-1}(A(\tilde{\boldsymbol{\theta}})), A^{-1}(A(\boldsymbol{\theta}))) &= d(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) \quad \text{(since } A(\cdot, \cdot) \text{ is locally invertible)} \\ &< \epsilon. \end{aligned}$$

Note that $A(\cdot,\cdot)$ is bijective with respect to $\boldsymbol{\theta}$ in a neighborhood of $\tilde{\boldsymbol{\theta}}$. Therefore, for any point $\mathbf{b}$ in a neighborhood of $A(\tilde{\boldsymbol{\theta}};\mathbf{q})$, there exists $\boldsymbol{\theta}$ such that $\mathbf{b} = A(\boldsymbol{\theta};\mathbf{q})$. Thus, $A^{-1}(\cdot,\cdot)$ is continuous, as required.

Second, we show that $A(\boldsymbol{\theta};\mathbf{q})$ is a proper map for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$. That is, we show that for any compact subset $Y \in \{A(\boldsymbol{\theta};\mathbf{q})|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$, the inverse image of Y, $A^{-1}(Y;\mathbf{q}) := \{\boldsymbol{\theta} \in \boldsymbol{\Theta}|A(\boldsymbol{\theta};\mathbf{q}) \in Y\}$, is also compact. Since $A(\boldsymbol{\theta};\mathbf{q})$ is continuous, $A(Y;\mathbf{q})$ is also closed for any closed set Y. Furthermore, $\boldsymbol{\Theta}$ is bounded. Therefore, by the definition of inverse image, $A^{-1}(Y;\mathbf{q})$ is a subset of $\boldsymbol{\theta}$. Therefore, $A^{-1}(Y;\mathbf{q})$ is bounded for all $\mathbf{q} \in \mathbf{Q}(\boldsymbol{\theta})$. Thus, $A(\boldsymbol{\theta};\mathbf{q})$ is a proper map.

Finally, we show that i) domain $\boldsymbol{\Theta}$ is arcwise connected and that ii) image $A(\boldsymbol{\Theta};\mathbf{q}) := \{A(\boldsymbol{\theta};\mathbf{q}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ is simply connected. We show i) first. In our model, $\boldsymbol{\Theta}$ is a Cartesian product of simply connected interval $[\underline{\theta}^k, \bar{\theta}^k]$ for all $k = 0, \ldots, K-1$. Thus, $\boldsymbol{\Theta}$ is obviously arcwise connected. Next, we show ii). Since $C(\mathbf{q},\boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q},\boldsymbol{\theta})$ are continuous, images $C(\mathbf{q},\boldsymbol{\Theta}) := \{C(\mathbf{q},\boldsymbol{\theta}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ and $C_{q^\ell}(\mathbf{q},\boldsymbol{\Theta}) := \{C_{q^\ell}(\mathbf{q},\boldsymbol{\theta}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ are simply connected for $\ell = 1, \ldots, L-1$. Thus, image $A(\boldsymbol{\Theta};\mathbf{q})$ is also simply connected. $\square$

Appendix H A Test for the Cost Function

First, we define exogenous variation in the number of bidders:

Definition 2 (Athey and Haile (2007))**.** *A bidding environment has exogenous variation in the number of bidders if, for all n', n'' such that $n' < n'' \leqq n$, $F(\cdot;n')$ is identical to $F(;n'')$.*

Then, consider the case in which the econometrician seeks to estimate $\boldsymbol{\theta}$ by using a cost function, $\widehat{C}(\mathbf{q},\boldsymbol{\theta})$, that differs from the true cost function – i.e., $\widehat{C}(\mathbf{q},\boldsymbol{\theta}) \neq C(\mathbf{q},\boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q}$. Then, let $\mathbf{b}^*(\boldsymbol{\theta}, n) = \{p^*(\boldsymbol{\theta}, n), \hat{G}^*(\boldsymbol{\theta}, n), \hat{g}^*(\boldsymbol{\theta}, n)\}$ and $\mathbf{q}^*(\boldsymbol{\theta}, n)$ denote observations implied by the bidder with type $\boldsymbol{\theta}$, given that the number of bidders in the auction is n. Let $\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \widehat{C}, n)$ denote the estimate. Then, the following two estimates:

$$\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \widehat{C}, n') \equiv A^{-1}(\mathbf{b}^*(\boldsymbol{\theta}, n'); \mathbf{q}^*(\boldsymbol{\theta}, n'), \widehat{C}) \text{ and}$$
$$\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \widehat{C}, n'') \equiv A^{-1}(\mathbf{b}^*(\boldsymbol{\theta}, n''); \mathbf{q}^*(\boldsymbol{\theta}, n''), \widehat{C})$$

generally differ for some or all $\boldsymbol{\theta}$. Then, values of $\widehat{F}(\boldsymbol{\theta})$ – i.e., the distribution of $\widehat{\boldsymbol{\theta}}$ – generally differ depending on n, which could give a testable implication because the true distribution of $\boldsymbol{\theta}$ is identical for all n.

In the following, we show that the test does not function if the scoring rule is QL. The observation of bidder type $\boldsymbol{\theta}$ in the scoring auction with n bidders implies that

$$C(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = p^*(\boldsymbol{\theta}, n) - \frac{1 - \hat{G}^*(\boldsymbol{\theta}, n)}{(n-1)\hat{g}^*(\boldsymbol{\theta}, n)}, \tag{A-8}$$

$$C_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = P_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta})) \text{ with } \ell = 1, \ldots, L-1. \tag{A-9}$$

We write $\mathbf{q}^*(\boldsymbol{\theta})$ instead of $\mathbf{q}^*(\boldsymbol{\theta}, n)$ because $\mathbf{q}^*(\cdot)$ is identical for all n under the QL scoring rule. Given that $C(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$ with $\ell = 1, \ldots, L-1$ are all identical for any n, the right-hand side in expression (A-8) is constant for any n. Then, $\boldsymbol{\theta}$ is recovered as:

$$\hat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \widehat{C}) = A^{-1}(\mathbf{b}^*(\boldsymbol{\theta}); \mathbf{q}^*(\boldsymbol{\theta}), \widehat{C}), \tag{A-10}$$

for all $\widehat{C}(\cdot)$ that satisfies Assumptions 1 through 4. It is easy to see that $\widehat{F}(\cdot; n') = \widehat{F}(\cdot; n'')$. This implies that the scoring auction model does not give a refutable restriction on observations under the exogenous variation in the number of bidders.

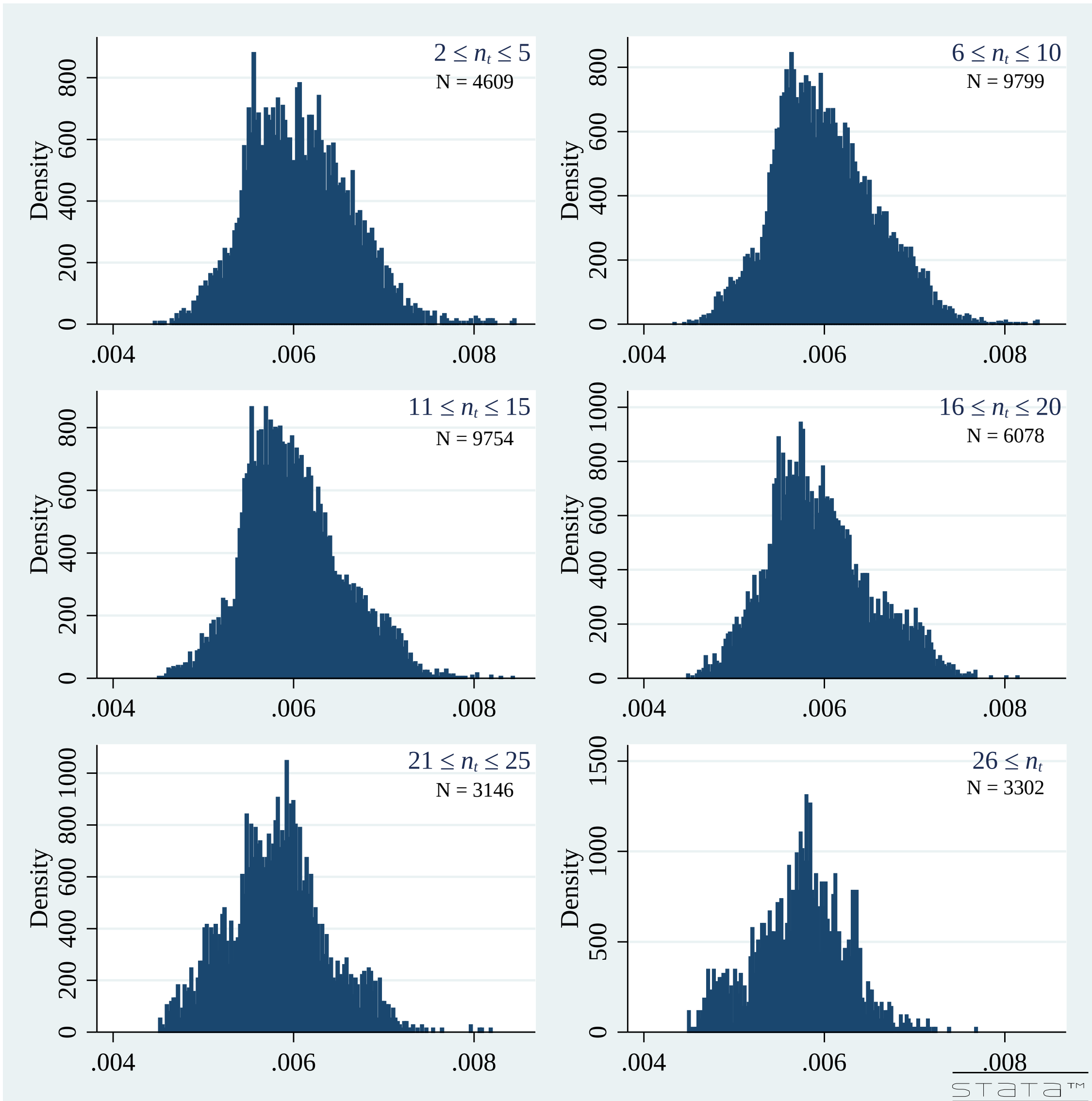

Figure 1: Distribution of Normalized Score for the set of auctions with the number of bidders ranging from 2 through 5 and from 6 through 10 (top row), from 11 through 15 and from 16 through 20 (middle row), and from 21 through 25 and from 26 through 30 (bottom row).

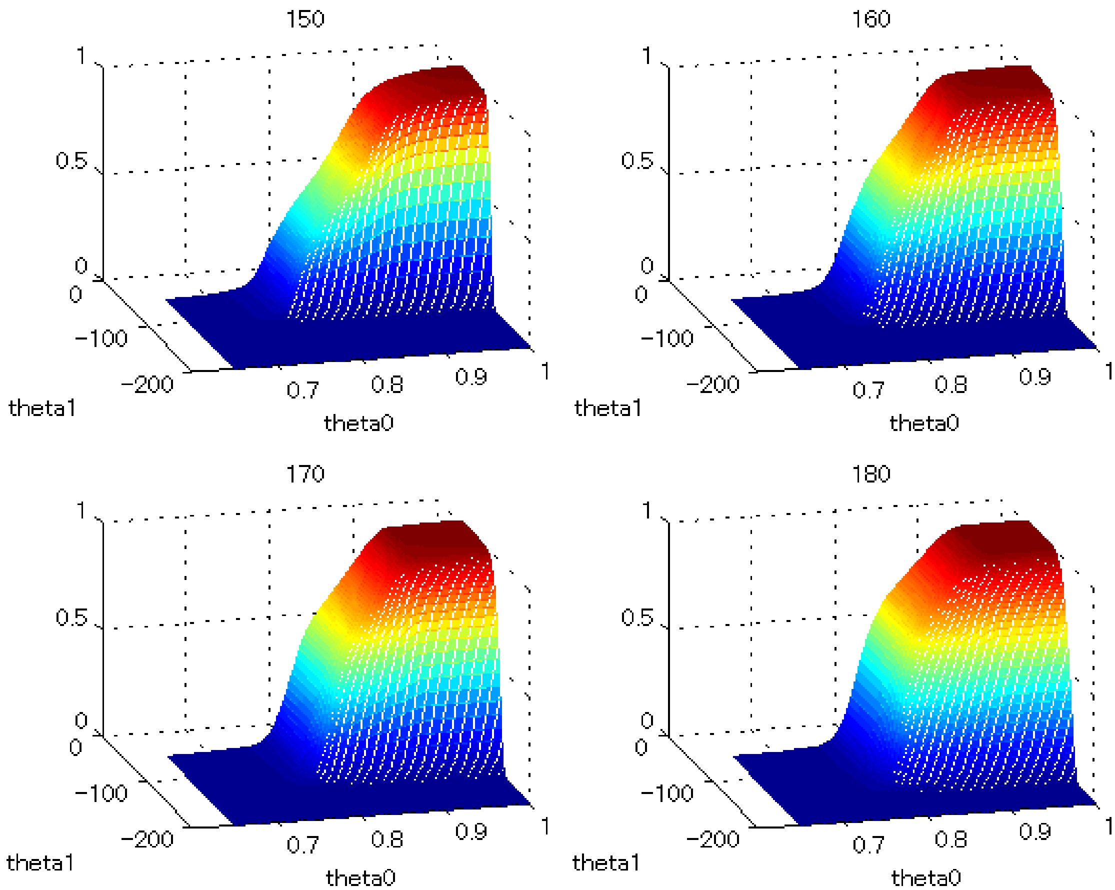

Figure 2: Estimated cdf of $\boldsymbol{\theta}$. Each panel corresponds to the scoring auctions with the quality upper bound equal to 150 (top left), 160 (top right), 170 (bottom left), and 180 (bottom right). The Gaussian kernel is used. The bandwidths for θ^0 and θ^1 for the quality upper bound: 160 are, e.g., .0022 and 0.7176, respectively.

Variable[*1]	Obs	Mean	SD	Min	Max
Number of Bidders	5,142	9.88	6.39	2	34
Engineers' Estimates[*2]	5,142	477.0	1,100.0	200.0	37,600
Win Price Bids[*2]	5,142	423.0	972.0	169.0	34,300
Win Quality-Bid Points	5,142	158.17	11.34	132.60	200.00
Win Scores	5,142	177.23	15.088	109.39	310.12
Price bids[*2]	36,688	531.0	984.0	160.0	37,100
Quality-Bid Points	36,688	153.19	11.11	101.50	200.00
Scores	36,688	180.19	15.315	109.39	310.12

[*1]The top five rows are the statistics for each auction; the bottom three rows are the statistics for each bid. [*2]Units are Yen million.

Table 1: Sample statistics

Quality-Bid Upper bound	Obs	Mean*1	SD*1	Min*1	Max*1
150	1,182	290.0	174.0	200.0	4,470.0
160	2,124	339.0	412.0	200.0	5,950.0
170	1,114	504.0	1,050.0	200.0	12,200
180	495	666.0	1,280.0	200.0	12,400
190	220	1,990.0	2,830.0	207.0	28,300
200	7	8,110.0	13,400	397.0	37,600
Total	5,142	477.0	1,100.0	200.0	37,600

*1Numbers represent the statistics regarding the engineers' estimates. Units are Yen million.

Table 2: Project sizes (by Quality-Bid Upper bound)

Form	$C(q,\boldsymbol{\theta})$	$\bar{q}$	Obs	Mean	SD	Min	Max	Change[*3]	
FS[*1]	-	-	5,142	177.2	15.09	109.4	310.1	-	
		130	5,142	163.7	14.94	98.21	580.4	-7.66%	(0.53%)
		140	5,142	171.3	16.58	95.53	589.0	-3.34%	(0.39%)
	Quadratic	150	5,142	175.3	18.58	91.86	631.1	-1.10%	(0.25%)
		160	5,142	175.5	20.65	87.89	640	-0.96%	(0.18%)
		170	5,142	172.5	22.22	83.83	576.5	-2.69%	(0.24%)
		130	5,142	163.2	15.15	96.21	572.7	-7.94%	(0.50%)
		140	5,142	170.9	16.71	90.26	605.9	-3.55%	(0.38%)
SP[*2]	Cubic	150	5,142	175.0	18.79	83.19	637.9	-1.24%	(0.24%)
		160	5,142	174.7	21.38	75.73	638	-1.44%	(0.37%)
		170	5,142	169.8	23.97	68.36	575.7	-4.20%	(0.53%)
		130	5,142	161.9	14.87	93.79	561.9	-8.68%	(0.51%)
		140	5,142	170.1	16.44	84.88	593.7	-4.00%	(0.40%)
	Quartic	150	5,142	174.5	18.74	74.47	626.1	-1.51%	(0.26%)
		160	5,142	173.7	22.07	63.90	636	-2.00%	(0.28%)
		170	5,142	166.9	25.89	54.08	574.8	-5.82%	(0.51%)

[*1]Observed FS auctions. [*2]Counterfactual second-price auctions. [*3]Change in mean from FS to SP auction; numbers in parentheses are standard deviations generated by bootstrapping samples. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 3: Buyer's Utilities (Price-only vs FS Auctions)

Form	$C(q, \boldsymbol{\theta})$	$\bar{q}$	Obs	Mean	SD	Min	Max	Change[*3]	
FS[*1]	-	-	5,142	0.059	0.075	0.000	0.762	-	
		130	5,142	0.040	0.067	0.000	0.674	-30.82%	(1.50%)
		140	5,142	0.046	0.071	0.000	0.716	-21.37%	(1.46%)
	Quadratic	150	5,142	0.053	0.076	0.000	0.745	-10.05%	(1.39%)
		160	5,142	0.060	0.083	0.000	0.774	2.58%	(1.38%)
		170	5,142	0.068	0.090	0.000	0.852	15.83%	(1.48%)
		130	5,142	0.041	0.067	0.000	0.685	-29.09%	(1.37%)
		140	5,142	0.046	0.072	0.000	0.722	-20.69%	(1.45%)
SP[*2]	Cubic	150	5,142	0.053	0.079	0.000	0.816	-8.82%	(1.55%)
		160	5,142	0.063	0.089	0.000	1.032	6.92%	(1.71%)
		170	5,142	0.074	0.103	0.000	1.275	26.57%	(2.01%)
		130	5,142	0.041	0.067	0.000	0.684	-29.25%	(1.37%)
		140	5,142	0.046	0.073	0.000	0.724	-21.15%	(1.54%)
	Quartic	150	5,142	0.054	0.082	0.000	1.027	-8.13%	(1.81%)
		160	5,142	0.065	0.098	0.000	1.413	11.52%	(2.23%)
		170	5,142	0.082	0.122	0.000	1.893	39.60%	(2.89%)

[*1]Observed FS auctions. [*2]Counterfactual second-price auctions. [*3]Change in mean from FS to SP auction; numbers in parentheses are standard deviations generated by bootstrapping samples. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 4: Bidders' Payoffs (Price-only vs FS auctions)

Form	$C(q, \boldsymbol{\theta})$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	177.23	15.088	109.39	310.12	-	
	Quadratic	5,142	178.51	21.292	98.26	646.5	0.72%	(0.12%)
SS[*2]	Cubic	5,142	178.50	21.232	98.26	644.9	0.71%	(0.12%)
	Quartic	5,142	178.48	21.135	98.26	636.8	0.71%	(0.12%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the PQR rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 5: Buyer's utilities (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	158.17	11.338	132.60	200.00	-	
	Quadratic	5,142	158.09	11.510	130.18	201.81	-0.05%	(0.02%)
SS[*2]	Cubic	5,142	158.09	11.457	131.93	201.56	-0.06%	(0.01%)
	Quartic	5,142	158.10	11.424	132.09	201.29	-0.05%	(0.01%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the PQR rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 6: Contracted Quality Levels (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	0.0585	0.0752	0.0000	0.7616	-	
	Quadratic	5,142	0.0572	0.0776	0.0000	0.8251	-2.32%	(1.09%)
SS[*2]	Cubic	5,142	0.0571	0.0775	0.0000	0.8248	-2.34%	(1.09%)
	Quartic	5,142	0.0571	0.0775	0.0000	0.8246	-2.34%	(1.09%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the PQR rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 7: Bidder's Payoffs (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS*1	-	5,142	177.23	15.088	109.39	310.12	-	
	Quadratic	5,142	178.46	20.041	103.25	547.73	0.69%	(0.11%)
SS*2	Cubic	5,142	178.46	20.094	102.79	549.31	0.69%	(0.11%)
	Quartic	5,142	178.47	20.132	102.67	550.68	0.70%	(0.11%)

*1Observed FS auctions (PQR). *2Hypothetical SS auctions with the QL rule. *Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 8: Buyer's Utilities (QL vs PQR Scoring Rules)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	0.0585	0.0752	0.0000	0.7616	-	
	Quadratic	5,142	0.0562	0.0752	0.0000	0.8157	-4.03%	(1.05%)
QL[*2]	Cubic	5,142	0.0562	0.0753	0.0000	0.8160	-3.89%	(1.05%)
	Quartic	5,142	0.0563	0.0754	0.0000	0.8161	-3.81%	(1.05%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the QL rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 9: Bidder Payoffs (QL vs PQR Scoring Rules)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	158.17	11.338	132.60	200.00	-	
	Quadratic	5,142	158.27	11.484	126.33	200.81	0.06%	(0.04%)
SS[*2]	Cubic	5,142	158.25	11.428	129.24	200.71	0.05%	(0.03%)
	Quartic	5,142	158.23	11.402	129.99	200.59	0.04%	(0.02%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the QL rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 10: Contracted Quality Levels (QL vs PQR Scoring Rules)

Online Appendix (Not For Publication)

Online Appendix I Proof of the Existence, Uniqueness, and Strict Monotonicity of the Solution to (7)

Proof. We first show that a solution to the maximization problem (7) exists for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-} \cap \boldsymbol{\Theta}_r$.

Suppose that $\boldsymbol{\theta} \in \boldsymbol{\Theta}_r$. Then, choosing $s < s^r$ results in a strictly negative payoff when winning (Recall that $u(s, \boldsymbol{\theta}) < 0$ for all $s < s^r$ if $\boldsymbol{\theta} \in \boldsymbol{\Theta}_r$). Moreover, choosing $s > s^r$ is weakly dominated, because it fails to meet the reserve score. Therefore, choosing s^r is optimal for bidder i if its type is $\boldsymbol{\theta} \in \boldsymbol{\Theta}_r$.

Suppose next that $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-}$. Then, the derivative of the objective function is given by

$$u_s(s, \boldsymbol{\theta})(1 - G(s)) - (n-1)u(s, \boldsymbol{\theta})g(s). \tag{OA-1}$$

For all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-}$, $z(\boldsymbol{\theta}) := \{s | u(s, \boldsymbol{\theta}) = 0\} < s^r$. It follows that $u(s^r, \boldsymbol{\theta}) > 0$. Recall (Guess), which addresses that $\lim_{s \to s^r} g(s) \to \infty$. The second term in (OA-1) is thus negative infinite. Because the first term is finite, (OA-1) is negative infinite if $s = s^r$. On the other hand, (OA-1) is strictly positive when i chooses $z(\boldsymbol{\theta})$, because the second term vanishes. Given that (7) is a smooth function of s, there exists s in the interior of $[z(\boldsymbol{\theta}), s^r]$ at which (OA-1) is equal to zero. Therefore, the solution exists for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-}$ in the interior of $[z(\boldsymbol{\theta}), s^r]$.

Next, we show that the solution is unique and strictly increasing in $\boldsymbol{\theta}$. Let s^* denote a solution to the maximization problem for some $\boldsymbol{\theta}$. Then, the sorting condition implies that

$$u_s(s^*, \hat{\boldsymbol{\theta}})(1 - G(s^*)) - (n-1)u(s^*, \hat{\boldsymbol{\theta}})g(s^*) \gtreqless 0, \tag{OA-2}$$

for all $\hat{\boldsymbol{\theta}} \gtreqless \boldsymbol{\theta}$. This suggests that s^* is suboptimal – i.e., too low (or too high) – for all $\hat{\boldsymbol{\theta}} \geq \boldsymbol{\theta}$ (or for all $\hat{\boldsymbol{\theta}} \leq \boldsymbol{\theta}$).[33] This, in turn, implies that, for all $\hat{\boldsymbol{\theta}} \geq \boldsymbol{\theta}$ (or for all $\hat{\boldsymbol{\theta}} \leq \boldsymbol{\theta}$), the solution is strictly greater (or smaller) than s^*. It follows that the solution is unique and strictly increasing in $\boldsymbol{\theta}$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-}$. Note that s^r is the optimal for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_r$. Hence,

[33] Here, "$\geq$" and "$\leq$" denote vector inequalities.

the optimal solution to (7) is unique and strictly increasing for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{r-} \cap \boldsymbol{\Theta}_r$ under (Guess). $\square$

Online Appendix II Generating counterfactual SS auction samples from the estimated parameters

Define

$$q^{FB}(\boldsymbol{\theta}) = q(z(\boldsymbol{\theta}), \boldsymbol{\theta}). \tag{OA-3}$$

Under the PQR scoring rule, $q^{FB}(\boldsymbol{\theta}) = \{q | C_q(q, \boldsymbol{\theta})q = C(q, \boldsymbol{\theta})\}$. Hence, solving the following polynomial:

$$\left(q^{FB} + \hat{\theta}^1_{i,t}\right)^\beta - q^{FB}\beta\left(q^{FB} + \hat{\theta}^1_{i,t}\right)^{\beta-1} + \hat{\theta}^0_{i,t} = 0,$$

gives us the estimate of q^{FB} under the PQR scoring rule. Using $\hat{q}^{FB}_{i,t}$, the break-even score, $z(\boldsymbol{\theta}) = C(\hat{q}^{FB}, \boldsymbol{\theta})/\hat{q}^{FB}$, is estimated as

$$\hat{z}(\hat{\boldsymbol{\theta}}_{i,t}) = \frac{1}{\hat{q}^{FB}_{i,t}}\left[\left(\hat{q}^{FB}_{i,t} + \hat{\theta}^1_{i,t}\right)^\beta + \hat{\theta}^0_{i,t}\right].$$

Given that the contract quality (as well as price) matches the second-lowest score $s_{(2)} = z(\boldsymbol{\theta}_{(2)})$, it is obtained as

$$\hat{q}^{post}_{\mathrm{II,t}} = \left(\frac{z(\hat{\boldsymbol{\theta}}_{(2),t})}{\beta}\right)^{\frac{1}{\beta-1}} - \hat{\theta}^1_{(1),t}.$$

The winner's payoff is, thus, given by

$$u(s_{(2),t}, \hat{\boldsymbol{\theta}}_{(1),t}) = \hat{q}^{post}_{\mathrm{II,t}} \cdot s_{(2),t} - \left(\hat{q}^{post}_{\mathrm{II,t}} + \hat{\theta}^1_{i,t}\right)^\beta - \hat{\theta}^0_{i,t}.$$

Online Appendix III Generating the counterfactual SS auctions with the QL scoring rule

Under the QL rule, the bidder's *pseudotype* is given by $k(\boldsymbol{\theta}) = \min_q C(q, \boldsymbol{\theta}) - \phi(\beta)q \equiv z(\boldsymbol{\theta})$. This implies that the minimizer is $q^{FB}(\boldsymbol{\theta}) \equiv q(z(\boldsymbol{\theta}), \boldsymbol{\theta})$ as defined in (OA-3). Because $P_q(s, q) = \phi(\beta)$ for all s and q, q^{FB} in the QL scoring rule is given by

$$\hat{q}^{FB}_{QL,i,t} = \left(\frac{\phi(\beta)}{\beta}\right)^{\frac{1}{\beta-1}} - \hat{\theta}^1_{i,t}. \tag{OA-4}$$

Hence, the bidder's pseudotype is estimated by

$$\hat{k}_{i,t} = \left(\hat{q}^{FB}_{QL,i,t} + \hat{\theta}^1_{i,t}\right)^{\beta} + \hat{\theta}^0_{i,t} - q^{FB}_{QL}(\hat{\boldsymbol{\theta}}_{i,t}). \tag{OA-5}$$

The lowest pseudotype bidder wins and receives the payment $p_{QL} = k(\boldsymbol{\theta}_{(2)}) + q^{FB}(\boldsymbol{\theta}_{(1)})$ in the SS auction with the QL scoring rule. Thus, both are estimated from observations. The buyer's utility from the contract is then estimated by

$$s_{QL,t} = \hat{p}_{QL,t} / q^{FB}_{QL}(\hat{\boldsymbol{\theta}}_{(1),t}).$$