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Abstract

We provide a microfoundation for the production function by using the concept of stochastic macro-equilibrium in Yoshikawa (“Stochastic Macro-equilibrium and Microfoundations for Keynesian Economics,” RIETI Discussion Paper, 2013). We consider an economy with multiple firms, with each firm possessing Leontief technology. We assume that the allocation of labor is determined by entropy maximization. We show that for selected productivity distribution, aggregate production is described by the popular Cobb-Douglas type.

Keywords: Production function, Microfoundation, Leontief technology, Stochastic macro-equilibrium

JEL classification: E10

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1 Introduction

In both theoretical and empirical macroeconomic analysis, it is popular to use the aggregate production function such as the Cobb-Douglas production function. A seminal paper of Solow (1956) constructs what is called as the neoclassical growth model by using the strictly concave and constant returns to scale production function. Some authors incorporate intertemporal utility function into the Solow model to investigate policy effects. Judd (1985, 1987) construct a neoclassical growth model with tax policy instruments and obtain the optimal capital tax rate. However, in a real economy, there is no representative firm nor the aggregate production function. The aggregate production function is simply a sum of individual production unit. Microfoundations for production function is therefore an important topic in modern macroeconomics. Jones (2005) claims that we need theoretical foundations of the aggregate production function.

Many authors have tried to give a microfoundation to the popular and empirically plausible production function such as CES production function. A pioneering paper of Houthakker (1955-1956) investigates a model that consists of production units, and each unit is maximizing its profit and has Leontief production function. He shows that if the production function of the individual firm is Leontief type and productivities are distributed according to Pareto, the aggregate production is Cobb-Douglas. ¹ Lagos (2003) adopts a similar framework and investigates how unemployment affects TFP. ²

Jones (2005) shows that if the distribution of ideas are Pareto, the global production function is also Cobb-Douglas. Dupuy (2012) constructs a model with heterogenous workers and jobs and obtains a similar results. A recent paper of Growiec (2013) is very close to Jones (2005), but he assumes that the productivity distribution follows Weibull distribution instead of Pareto and obtains the CES production function. Marco (2003)

¹Existing literature on the aggregation of the production function includes Sato (1975).

²A recent paper of Petrosky-Nadeau (2013) incorporate financial friction into Lagos (2006).

uses NBER Patent Citation Datafile, and performs a nonparametric hazard estimation. He then finds that the patent citations follows Weibull distribution. As Felipe and Fisher (2003) argue, the conditions under which a popular aggregate production function such as the Cobb-Douglas can be derived are very much limited. Therefore aggregate production function still do not have a sound theoretical foundations.

We adopt a novel approach and provide a microfoundation for the production function by using the concept of stochastic macro-equilibrium in Aoki and Yoshikawa (2007) and Yoshikawa (2013). The concept is based on the statistical physics theory. As in Houthakker (1955-1956) and Jones (2005), we consider an economy with multiple firms, and each firm has Leontief technology. In Houthakker (1955-1956), the allocation is determined by the profit maximization of firms, but here we assume that it is determined by probability maximization problem. Although the probability function has a complex form, it is approximated by the entropy if the number of firms is large. We then study the entropy maximization problem and show that for some productivity distribution, the aggregate production becomes Cobb-Douglas type. We also show that our model may derive convex-concave production function, that exhibits increasing returns to scale as long as capital is small, and then has constant returns to scale. It is true that many authors including Skiba (1978) investigate the optimal growth model with the nonconvex technology, they simply assume the shape of the production function and provide no foundation. We finally show that the production function may be CES production function which is now becoming popular in macroeconomic analysis.

The shape of the aggregate production function plays a crucial role in the analysis of growth and inequality. A prominent book of Piketty (2014) argues that the difference between the rate of return on capital (r) and the economic growth rate (g) plays a crucial role in determining the inequality. Especially he argues that as the economic growth rate lowers, the value of $r - g$ rises and then the degree of inequality becomes heavier than

before. As Piketty (2014), Jones (2014) and Krusell and Smith (2014) argue, the elasticity of substitution between capital and labor plays an important role on determining the effect of the economic growth on $r - g$. As is clear from these arguments, precise measurement of the aggregate production function is very much important in the analysis of economic inequality.

The paper proceeds as follows. Section 2 describes the model. Section 3 characterizes the shape of the production function. Section 4 studies a case with non-concave production function. Section 5 concludes. The Appendix contains proofs of the propositions.

2 Model

In this section, we provide our model.

2.1 Set-up

The set-up is very close to Houthakker (1955-1956) and Jones (2005). There are I firms, where I is a positive integer. We assume that $I > 4$. Total number of workers is N and aggregate capital is K . Here we assume that both N and K are given. Firm i employs N_i workers, and K_i units of capital. Then the aggregate labor demand and capital demand are respectively equal to $\sum_{i=1}^I N_i$ and $\sum_{i=1}^I K_i$. The production function is of the Leontief-type, and to produce one unit of the final good, the firm needs $1/B_i$ units of workers and $1/A_i$ units of capital, where $A_i > 0$ is capital efficiency, $B_i > 0$ is labor efficiency. The Leontief production function is assumed in Gorton and Ordóñez (2014).

If we let $b_i = \frac{B_i}{A_i}$ denote the relative productivity of capital, it satisfies $K_i/N_i = b_i$ where K_i is capital of firm i . We denote the output of the firm i by y_i . The firm's output is given by $y_i = A_i K_i = B_i N_i$. This is re-written as

$$y_i = A_i b_i N_i. \tag{1}$$

The total output for this economy is equal to $Y = \sum_{i=1}^I y_i$. The resource constraints on capital and labor are respectively given by

$$\sum_{i=1}^I K_i = \sum_{i=1}^I b_i N_i \leq K, \quad (2)$$

$$\sum_{i=1}^I N_i \leq N. \quad (3)$$

In what follows, we focus on the case where that both constraints are binding. (Later we obtain a condition to ensure that the inequalities holds with equality.)

2.2 Allocation

In Houtakker (1955-1956), the allocation of labor is determined by the profit maximization problem of each firm. In this paper, we follow a pioneering paper of Yoshikawa (2013), and determine the allocation of labor,

$$\mathbf{N} = (N_1, N_2, \dots, N_I) \in \mathbb{R}^{I+}$$

by the maximum likelihood method. Here we assume that each worker is information-constrained and does not know which is the most efficient firms. Let $P_{\mathbf{N}}$ be the probability of getting the allocation \mathbf{N} . Yoshikawa (2013) finds that it is represented as

$$P_{\mathbf{N}} = \frac{1}{I^N} \frac{N!}{\sum_{i=1}^I n_i!}.$$

Yoshikawa (2013) then uses Stirling formula

$$\ln x! \simeq x(\ln x - 1).$$

for large x , and he simplifies the probability as

$$\ln P_{\mathbf{N}} = - \sum_{i=1}^I n_i \ln n_i + \text{constant}. \quad (4)$$

where $n_i = N_i/N$. By definition, the relative labor supply satisfies $\sum_{i=1}^I n_i = 1$.

If we let $k = K/N$ denote the capital-labor ratio, the resource constraints (2) and (3) can be simplified as the constraints on the labor supply n_i :

$$\sum_{i=1}^I b_i n_i \leq k, \quad (5)$$

$$\sum_{i=1}^I n_i \leq 1. \quad (6)$$

The term $b_i n_i$ represents per capita capital. We then follow Yoshikawa (2010) and suppose that the allocation of labor maximizes the entropy

$$S = - \sum_{i=1}^I n_i \ln n_i$$

subject to the constraints (5) and (6). The function $-x \ln x$ is strictly concave function of x as long as x is less than one. The Lagrangian of the problem is

$$L = - \sum_{i=1}^I n_i \ln n_i + \lambda \left(1 - \sum_{i=1}^I n_i\right) + \mu \left(k - \sum_{i=1}^I b_i n_i\right), \quad (7)$$

where $\lambda \geq 0$ and $\mu \geq 0$ are the Lagrange multipliers. As long as $n_i < 1$, the function is strictly concave. The FOCs on n_i are given by

$$\frac{\partial L}{\partial n_i} = 1 + \ln n_i + \lambda + \mu b_i = 0. \quad (8)$$

Therefore $n_i = 1/p \exp(-\mu b_i)$ where $p = \exp(1 + \lambda)$. Substituting $n_i = p^{-1} \exp(-\mu b_i)$ into the Lagrangian yields

$$\lambda \left[p - \sum_{i=1}^I \exp(-\mu b_i) \right] = 0, \quad (9)$$

$$\mu \left[pk - \sum_{i=1}^I b_i \exp(-\mu b_i) \right] = 0. \quad (10)$$

We now obtain a condition that the two constraints are binding and the multipliers are strictly positive. If they are, the multiplier μ solves

$$p = \sum_{i=1}^I \exp(-\mu b_i) = \frac{1}{k} \sum_{i=1}^I b_i \exp(-\mu b_i)$$

Now we follow Yoshikawa (2013) and suppose that the relative productivity parameter satisfies

$$b_i = bi,$$

for some $b > 0$ and that $I > 0$ is sufficiently large. The following lemma describes the condition that the two constraints are binding.

Lemma 1 *Suppose that $b_i = bi$. If the following inequalities hold, then (5) and (6) are binding, and the multipliers are strictly positive.*

$$\frac{I+1}{2}b > k, \tag{11}$$

$$\sqrt{\frac{ek}{b}} \left[\left(1 - \sqrt{\frac{b}{ek}}\right) - \left(1 - \sqrt{\frac{b}{ek}}\right)^{I+1} \right] > 1. \tag{12}$$

The two inequalities hold if $k > \frac{4b}{e}$ and I is sufficiently large.

Proof. See the Appendix. ■

As $n_i = p^{-1} \exp(-\mu b_i)$ and $p = \sum_{i=1}^I \exp(-\mu b_i)$, we have

$$n_i = \frac{N_i}{N} = \frac{\exp(-\mu b_i)}{\sum_{i=1}^I \exp(-\mu b_i)}. \tag{13}$$

This equation implies that the relative allocation of labor is inversely related to the labor efficiency parameter b_i .

3 Production function

In this section, we characterize the production function.

3.1 Per-capital production function

When we maximize the entropy subject to the resource constraint, the labor allocation also depends on the aggregate capital and labor. Thus the total output $Y = \sum y_i$ also

depends on K and N . Now let us define the aggregate production function as $F(K, N)$. We have the following proposition on the shape of the aggregate production function.

Proposition 1 *The aggregate production function has a constant returns to scale.*

Proof. The function satisfies

$$F(K, N) = N \sum_{i=1}^I A_i n_i(k),$$

where $n_i(k)$ solve the maximization problem above. Since $k = K/N$, if we let $f(k) = \sum_{i=1}^I A_i n_i(k)$, $F(K, N) = Nf(K/N)$. ■

If we let $x = \exp(-\mu b) \in (0, 1)$, we can express the capital-labor ratio k as an increasing function of x :

$$k = \frac{b \sum_{i=1}^I i \exp(-\mu b i)}{\sum_{i=1}^I \exp(-\mu b i)} = \frac{b \sum_{i=1}^I i x^i}{\sum_{i=1}^I x^i} \simeq b \frac{1}{1-x}, \quad (14)$$

or equivalently $x = 1 - b/k$. This equality shows that capital labor ratio must be less than b . The aggregate production function is approximately equal to

$$\sum_{i=1}^I y_i = N \sum_{i=1}^I A_i b_i n_i \simeq N \frac{b \sum_{i=1}^I A_i i x^{i-1}}{\sum_{i=1}^I x^{i-1}}. \quad (15)$$

Thus the per-capita production function is

$$f(k) = \frac{F(K, N)}{N} = \frac{1-x}{1-x^I} \sum_{i=1}^I A_i i x^{i-1}. \quad (16)$$

Now suppose that $I = +\infty$. In this case,

$$f(k) = (1-x) \sum_{i=1}^{\infty} a_i x^{i-1}. \quad (17)$$

with $a_i = iA_i$.

3.2 Cobb-Douglas

In this section we show that for some parameter values, the aggregate production becomes Cobb-Douglas. Thus

$$f(k) = \frac{1}{k} \sum_{i=1}^I A_i i \left(1 - \frac{b}{k}\right)^{i-1} \quad (18)$$

It is well-known that we can expand the function $(1-x)^{-s}$ as

$$\frac{1}{(1-x)^s} = \sum_{i=1}^{\infty} \binom{s+i-2}{i-1} x^{i-1}, \quad (19)$$

where $s > 0$ and $\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$. Therefore if we let

$$A_i = \frac{1}{i} \binom{i-\alpha-1}{i-1} > 0,$$

the per-capita production function becomes the Cobb-Douglas production function.

$$f(k) = g(1-x)^{-\alpha} = gk^\alpha, \quad (20)$$

where $g = b^{-\alpha}$ is a positive constant. We finally have the following proposition.

Proposition 2 *If the productivity parameters are $A_i = \frac{1}{i} \binom{i-\alpha-1}{i-1}$ and $B_i = b \binom{i-\alpha-1}{i-1}$, the aggregate production is the Cobb-Douglas type.*

Proof. If $b_i = bi$ and $A_i = \frac{1}{i} \binom{i-\alpha-1}{i-1}$, then $B_i = b_i A_i = \binom{i-\alpha-1}{i-1}$. ■

Figure 1 shows the technological coefficients A_i and B_i as a function of i

3.3 CES

We next show that the aggregate production may be CES. The CES production function is now becoming popular in dynamic macroeconomic analysis.³ First note that for any

³Several papers including Krump et al. (2011) empirically show the advantage of CES production function over the Cobb-Douglas type. Klump and La Grandville (2000) show how to calibrate the coefficient of the production function.

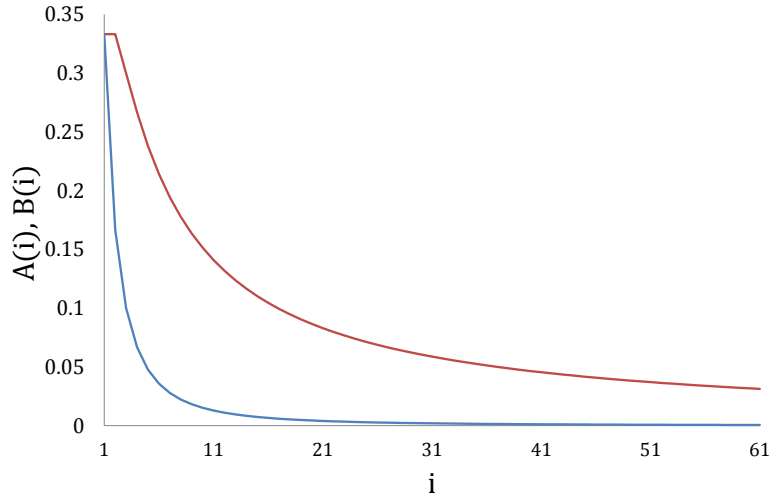


Figure 1: production coefficients

$x \in (0, 1)$,

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + \dots$$

Therefore if we set $a_i = 1/i$, then

$$f(k) = b(1-x) \frac{1+x}{1-x} = b(1+x) = b \left(1 - \frac{b}{2k} \right).$$

Then the aggregate production function is

$$F(K, N) = \left(\frac{a}{K} + \frac{b}{N} \right)^{-1}.$$

Thus the production function is of the CES type $A(K^{-\sigma} + L^{-\sigma})^{1/\sigma}$ where the parameter σ is equal to one. We finally have the following proposition.

Proposition 3 *If the technological coefficients are $a_i = 1/i$ and $b_i = bi$, the aggregate production is the CES type.*

3.4 Non-concave production function

So far we have focused on the Cobb-Douglas type production function. In this section, we show that for some parameter values, the aggregate production function can exhibit

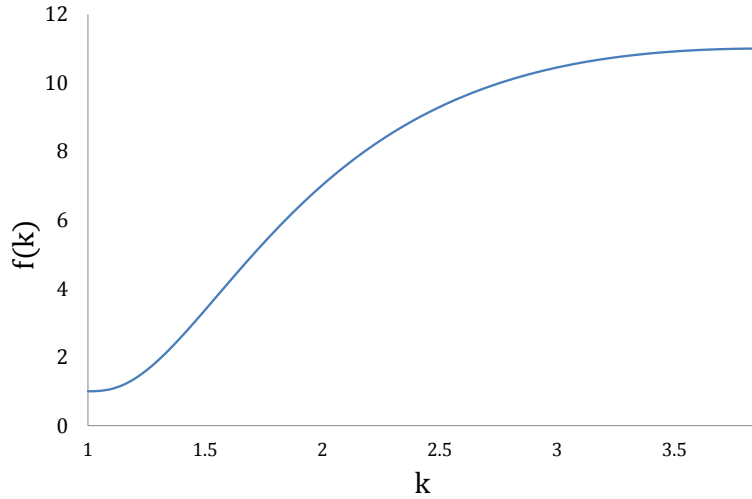


Figure 2: Convex-concave production function

increasing returns to scale.

First of all, if we suppose that the efficiency coefficient is defined as

$$A_i = (i - 1)(i - 2) \cdots (i - M),$$

where $M > 3$, then

$$\sum_{i=1}^{\infty} A_i x^{i-1} = \sum_{i=1}^{\infty} i(i-1)(i-2)x^{i-1} = \frac{d^M}{(dx)^M} \left[\sum_{i=1}^{\infty} x^i \right] = \kappa \frac{1}{(1-x)^M}$$

where κ is a positive constant. Thus the per-capital production function is determined by

$$f(k) = (1-x) \sum_{i=1}^{\infty} A_i x^{i-1} = Ak^M.$$

This implies that the production function exhibits increasing returns to scale.

Next we show that the model may imply the concave-convex production function. Such a production function is studied by Skiba (1978), Majumdar and Mitra (1982) and Dockner and Nishimura (2005). However, they do not provide a microfoundation for the non-concave production function.

Proposition 4 *For some parameter values, the production function is convex-concave.*

Proof. See the Appendix. ■

The figure shows the case of concave-convex production function in the case where $a_1 = 1$, $a_2 = 1.1$, $a_3 = 0$ and $a_4 = 100$

4 Conclusion

In this paper, we provide a microfoundation for the production function by using the concept of stochastic macro-equilibrium. We consider an economy with multiple firms, and each firm has Leontief technology. Instead of utility maximization, the allocations of labor and capital are determined by entropy maximization. We show that for some productivity distribution, the aggregate production becomes Cobb-Douglas type.

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Appendix

The Appendix provides a proof of Lemmas and Propositions.

A Proof of Lemma 1

We show that when we ignore either one of two inequalities and maximizes the entropy, the solution does not satisfy the remaining constraint.

First, if we ignore (6), then the solution satisfies $n_i = \exp(-1 - \mu bi)$. If we let $x = \exp(-\mu b)$, we have

$$n_i = \frac{x^i}{e}.$$

As the resource constraint is binding, $\sum_{i=1}^I b_i n_i = k$ and we have

$$\sum_{i=1}^{\infty} b_i n_i = \frac{b}{e} \sum_{i=1}^{\infty} i x^i = \frac{b}{e} \frac{x}{(1-x)^2} = k$$

As $x < 1$, this implies

$$\frac{1}{(1-x)^2} > \frac{x}{(1-x)^2} = \frac{ek}{b}$$

Or equivalently $x > 1 - \sqrt{\frac{b}{ek}}$. We have

$$\sum_{i=1}^I n_i = \frac{1}{e} \sum_{i=1}^I x^i = \frac{x - x^{I+1}}{1-x} > \sqrt{\frac{ek}{b}} \left\{ \left(1 - \sqrt{\frac{b}{ek}}\right) - \left(1 - \sqrt{\frac{b}{ek}}\right)^{I+1} \right\}.$$

Thus if (12) holds, the first inequality does not hold.

Second, if we ignore the second inequality, then the solution is $n_i = 1/I$. Thus if (11) holds,

$$\frac{1}{I} \sum_{i=1}^I b_i > k,$$

and then (5) does not hold. If this holds, we cannot ignore (5) when we maximize S .

First, if I is sufficiently large, (11) holds. Next, if $k > \frac{4b}{e}$,

$$\sqrt{\frac{ek}{b}} \left(1 - \sqrt{\frac{b}{ek}}\right) = \sqrt{\frac{ek}{b}} - 1 > 1.$$

Thus for sufficiently large I , both two inequalities holds.

B Proof of proposition 2

Assume that $a_1 = 1$, $a_2 = 2$, $a_3 = 5$ and $a_i = 0$ for $i > 4$. In this case,

$$f(k) = (1 - x)(1 + 2x + 5x^2) \equiv g(x)$$

with $x = 1 - 1/k$. We have

$$f'(k) = \frac{dx}{dk} g'(x) = (1 - x)^2 (3 + 10x - 5x^2) \equiv \hat{g}(x)$$

We now show that $f''(k) > 0$ if k is around 1. It is sufficient to show that $\hat{g}(x)$ is increasing function of x when $x = 1$. One can easily show that $\hat{g}(x)$ is a strictly increasing function of x and is also k when $k = 1$. This implies $f''(1) > 0$. and for some $\bar{x} > 1$, $\hat{g}(\bar{x}) = 0$. This implies $f'(\bar{k}) = 0$. Thus as long as $k \in (1, \bar{k})$, then the production function is convex-concave.

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