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Abstract

Income distribution (except for very high incomes) is widely understood to be well described by a log-normal distribution. Existing research has modeled an individual's income as an independent stochastic process to explain the observed log-normality. In this paper, I propose a stochastic model whereby an individual's income is not independent, but instead depends crucially on the incomes of other members of the economy. The model clarifies how the effects of economic interactions work. It turns out that they are favorable toward the wealthy as they enable them to keep their status with high probability. This represents a universal structure of economic systems.

Keywords: Income distribution; Fokker-Planck equation; Interaction effect

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1. Introduction

The income distribution among individuals is very important and is one of the main themes in economics. In fact, the study of income distribution has a long history. Pareto (1897) observed that the distribution of very high incomes obeys the universal power law. This power law behavior known as Pareto's law has been confirmed time and again for different economies, and also for different periods (Dragulescu and Yakovenko (2001), Montroll and Shlesinger (1983)). For explaining Pareto's law, various models have been proposed. For example, Champernowne (1953) models the evolution of incomes as a stochastic process. We must note, however, that Pareto's law is only applicable to the tail of income distribution, i.e. for extremely high incomes. By far the majority of incomes actually follows the log-normal distribution, not power laws. In this paper, I focus on this segment of income distribution, namely log-normal distribution.

Gibrat (1931) models income as an accumulation of random multiplicative shocks, which result in a log-normal distribution. It is now called Gibrat's law. In fact, the two parameters' log-normal distribution has been used to describe income distribution. It turns out that this functional form is very useful to estimate various measures of poverty and inequality (Pinkovskiy and Sala-i-Martin (2009)). It is reasonable to assume that the income distribution is the log-normal distribution with the Pareto upper tail, i.e. fat tail. This dichotomy can be justified by recent studies which show that the main income sources of low-middle income earners, who account for 97-99% of all the income earners, are wages and salaries and for high income earners whose distribution follows the power law, capital gains account for significant part of their incomes. It suggests that the income generating process of the rich based on asset accumulation requires a different theoretical model from the one for explaining the log-normal distribution. In fact, many models have been already proposed and succeed in reproducing the observed power law(Levy (2003), Gabaix (1999)).

In this paper, I focus on the distribution of incomes for the majority of people which is well fitted by a log-normal distribution, and explore the implications of the distribution. Actually, many attempts to reproduce the log-normal distribution have been done to date. For example, Sargan (1957) and Pestieau and Possen (1979) developed models with economic interpretations.

The main idea of the mechanism generating log-normal is quite simple. Let X_t be an individual income at time period t and assume that X_t is described by the following stochastic process,

$$X_t = A_t X_{t-1}$$

where A_t is an i.i.d. random shock which has a finite second moment. It implies that the growth rate X_t/X_{t-1} is an independent random variable, and that can be X_t is written as a product of random shocks, i.e.

$$X_t = A_t A_{t-1} \dots X_0$$

where X_0 is an initial value. Taking the logarithm of this equation, we have

$$\log X_t = \log A_t + \log A_{t-1} + \dots + \log X_0$$

When t is large, we would be able to apply the central limit theorem, and obtain the normal distribution of $\log X_t$. In recent studies, for example, Battistin et al. (2009), this mechanism plays a central role to generate the observed lognormal distribution.

Though this argument is straightforward, it is actually unsatisfactory. As pointed out by Kalecki (1945), the variance of $\log X_t$ increases indefinitely with time. In other words, as time goes by, inequality in income becomes greater in monotone. This is inconsistent with empirical data; distribution of income has a finite, fairly stable variance. The straight forward application of the central limit theorem, therefore, has a serious problem. One might think that one man's lifetime is finite so that the variance never diverges. However, in that case, the aggregate distribution of income is composed of distributions with different variances reflecting different ages of individuals, and, therefore, the central limit theorem does not hold true in the first place. As a result, we cannot expect that the income distribution is log-normal in such a case.

Furthermore, the proposed models to date implicitly assume that an individual's income process is independent of those of others. It is plainly unrealistic to assume that the evolution of an individual income is not affected by others'

incomes. We reasonably expect that a worker's income increases when his/her customers, employers or fellow people in his/her community become rich. Otherwise, workers would never migrate from poor countries to seek jobs in rich countries. Obviously "interaction effects", a kind of externality, are important. The purpose of the present paper is to investigate the interaction effects by way of stochastic methods.

The rest of the paper is organized as follows. In Section 2, I present the model and examine a stationary state of the income process. In Section 3, I summarize the implications of the model in Section 2 and show its relationship with inequality. Section 4 concludes the paper.

2. The Model

2.1. Income Process

Consider an economy populated by N workers. Let $X_t^{(n)}$ be the income of n th worker at time t . As mentioned in Introduction, the evolution of $X_t^{(n)}$ is assumed to be affected by others' incomes. It depends on two components, one the interaction effects which are a function of $X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}, \dots, X_t^{(N)}$, and the other a random component.

The differential equation of $X_t^{(n)}$, therefore, is

$$dX_t^{(n)} = F^{(n)}(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(N)}, t)dt + DdW_t^{(n)} \quad (1)$$

where W_t is Brownian motion representing random shocks and D is constant. Equation(1) is a starting point of our analysis. Before we proceed, it is useful to discuss the relation between stochastic process generating income distribution and optimization on the part of an individual economic agent. In the economic literature, the optimal choices of agents plays a central role in accounting for inequality. For example, Castañeda et al. (2003) model the labor decision explicitly and incorporate the life cycle profile of earnings and intergenerational transmission of earnings to account for the U.S. earnings and wealth inequality.

Some economists who emphasize the role of optimization of agents have criticized stochastic models on the ground that they do not incorporate individual optimizing behavior. Sahota (1978), for example, states as follows:

"...in Friedman's individual choice theory, chance and luck (including genetic luck) play their roles, but only subsidiary ones, while economic behavior (in the face of uncertainty) is the basic determinant of income inequalities. ... In the face of newly budding theories, the stochastic theory seems to be a "degenerate problem-shift.""

The stochastic theory is to be regarded as a complement to the optimization approach. In fact, the evolution of X_t does not contradict to the maximization of individuals. The point is that one can never observe stochastically changing economic decisions of N workers exactly. It lies simply beyond the boundary of our knowledge. Thus, we need "coarse graining". Note that we are not concerned with the income of a particular individual, but rather income distribution, a macroscopic phenomena, the details of the workers are simply irrelevant. All we do know is the information indispensable to the macroscopic behavior of the system. It enables us to analyze distribution of income without any dubious assumptions.

Let us go back to Equation(1). To proceed further, assume that F is common to all workers ($F^{(n)} = F$), and does not depend on t explicitly. Furthermore, F is assumed to be invariant with respect to the permutation of incomes except his/her own income $X_t^{(n)}$;

$$\begin{aligned} & F(\dots, X_t^{(i-1)}, X_t^{(i)}, X_t^{(i+1)}, \dots, X_t^{(j-1)}, X_t^{(j)}, X_t^{(j+1)}, \dots) \\ &= F(\dots, X_t^{(i-1)}, X_t^{(j)}, X_t^{(i+1)}, \dots, X_t^{(j-1)}, X_t^{(i)}, X_t^{(j+1)}, \dots) \end{aligned}$$

It follows that the interaction effects depend only on the income distribution, not incomes of particular persons. Finally, we take so that the total number of workers N goes to infinity. Under these conditions, Equation(1) may be written as

$$dX_t^{(n)} = F(X_t^{(n)}, g_t)dt + DdW_t^{(n)} \quad (2)$$

where g_t is the distribution of incomes at time period t . $dX_t^{(n)}$ depends on his/her own income $X_t^{(n)}$ and the income distribution g_t in the economy as a whole. Note, however, that the income distribution, g_t , is not given, but is the

consequence of the evolution of incomes $X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}, \dots$. This simultaneity is the difficulty of the interaction effects among workers. Unfortunately, it is intractable to solve Equation(2) in the general case. However, we can analyze the stationary case. Let g be the stationary state of g_t ; by definition, g_t does not depend on time t in the stationary state.

Let us denote $F(x, g)$ by $f(x)$. Equation(2) then becomes

$$dX_t^{(n)} = f(X_t^{(n)})dt + DdW_t^{(n)} \quad (3)$$

Because the empirical distributions of incomes are well approximated by the log-normal distribution and stable, the resulting stationary distribution of Equation(2) must be log-normal. Thus, the goal of our analysis is to find the function f for which the stationary distribution derived from equation(3) is log-normal.

2.2. Fokker-Planck Equation

The probability density of X_t , P , derived from equation(3) changes over time. Its evolution is described by the Fokker-Planck equation. Following Risken (1996) and Kubo et al. (1983-85), we first derive the Fokker-Planck equation and solve it in a stationary case.

We start with the Chapman-Kolmogorov equation

$$\frac{\partial}{\partial t}P(x, t) = - \int w(x, r)drP(x, t) + \int w(x - r, r)drP(x - r, t) \quad (4)$$

where $w(x, r)$ is the probability of transitioning from x to $x' \equiv x + r$. Equation(4) describes how P changes over time. The right hand side of equation(4) is simply the sum of "in" and "out" of probability mass. Namely, the first term of the right hand side represents fluxes out of x due to transitions from x to another point. The second term represents fluxes into x due to transitions from other points to x .

The integrand in Equation(4) can be expanded in Taylor series,

$$\begin{aligned} \frac{\partial}{\partial t}P(x, t) &= - \int w(x, r)drP(x, t) + \int dr \sum_{n=0}^{\infty} \frac{1}{n!} r^n (-\frac{\partial}{\partial x})^n w(x, r) P(x, t) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial x})^n \int dr r^n w(x, r) P(x, t) \end{aligned} \quad (5)$$

Let $M_n(x)$ denote the n th moment, which is;

$$M_n(x) = \int r^n w(x, r) dr$$

Equation(4) now becomes

$$\frac{\partial}{\partial t}P(x, t) = \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial x})^n M_n(x) P(x, t) \quad (6)$$

Equation(6) is called the Krammers-Moyal expansion. Using Ito's lemma, we can calculate $M_n(x)$ explicitly in our model. Equation(3) implies

$$\begin{aligned} M_1(x) &= f(x) \\ M_2(x) &= D^2 \\ M_n(x) &= 0(n \geq 3) \end{aligned}$$

Therefore, we finally obtain the Fokker-Planck equation,

$$\begin{aligned} \frac{\partial P(x, t)}{\partial t} &= L_{FP}P(x, t) \\ L_{FP}(x) &= -\frac{\partial}{\partial x}f(x) + \frac{D^2}{2}\frac{\partial^2}{\partial x^2} \end{aligned} \quad (7)$$

2.3. The Stationary State

Instead of solving Equation(7) directly, it is transformed into

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial S(x, t)}{\partial x} \quad (8)$$

$$S(x, t) \equiv f(x)P(x, t) - \frac{D^2}{2} \frac{\partial}{\partial x} P(x, t) \quad (9)$$

This equation is called a continuity equation in physics. Continuity equations are the local form of Conservation laws. In our case, the conserved quantity is the probability because the total of the probability must be 1. A conserved quantity can only move from place to place. Suppose that $P(x, t)\Delta x$ is the amount of the quantity in the region between x and $x + \Delta x$ at time t . Let $S(x, t)$ denote a rate of a flow of the quantity through the boundary of x . The quantity flux into the region must be (quantity flow in) - (quantity flow out), which is

$$(S(x, t) - S(x + \Delta x, t))\Delta t \quad (10)$$

Therefore, the conservation of the probability in the region may be expressed as

$$\begin{aligned} (P(x, t + \Delta t) - P(x, t))\Delta x &= -(S(x + \Delta x, t) - S(x, t))\Delta t \\ \frac{(P(x, t + \Delta t) - P(x, t))}{\Delta t} &= \frac{-(S(x + \Delta x, t) - S(x, t))}{\Delta x} \end{aligned}$$

This is nothing but Equation(8). Thus, S is called "probability current". It follows that the probability current must be zero in a stationary state. We thus have

$$f(x)P_{st}(x) = \frac{D^2}{2} \frac{\partial}{\partial x} P_{st}(x) \quad (11)$$

Imagine that P_{st} can be written as $N \exp Q$ (N is constant). Substituting it into Equation(11), we obtain

$$f(x) = \frac{D^2}{2} \frac{dQ(x)}{dx} \quad (12)$$

As mentioned in Introduction, the distribution of incomes is well described by log-normal distribution;

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(\frac{-(\ln x - \mu)^2}{2\sigma^2}\right) \quad (13)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp Q, \quad Q = \frac{-(\ln x - \mu)^2}{2\sigma^2} - \log x \quad (14)$$

where μ is location parameter and σ is scale parameter. Substituting the log-normal distribution of Equation(14) into Equation(12), we finally obtain the following proposition.

Proposition

With the assumptions above, the function f in equation (3) consistent with empirically observed stationary log-normal distribution of X_t is as follows:

$$f(x) = -\frac{D^2}{2} \left(\frac{1}{x} - \frac{\mu}{\sigma^2 x} + \frac{\log(x)}{\sigma^2 x} \right) \quad (15)$$

In this way, we can derive the interaction effect $f(x)$. It should be recognized that Equation(15) is not based on assumptions of preferences or abilities, but it derives from the fact that the income distribution is close to the log-normal distribution. Therefore, the condition(15) must hold in any countries and it shows an universal structure of the economic system.

3. Implications

Figure 1 shows the graph of Equation(15). A positive(negative) value of $f(x)$ means that the expected value of dX_t is positive(negative). Because the interaction effects f must compensate for the dissipative force, it must hold that $f(x) > (<)0$ at lower(larger) x . A large positive value implies that a worker's income in the next period is more likely to be higher than the current level and vice versa. Therefore it is reasonable to be a large positive value at lower x because a poor worker may have plenty of opportunities to get a much better job, given that his/her neighbors are rich. In the right of the graph, i.e. for incomes of middle class, however, a curious property emerges. Though f is negative, it has its minimum value and then slightly increases as x becomes large. Since the negative value of $f(x)$ means a downward pressure on X_t , it shows that a relatively rich worker has a tendency to decrease his/her income if others' incomes are much smaller than his/her own income, which is consistent with our intuition. According to this intuition, we might expect that f is just downward sloping. But there exists the minimum value. It makes the interpretation more puzzling. It implies that the downward pressure on X_t is not exerted strongly compared to the upward pressure and slightly weaken as X_t becomes large and goes beyond the critical value where $f(x)$ takes its minimum value. Interestingly enough, it implies that once workers happens to be rich, the structure of the economy is not willing to bring them back. From a political point of view, if the degree of inequality is defined in terms of the permanent income instead of current income, it indicates the need for higher tax rates on the well-off.

Let us take a close look at the function $f(x)$. we can calculate important values of x at which $f(x)$ takes the characteristic value. Setting $f(x)$ in Equation(15) to be zero and elementary algebra shows that $f(x)$ takes zero at

$$x_{mode} = e^{\mu - \sigma^2}$$

Recall x_{mode} is nothing but the mode of the log-normal distribution. At x_{mode} , no pressure on X_t works. Differentiating $f(x)$ with respect to x , it can be shown that $f(x)$ takes its minimum value at

$$x_{min} = e^{1+\mu - \sigma^2}$$

Thus, it is recognized that $x_{min} = 2.71... \times x_{mode}$ and the existence of the bottom has a significant meaning. Once X_t reaches the certain level, x_{min} , the surroundings where workers earn incomes improve. The mechanism underlying the income distribution, therefore, turns out to be somewhat "generous" to the rich.

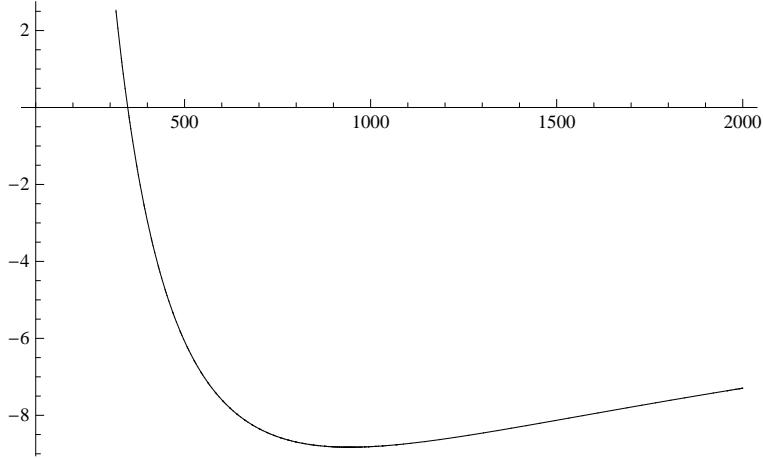


Figure 1: the graph of the function $f(x)$

The present analysis has other implications for poverty and inequality. Since the distribution is described to be log-normal, the degree of inequality can be measured by its scale parameter σ . In fact, Gini coefficient of the lognormal distribution is written as

$$\text{Gini coefficient} = 2\Phi(\sigma/\sqrt{2}) - 1$$

where $\Phi()$ is the cumulative distribution function of the standard normal distribution. It is one of the advantages of log-normal distributions to describe income distributions. This is why log-normal distributions are widely used.

For simplicity, assume that the location parameter μ is fixed. A larger value of σ definitely implies a greater inequality, so we examine the relationship between σ and the interaction effect $f(x)$. Differentiating $f(x)$ with respect to σ^2 , we thus have

$$df/d\sigma^2 = -\frac{D^2}{2x\sigma^2}(\mu - \log x) \quad (16)$$

Hence,

$$\begin{aligned} df/d\sigma^2 &< 0 \quad \text{if } x < e^\mu \\ df/d\sigma^2 &> 0 \quad \text{if } x > e^\mu \end{aligned} \quad (17)$$

Here, it should be noted that e^μ is the median of the log-normal distribution. What Equation(17) means is quite straightforward. Workers whose incomes are in the lower half suffer from an increase in σ , i.e. the Gini coefficient because the interaction effect $f(x)$ decrease, in other words, their prospects of higher incomes become gloomy. This simple relationship is consistent with our interpretation about the function $f(x)$. It contains all information about how workers' incomes are affected by the economy and their prospects of the future.

4. Concluding Remarks

It is well known that the income distribution is approximately log-normal except for the super rich. In this paper, I have explored implications of this empirical fact using the stochastic approach. Because an economy consists of a large number of individuals and very complicated due to interactions, it is necessary to abstract only the information indispensable for explaining macroscopic phenomena and to throw away unimportant details. Thus, our model is based on the stochastic approach by which the evolution of the income process is described instead of the formulation of their utility maximization. This enables us to derive general conclusions about the income distribution.

We have seen that the interaction effect has characteristic features. The function $f(x)$ representing the effect is not just a downward sloping curve but takes its minimum value. It implies that once an individual becomes rich by a lucky chance, he/she is likely to stay there for a while. Recall that this conclusion depends only on the empirical fact that income distributions are well fitted by log-normal distributions. Therefore, it shows an universal structure of economic systems. For this reason, this finding has robust and important implications for the theory of the income distribution as well as the analysis of poverty and inequality.

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