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Abstract

This paper investigates the effects of taxation on the distributions of income and wealth and on the welfare of heterogeneous households. I first demonstrate that the tails of income and wealth distributions converge to a Pareto distribution in a Bewley model in which households bear idiosyncratic investment shocks. This result extends the previous analysis in Nirei (2009). Thereafter, a non-distortionary tax and flat-rate taxes on capital income and consumption are introduced, and their impacts on aggregate wealth, the inequality index, households' welfare, and transition paths are quantitatively investigated. When the tax rate is set to generate the same GDP-government expenditure ratio, the model with capital tax generates smaller aggregate wealth and a smaller inequality index than the case with consumption tax or non-distortionary tax.

Keywords: investment risk, Pareto distribution, Gini coefficient, and capital tax.
JEL classification: E20

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1 Introduction

Mechanisms underlying the distributions of income and wealth and the effects of tax on the distributions have attracted renewed interests. Empirical studies by Feenberg and Poterba [11] and Piketty and Saez [20] investigated the time series of top income share in the U.S. by using tax returns data, and suggested that the tax reform play an important role in determining inequality among households. Guvenen, Kuruscu, and Ozkan [13] investigated the impact of tax on wage inequality. Cagetti and De Nardi [6] studied the effect of estate tax on wealth inequality.

Along with the empirical studies, there has been a rapid development on the theoretical framework that accommodates the household heterogeneity and thereby income and wealth inequality. For example, Numerical studies by Huggett [14] and Castañeda, Díaz-Giménez, and Ríos-Rull [9] successfully capture the overall shape of earnings and wealth distributions in a dynamic general equilibrium model. Quadrini [21] and Cagetti and De Nardi [6] match the wealth distribution in a rich model that explicitly formulates entrepreneurs.

The present paper complements theirs by concentrating on the the tail distributions of income and wealth, which covers a small fraction of population but has a large impact on the inequality measures because of the tail's large shares in income and wealth. This motivation is shared with Benhabib and Bisin [4] who investigated the wealth distribution in an overlapping generations model.

In this paper, I extend my previous study [19] which developed a simple analytical theory of income distribution in the Solow and Ramsey growth models. The previous paper incorporated an idiosyncratic asset return shock, and showed that the Solow model generates a stationary Pareto distribution for the detrended household income at the balanced growth path. The paper analytically derived the Pareto exponent from fundamental parameters. The intuition for the Pareto distribution was analogous to Gabaix [12]: the wealth accumulation follows a multiplicative process with a reflective lower bound in which wage income serves as the reflective bound. The Pareto exponent was determined by the balance of the savings from wage income that pushes up the bottom of the wealth distribution and the inequalizing diffusion effect that is attributed to risk taking behaviors of the top wealth holders.

In the present paper, I show that the similar analysis applies to a Bewley model with idiosyncratic investment shocks and borrowing constraints. Precautionary savings due to the borrowing constraints and uninsurable shocks serve as the lower bound of the wealth accumulation process. Using numerical computation, I show that the model generates the Pareto distribution, and that the Pareto exponent is decreased by a capital tax. Then, I introduce taxes on labor income and consumption as well, and compare various taxation scheme that generate the same government expenditure-GDP ratio. I find that the capital tax generates a more egalitarian distribution than consumption tax, whereas the effects of consumption tax are similar to that of non-distortionary tax.

The rest of the paper is organized as follows. In Section 2, I present a variation of the Bewley model with idiosyncratic investment risks and borrowing constraints and show the stationary distributions of income and wealth. I compare the result to the same model without borrowing constraints, which generates a non-stationary log-normal process for individual wealth. In Section 3, I introduce taxation other than capital tax, compute the stationary equilibrium and compare the results across taxation schemes. Section 4 concludes.

2 Quantitative demonstration of the Pareto distribution

2.1 Bewley model with investment shocks and borrowing constraints

In this section, a Bewley model with idiosyncratic investment shocks and borrowing constraints is presented. The model is a variation of Covas [10] with a pension program and exogenous growth. Consider a continuum of infinitely living households $i \in [0, 1]$. Each household inelastically supplies one unit of labor. Household i is also endowed with initial capital $k_{i,0}$ and a "backyard" production technology that is specified by a Cobb-Douglas production function:

$$y_{i,t} = k_{i,t}^{\alpha}(a_{i,t}l_{i,t})^{1-\alpha}, \tag{1}$$

where $l_{i,t}$ is the labor employed by i and $k_{i,t}$ is the capital owned by i. The labor-augmenting productivity $a_{i,t}$ has a common trend $\gamma > 1$:

$$a_{i,t} = \gamma^t \epsilon_{i,t}, \tag{2}$$

where $\epsilon_{i,t}$ is a temporary productivity shock. I assume that $\epsilon_{i,t}$ follows a two-state Markov process. The households do not have the means to insure against the productivity shock $\epsilon_{i,t}$ except for their own savings.

Each household lineage is discontinued with a small probability μ in each period. At this event, a new household is formed at the same index i with no wealth. Following the perpetual youth model [5], I assume that the households participate in a pension program. The households contract all of the non-human wealth to be confiscated by the pension program at the discontinuation of the lineage, and they receive in return the premium in each period of continued lineage at rate p per unit of non-human wealth they own. The pension program is a pure redistribution system, and must satisfy the zero-profit condition: $(1-\mu)p = \mu$. Thus, the pension premium rate is determined as

$$p = \mu/(1-\mu). \tag{3}$$

The households can hold assets in the form of physical capital $k_{i,t}$ and bonds $b_{i,t}$. The bond bears a risk-free interest R_t. The households can engage in lending and borrowing through bonds. There is a limit on the borrowing. I assume that the households can borrow only up to a fraction $\underline{\theta}$ of their total wealth $k_{i,t}+b_{i,t}$. Capital income is taxed at flat-rate τ_k. I assume that the tax proceeds are spent on the unproductive government purchase of goods. In the following notation, consumption $c_{i,t}$, assets $k_{i,t}$ and $b_{i,t}$, and real wage w_t are detrended at the rate of technical progress γ.

In each period, a household maximizes its profit from physical capital $\pi_{i,t} = y_{i,t} - w_t l_{i,t}$ subject to the production function (1). Labor can be hired at wage w_t, and the labor contract is struck after the realization of the technology shock $a_{i,t}$. The first-order condition of profit maximization yields the labor demand function:

$$l_{i,t} = (1-\alpha)^{1/\alpha} a_{i,t}^{(1-\alpha)/\alpha} w_t^{-1/\alpha} k_{i,t}. \tag{4}$$

Plugging into the production function, I obtain the goods supply function:

$$y_{i,t} = (1-\alpha)^{1/\alpha-1} a_{i,t}^{(1-\alpha)/\alpha} w_t^{1-1/\alpha} k_{i,t}. \tag{5}$$

Then, I obtain $\pi_{i,t} = \alpha y_{i,t}$ and $w_t l_{i,t} = (1-\alpha) y_{i,t}$. At the optimal labor hiring $l_{i,t}$, the return to capital is defined as

$$r_{i,t} \equiv \pi_{i,t}/k_{i,t} + 1 - \delta = \alpha(1-\alpha)^{(1-\alpha)/\alpha}(a_{i,t}/w_t)^{(1-\alpha)/\alpha} + 1 - \delta. \tag{6}$$

Given the optimal operation of physical capital in each period, the households solve the following dynamic programming:

$$V(W_i, \epsilon_i) = \max_{c_i, k_i', b_i', W_i'} \frac{c_i^{1-\sigma}}{1-\sigma} + \tilde{\beta} E\left(V(W_i', \epsilon_i') \mid \epsilon_i\right) \tag{7}$$

subject to

$$W_i = c_i + \gamma(k_i' + b_i'), \tag{8}$$

$$W_i = (1+p)(r_i k_i + R b_i - \tau_k((r_i - 1)k_i + (R-1)b_i)) + w, \tag{9}$$

$$b_i'/(k_i' + b_i') > -\underline{\theta}, \tag{10}$$

$$k_i', k_i' + b_i' > 0, \tag{11}$$

where $\tilde{\beta}$ is a modified discount factor $\tilde{\beta} \equiv \beta\gamma^{1-\sigma}(1-\mu)$. $W_{i,t}$ denotes the total resources available to i at t (the cash-at-hand). The control variables k_i and b_i can be equivalently expressed by i's total financial assets $x_i \equiv k_i + b_i$ and portfolio $\theta_i \equiv b_i/x_i$. Thus, the dynamic programming solves the optimal savings problem for x_i and the portfolio choice for θ_i.

An equilibrium is defined as a value function V, policy functions (x, θ), price functions (w, R), a joint distribution function Λ, and the law of motion Γ for Λ such that $V(W_i, \epsilon_i; \Lambda)$, $x(W_i, \epsilon_i; \Lambda)$, and $\theta(W_i, \epsilon_i; \Lambda)$ solve the household's dynamic programming, such that prices $w(\Lambda)$ and $R(\Lambda)$ clear the markets for labor $\int_0^1 l_{i,t} di = 1$, goods, and bonds $\int_0^1 b_{i,t} di = 0$, and such that the policy functions and the exogenous Markov process of ϵ_i constitutes Γ that maps the joint distribution of $\Lambda(W_i, \epsilon_i)$ to that in the next period. A stationary equilibrium is defined as a particular equilibrium in which Λ is a fixed point of Γ.

With an autocorrelation in productivity $\epsilon_{i,t}$, the households with high productivity will invest in capital with a high rate of borrowing, while the households with low productivity will shift their assets to risk-free bonds. Thus, as Covas [10] argued, this model captures an economy in which a fraction of the households choose to become entrepreneurs while the other households rely on wage and the returns from safe assets as their main income source. Since the entrepreneurs bear the investment shocks that generate the fat tail of wealth distribution in this model, I will observe that the tail population largely consists of current and past entrepreneurs. As a model of entrepreneurship, the model presented here is not as rich as the one with occupational choice (see Quadrini [22] for a survey). Nonetheless, in this model, the entrepreneurs (households with high productivity) do not diversify much of their investment risks while workers choose to bear substantially smaller risks.

2.2 Borrowing constraints and Pareto distribution

I numerically solve for a stationary equilibrium of this economy. This model features a multiplicative investment shock instead of an endowment shock that enters the wealth accumulation process additively as in the benchmark model of Aiyagari [1]. Thus, the stationary wealth distribution has a fat tail unlike in the Aiyagari economy. This means that the simulation of wealth accumulation process suffers a slow convergence of aggregate wealth, due to the fact that the aggregated noise in a fat tail does not decrease as quickly as the simulated population increases. However, if the wealth state is discretized in logarithmic space, the stationary distribution can be computed well simply by iterating the multiplication of the Markov transition matrix. Intuitively, this is because the logarithm of a multiplicative process falls back to an additive process. To manage the computation of portfolio choice, I follow a two-step approach similar to Barillas and Fernández-Villaverde [3], who solve the neoclassical growth model with labor choice using the endogenous gridpoints method by Carroll [7] for the savings problem and the standard value function iteration for the labor choice. Further details in the computation are deferred to Appendix A.

I compute the stationary equilibrium distributions of after-tax income and wealth under tax rates $\tau_k = 0.5$ and $\tau_k = 0.28$. The tax rates are chosen to emulate the change in the top marginal income tax rate in 1986 in the U.S., which is discussed in Section 3.3). The wealth corresponds to $W_{i,t}$ and the after-tax income is $w_t + (1-\tau_k)((r_{i,t} - 1)k_{i,t} + (R_t - 1)b_{i,t})$. The transition matrix Π for the investment shock ϵ_i is set by $\pi_{11} = 0.9723$ and $\pi_{22} = 0.8$, for which the stationary fraction of households with high productivity is 12% and the average exit rate from the high productivity group is 20%. These numbers correspond to the fraction and exit rate of the entrepreneurs in the U.S. data (Kitao [15]). The states of the Markov process $\epsilon_{i,t}$ are set at $\{0.95, 1.05\}$. The states are chosen so that the stationary wealth distribution in the model with tax rate $\tau_k = 0.5$ has a Pareto exponent 2.2, which roughly matches with the U.S. level right before the tax cut in 1986. The lineage discontinuation rate μ is set at 1%. The borrowing constraint $\underline{\theta}$ is set at 0.5. The parameters on technology and preferences are set at standard values: $\alpha = 0.36$, $\delta = 0.1$, $\sigma = 3$, $\beta = 0.96$, and $\gamma = 1.02$.

Figure 1 plots the distributions of income and wealth at stationary equilibrium for $\tau_k = 0.5$ and 0.28. Pareto tails are clearly observed in both income and wealth distributions. The Pareto exponents for income and wealth coincide, since high income earners earn most of the income from capital in this model. The Pareto tail is significantly flatter in the low tax regime than in the high tax regime: 2.22 for $\tau_k = 0.5$ and 1.96 for $\tau_k = 0.28$. The simulations show that the Pareto distributions are obtained in this model even when $\mu = 0$, i.e., the households live indefinitely. I also compute a transition path from the stationary distribution under $\tau_k = 0.5$ to that under $\tau_k = 0.28$. In Figure 1, the plot shown by dots shows the transitional distribution 10 years after

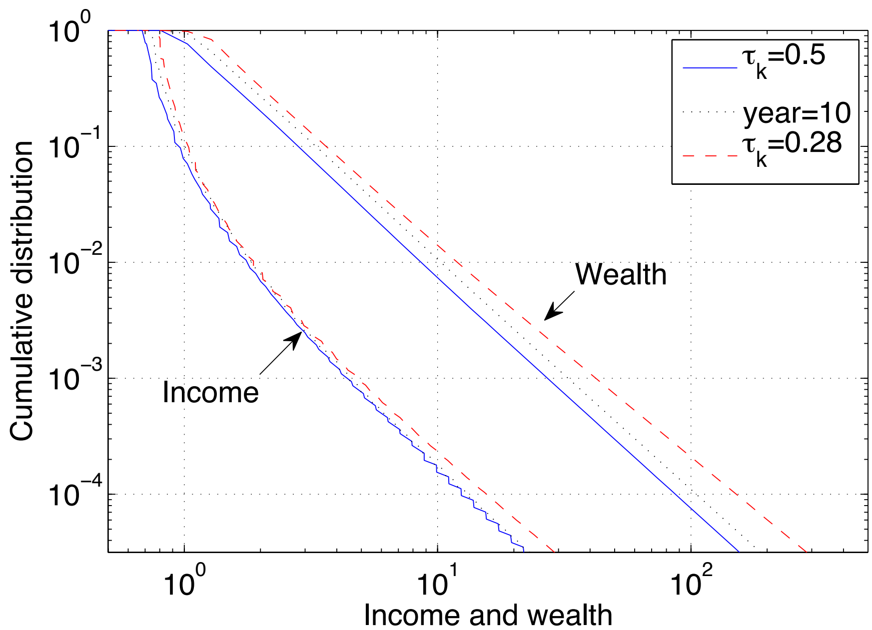

Figure 1: Stationary distributions of income (left) and wealth (right) for capital tax rates $\tau_k = 0.5$ and $\tau_k = 0.28$

the tax cut is introduced.

2.3 Case without borrowing constraints

In this section, I analytically investigate the case without borrowing constraints. I show that the wealth follows a log-normal process if there is no limit on borrowing, no tax, $\mu = 0$, and if $\epsilon_{i,t}$ is independent across t. This log-normal process implies that no stationary distribution of relative wealth exists. When $\mu > 0$, the stationary distribution of wealth has a Pareto tail, and the Pareto exponent is analytically derived.

Since this model features a utility exhibiting constant relative risk aversion, the savings rate and portfolio decisions are independent of wealth levels if there is no limit on borrowing (Samuelson [23], Merton [18]). Here I draw on Angeletos' [2] analysis of a Bewley model with idiosyncratic investment risks. I set the capital tax to be zero: $\tau_k = 0$. Let H_t denote the human wealth, defined as the expected discounted present value of future wage income stream:

$$H_t \equiv \sum_{\tau=t}^{\infty} w_\tau (1-\mu)^{\tau-t} \prod_{s=t+1}^{\tau} R_s^{-1}. \tag{12}$$

The evolution of the human wealth satisfies $H_t = w_t + (1-\mu) R_{t+1}^{-1} H_{t+1}$. We define a household's total wealth as

$$W_{i,t} = (1+p)(r_{i,t} k_{i,t} + R_t b_{i,t}) + H_t. \tag{13}$$

Note that the wage w_t, human wealth H_t, and total wealth $W_{i,t}$ are detrended by the growth factor γ.

Consider a balanced growth path at which R_t, w_t, and H_t are constant over time. Then, the household's problem is formulated in a recursive form:

$$V(W_i) = \max_{c_i, k_i', b_i', W_i'} \frac{c_i^{1-\sigma}}{1-\sigma} + \tilde{\beta} E\left(V(W_i')\right) \tag{14}$$

subject to

$$W_i = c_i + \gamma(k_i' + b_i') + (1-\mu)R^{-1}H, \tag{15}$$

$$W_i = (1+p)(r_i k_i + R b_i) + H. \tag{16}$$

This dynamic programming allows the following solution with constants s and ϕ (see Appendix B for derivation):

$$c = (1-s)W, \tag{17}$$

$$k' = \phi s W, \tag{18}$$

$$b' = (1-\phi)sW - (1-\mu)R^{-1}H. \tag{19}$$

By substituting the policy functions in the definition of wealth (13), and by noting that $(1-\mu)(1+p) = 1$ holds from the zero-profit condition for the pension program (3), I obtain the equation of motion for the detrended individual total wealth:

$$W_{i,t+1} = \begin{cases} \tilde{g}_{i,t+1} W_{i,t} & \text{with prob. } 1-\mu \\ \bar{H} & \text{with prob. } \mu, \end{cases} \tag{20}$$

where the growth rate is defined as

$$\hat{g}_i' \equiv \frac{(\phi r_i' + (1-\phi)R)s}{1-\mu}. \tag{21}$$

Thus, at the balanced growth path, the household wealth evolves multiplicatively according to (20) as long as the household lineage is continued. When the lineage is discontinued, a new household with initial wealth $W_i = H$ replaces the old one. Therefore, the individual wealth W_i follows a log-normal process with random reset events where H is the resetting point. Using the result of Manrubia and Zanette [17], the Pareto exponent of the wealth distribution is determined as follows.

Proposition 1 *A household's detrended total wealth $W_{i,t}$ has a stationary Pareto distribution with exponent λ that is determined by*

$$(1-\mu)\mathrm{E}(\hat{g}_{i,t}^{\lambda}) = 1 \tag{22}$$

if $\mu > 0$. If $\mu = 0$, $W_{i,t}$ has no stationary distribution and asymptotically follows a log-normal distribution with diverging variance.

Proof: See Appendix C.

As seen in (22), the Pareto exponent is large when μ is large. If there is no discontinuation event (i.e., $\mu = 0$), then the individual wealth follows a log-normal process as in [2]. In that case, the relative wealth $W_{i,t}/\int W_{j,t}dj$ does not have a stationary distribution. Eventually, a vanishingly small fraction of individuals possesses almost all the wealth. This is not consistent with the empirical observations.

The log-normal process, and thus the diverging variance, do not occur in the model with borrowing constraints even if $\mu = 0$. The difference occurs from the fact that the consumption function is nonlinear in wealth when there is a borrowing constraint whereas it is linear without borrowing constraints. The linear consumption function arises in a quite narrow specification of the Ramsey model as Carroll and Kimball [8] argue. For example, a concave consumption function with respect to wealth arises when the labor income is uncertain or when the household's borrowing is constrained. This implies that the log-normal process of wealth is a special case whereas the Pareto distribution characterizes a wide class of specifications.

In sum, this section demonstrated that the Pareto distribution of income and wealth naturally arises as a stationary distribution in the Bewley model with idiosyncratic investment shocks and borrowing constraints. I show that the precautionary savings that arise from the borrowing constraints and the discontinuation of household lineage generate the stationary Pareto distribution, whereas without these two factors the model will generate a non-stationary log-normal development of wealth distribution. This section also showed that the capital taxation reduces the stationary Pareto exponent. A complete analysis of the effect of the savings on the Pareto exponent is provided in a separate paper (Nirei [19]).

3 Effects of various taxation

3.1 Modified model with consumption, capital, and labor taxes

In the last section, I conducted an experiment of a tax cut in an environment where the government expenditure is passively determined by the tax proceeds. In this section, I fix the government expenditure as a fraction of GDP, and the tax rate is determined so that the tax proceeds in a stationary equilibrium meets the required government expenditure. By this way, I can compare the effects of various taxation schemes given the constraint that the government must finance an exogenously fixed expenditure-GDP ratio.

I modify the budget constraints of the household at the balanced growth path as follows:

$$
\begin{aligned}
W_i &= (1+\tau_c)c_i + k_i' + b_i', && (23)\\
W_i &= (1+p)(1-\tau_k)(r_i k_i + R b_i) + (1-\tau_l)w, && (24)\\
b_i'/(b_i' + k_i') &> -\underline{\theta}, && (25)\\
k_i', k_i' + b_i' &> 0 && (26)
\end{aligned}
$$

where τ_l, τ_k, and τ_c denote flat tax rates on labor income, capital income, and consumption, respectively. Note that the labor tax is non-distortionary, since labor is supplied inelastically in this model.

The exogenous ratio of government expenditure to GDP is denoted by $\bar{g}$. The government budget constraint requires:

$$\bar{g}\int y_i di = \tau_c \int c_i + \tau_k(r_i k_i + Rb_i)di + \tau_l w \tag{27}$$

The stationary equilibrium is defined similarly to that in the previous section, with an addition of the government budget constraint. The stationary equilibrium is solved numerically with the algorithm similar to that in the previous section. One difference is that I use simulations here rather than the multiplication of transition matrix to compute the stationary distribution of wealth.

3.2 Numerical results

The parameter values are set as follows. The investment shock has two states, $\epsilon_{i,t} \in \{0.85, 1.15\}$. These values are chosen so that the Pareto exponent of the simulated income distribution falls in the empirical range. The transition matrix for $a_{i,t}$ is set so that $\Pr(\epsilon_{i,t} = 1.15 \mid \epsilon_{i,t-1} = 1.15) = 0.95$ and $\Pr(\epsilon_{i,t} = 0.85 \mid \epsilon_{i,t-1} = 0.85) = 0.5$. This transition probability is chosen so that at the stationary distribution 10% of the households have a high shock. The government expenditure is set at $g = 0.1$ of GDP. Tax rates for labor income, capital income, or consumption are set so that each tax scheme raises the tax proceeds equal to the government expenditure. The government expenditure is assumed unproductive. Other parameters are the same as in the previous section: $\alpha = 0.36$, $\delta = 0.1$, $\sigma = 3$, $\beta = 0.96$, $\gamma = 1.02$, and $\mu = 0.01$. For the computation, the resource available to the households (the cash-at-hand) is discretized by 100 grids which are equally spaced in logarithmic scale, and a population of 100,000 households are simulated.

Figure 2 plots the policy functions for savings (wealth next period) for the house-

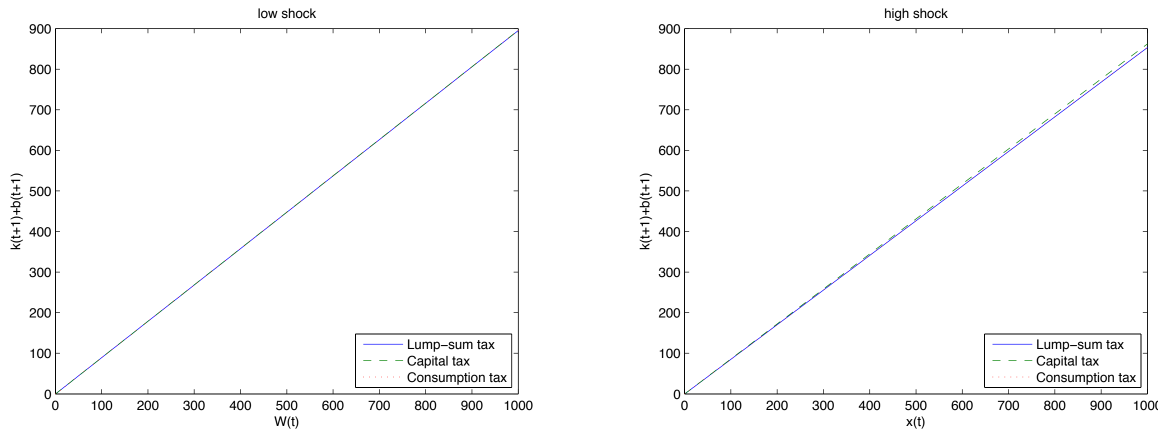

Figure 2: Optimal savings $k_i' + b_i'$ as a function of W_i for the households with a low shock (left) and a high shock (right)

holds with a low shock and a high shock. We observe that the functions are almost linear except for around the lower bound of wealth. The households with high shocks consume more than those with low shocks. The households save more under the capital tax regime than other tax regimes.

Figure 3 plots the policy functions for portfolio (the risk-free bond's share of total wealth) for the households with a low and a high shock. The households with low shocks tend to hold wealth in the form of risk-free bond, while the households with high shocks always borrow through bonds. The portfolio of the households with high wealth and low shocks converges to a constant, which is consistent with the implication of the Ramsey model without borrowing constraints. When the households are far from the borrowing constraint point, their behavior becomes similar to that without borrowing constraints. The households near the lower bound of wealth tend to choose risky portfolio.

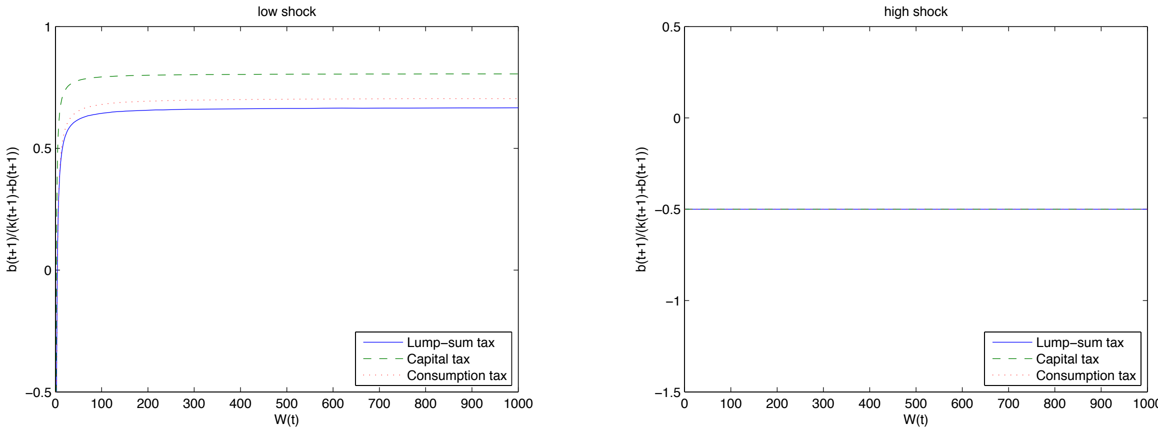

Figure 3: Optimal portfolio $b'_i/(k'_i + b'_i)$ as a function of W_i for the households with a low shock (left) and a high shock (right)

Figure 4 plots the value functions for the households with a low and a high shock under various tax schemes. It is observed that the labor income tax achieves a higher value for each wealth level than the capital income tax and consumption tax. This is because the labor income tax in our model is equivalent to lump-sum tax, since labor is supplied inelastically, and thus the labor tax is non distortionary. I observe that the value functions under capital income tax and consumption tax do not differ very much.

Figure 5 plots the cumulative distribution of wealth across households in log-log scale. Pareto distributions are observed in the tail. Pareto exponent is clearly higher under the capital income tax scheme than under the labor income tax or the consumption tax. This is because the capital income tax reduces the variance of the after-tax rate of return to capital. In addition to the direct effect of tax on the after-tax rate of return, households shift their wealth from risky assets to risk-free asset as shown in Figure 3, because the reduced variance of after-tax rate decreases the contribution

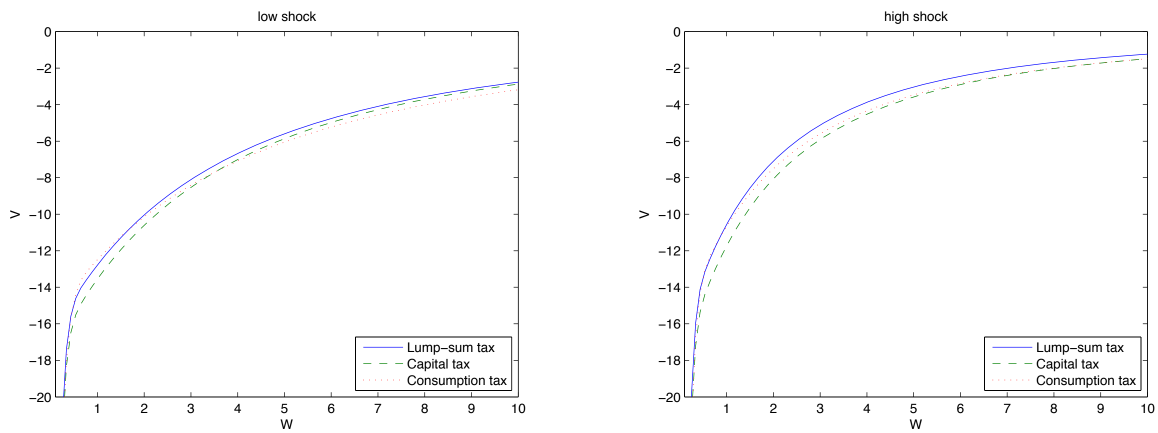

Figure 4: Value functions for the households with a low shock (left) and a high shock (right)

of diffusion effects to the asset growth. Hence, the capital tax lowers the households' risk-taking and results in the steepened tail distribution of wealth.

Table 1 shows the aggregate quantities and prices under various tax schemes. It is clear that the capital taxation generates lower capital, output, consumption, and wage. The after-tax return on capital is equilibrated across tax schemes, because the Euler equation dictates the marginal rate of intertemporal substitution to be equal to the after-tax return. The consumption tax produces almost the same outcome as the labor (non-distortionary) tax. This results confirm the similar results known in homogeneous agent models.

Table 2 shows the median household's status and the overall inequality measures. The median of x (the wealth available in a period), k (the physical capital), consumption, and value v are reported. The Gini coefficient is computed for wealth x. The Pareto exponent is estimated for the top 1% wealth holders by Hill's estimator. I ob-

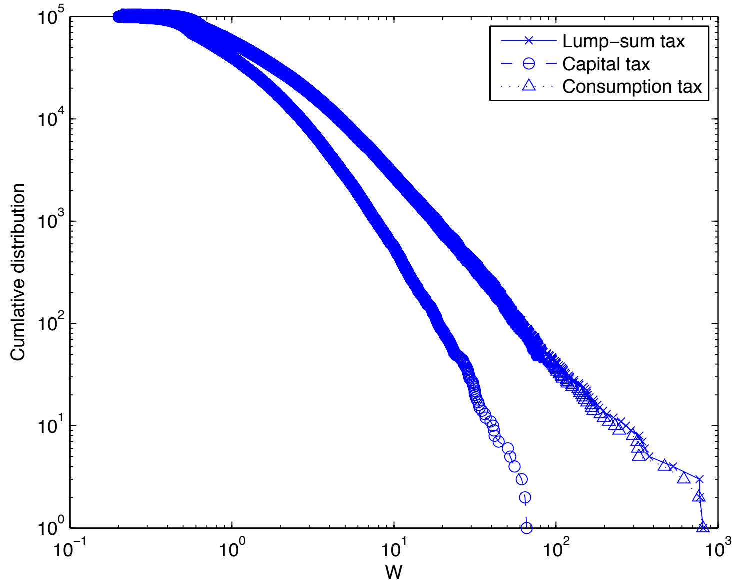

Figure 5: Simulated stationary distributions of wealth

	K	Y	C	R-1	r(1)-1	r(2)-1	w
$\tau_l = 0.16$	2.32	1.39	0.88	0.105	0.099	0.24	0.76
$\tau_c = 0.15$	2.34	1.41	0.87	0.104	0.098	0.24	0.76
$\tau_k = 0.47$	1.32	1.13	0.75	0.200	0.193	0.61	0.40

Table 1: Simulation results on aggregate quantities and prices

	med x	med k	med c	med v	Gini	Pareto
$\tau_l = 0.16$	1.88	1.11	0.72	-7.19	0.57	1.84
$\tau_c = 0.15$	2.05	1.16	0.72	-6.99	0.58	1.90
$\tau_k = 0.47$	1.51	0.80	0.65	-9.72	0.46	2.58

Table 2: Simulation results on median household and inequality indices

serve that the median consumption and welfare value is lower under the capital tax. The inequality measures decrease under the capital tax. This is consistent with previous results in this paper. The capital tax steepens the slope of the tail distribution, i.e. increases the Pareto exponent. Even though the capital tax also lowers the wage rate, the equalization of the wealth dominates the effect and the overall inequality measure, i.e. the Gini coefficient, goes down.

Finally, I conduct a welfare comparison. I consider a household with wealth x. The value for this household located in the stationary economy under capital tax is denoted by $V_{\text{capital}}(x)$, while that under consumption tax is $V_{\text{consumption}}(x)$. I define a compensated wealth $\phi(x)$ as:

$$V_{\text{capital}}((1+\phi)x) = V_{\text{consumption}}(x) \tag{28}$$

Namely, the household with x is indifferent between the two economies if ϕ fraction of wealth x is compensated for living under the capital tax regime.

Figure 6 plots the compensation ϕ as a function of percentile in wealth x and a productivity shock status. I observe that the capital tax is preferred by almost all the households with high shocks and 90 percent of the households with low shocks. This implies that, in this model, the benefits of the high wage and high returns to capital outweigh the cost of consumption tax for most of the households, except for

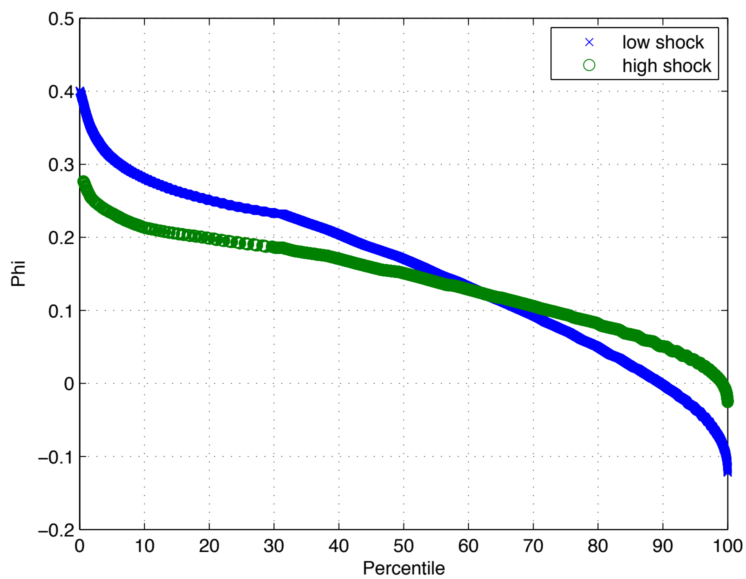

Figure 6: Compensated wealth ϕ for the relocation from the consumption tax economy to the capital tax economy for a household with a relative wealth shown in the percentile of x

the households with low shocks and high wealth. This is intuitive: the households with low shocks and high wealth finance their high consumption by dissaving, and thus the benefit of high wage and returns is outweighed by the high cost of consumption.

3.3 Transition effects of a tax cut

Taxation has a direct effect on wealth accumulation by lowering the after-tax increment of wealth and the effect through the altered incentives that the households face. The impact of income tax legislation in the 1980s in the U.S. has been extensively discussed in the context of the recent U.S. inequalization. As studied by Feenberg and Poterba [11], an unprecedented decline in the Pareto exponent occurred right after the Tax Reform Act in 1986. Although the stable exponent after the downward leap suggests that the sudden decline was partly due to the tax-saving behavior, the steady decline of the Pareto exponent in the 1990s may suggest more persistent effects of the tax act. Piketty and Saez [20] suggests that the imposition of progressive tax around the Second World War was the possible cause for the top income share to decline during this period and stay at its low level for a long time until the 1980s.

The simulated results shown in Figure 1 and the above analytical results confirm that the tax cut that affects capital income reduces the Pareto exponent. However, the simulations of the transition path from $\tau_k = 0.5$ to 0.28 show that the transition in the slope of the tail takes a very long time while the mean of income and wealth converges relatively fast. Given a tax cut, the households choose to consume more in the present, and thus their accumulation of wealth becomes slower than what the Solow model predicts as in [19]. This postponement effect is greater for the households with higher wealth.

This point is shown in Figure 7, in which the convergence to the stationary distri-

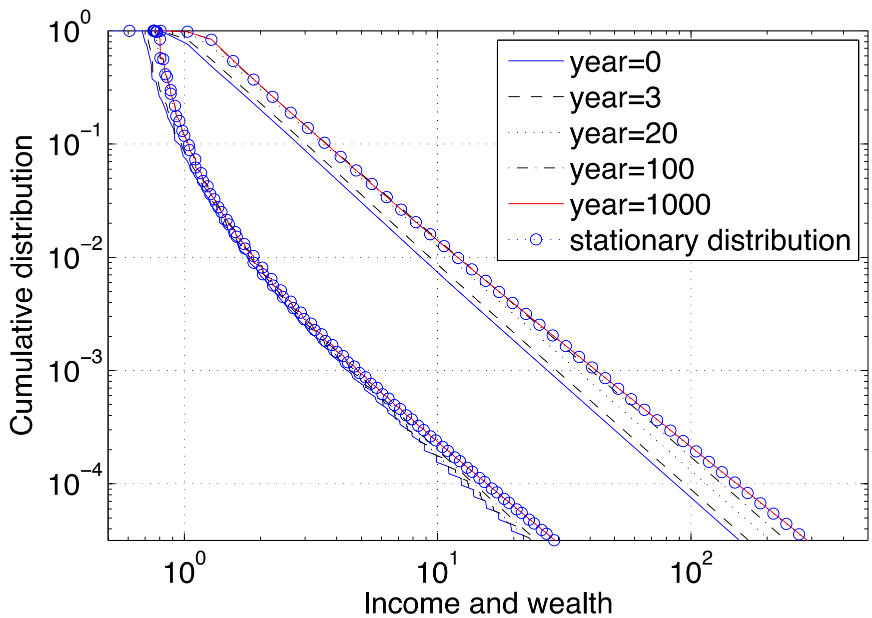

Figure 7: Transition path from the stationary equilibrium with $\tau_k = 0.5$ to that with $\tau_k = 0.28$

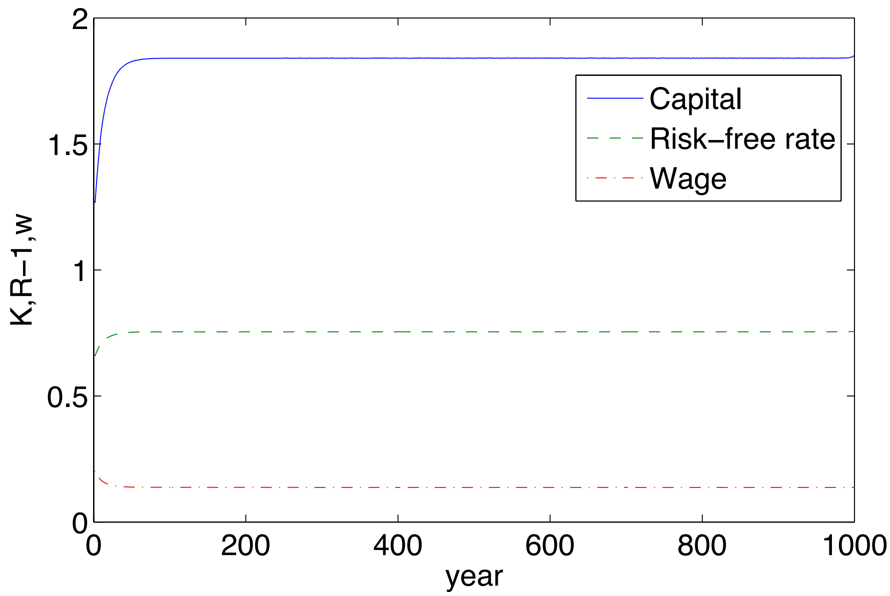

Figure 8: Transition paths from the stationary equilibrium with $\tau_k = 0.5$ to that with $\tau_k = 0.28$

bution of wealth is faster for the low-wealth group than the high-wealth group. Thus, even though the aggregate capital adjusts toward its new level steadily (as shown in Figure 8), the effect on slope takes much longer. This is an important departure from the Solow model.

The transition path in the simulated model is consistent with the fact that the top share of income has increased after the tax cut [20], since the aggregate capital and the top income shares converge to a new stationary level relatively quickly. However, it is not consistent with the observation that the Pareto exponent has also decreased in the years after the cut. One possible explanation is that the Pareto exponent I estimated at the top 1 percentile of income [19] does not correspond to the Pareto exponent I observe at the very end of our computed distribution. In Figure 1, I observe that the slope at the 1 percentile is steeper than the slope further at the tail. This is because the contribution of wage income is non-negligible for this level of income.

3.4 Effects of riskiness

Finally, I explore an alternative explanation for the U.S. historical experience. The Pareto exponent has experienced a large change in drift around 1970s in the U.S. It has been recognized that the income distribution has become inegalitarian since 1980s ([20], [24]), and it has come to much public notice. The flattened tail constitutes an important part of this inequalization process. Our model (as in [19]) identifies several fundamental parameters as the determinant of the Pareto exponent. Let us focus on the key variable, $(z/\bar{x})/(\mathrm{Var}(g)/2)$, that measures the ratio between the contributions of the savings from labor income and of the diffusion effect. The numerator can be decomposed into the personal savings rate, the labor share of income, and the output-capital ratio. The latter two factors are fairly stable, while the personal savings rate shows a steady decline in these years. Figure 9 plots the historical personal savings rate in the NIPA statistics. The denominator, the variance of the asset growth rate for individuals, is hard to measure. Figure 9 plots the time series of the annual excess returns of the S&P500 index relative to the treasury bills (smoothened by 7-year moving average). The plotted excess returns measure our $\mathrm{Var}(g)/2$ under the assumptions that the individual wealth experiences the same volatility as the S&P500 index, and that the mean of the logarithmic instantaneous returns coincide between the S&P500 index and the Treasury Bills. Under the assumptions, the logarithm of the annual return of the S&P500 is equal to the returns on the treasury bills plus a half of the logarithmic variance of the S&P500 returns. I note that the excess returns experienced a marked

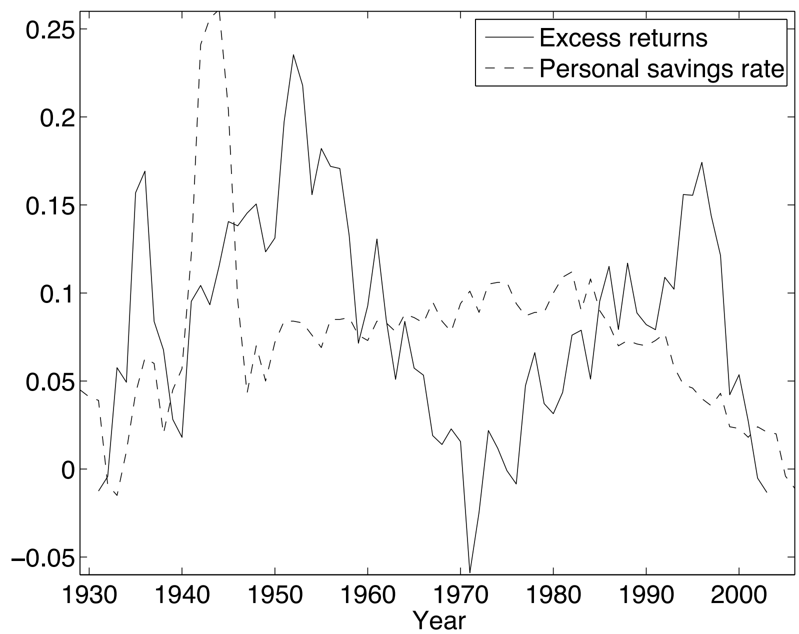

Figure 9: Excess returns measured by S&P500 index returns minus T-Bills returns (left axis) and the personal savings rate (right axis). The excess returns are smoothed by 7-year moving average.

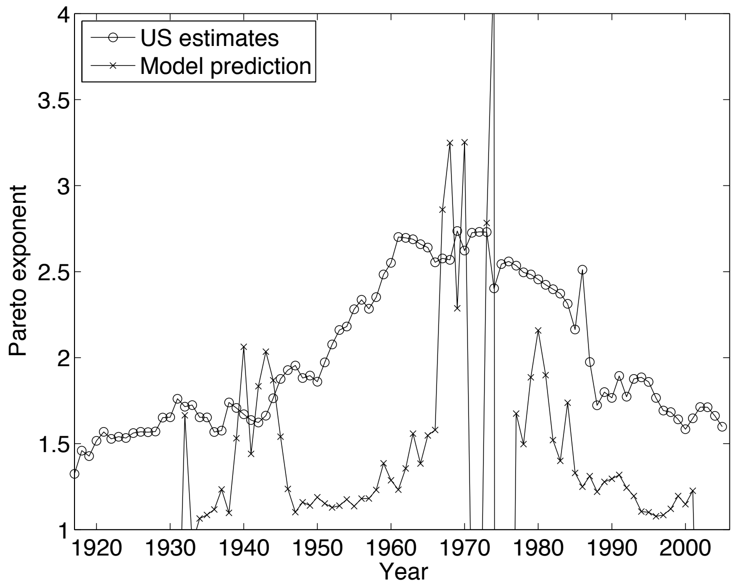

Figure 10: U.S. Pareto exponent (estimated in the range of the 99 percentile to the 99.9 percentile income) and the model prediction when the excess returns of S&P500 index is used as the proxy for Var(g)/2.

trough around 1970. Figure 10 shows the model prediction when the excess return is used for $\mathrm{Var}(g)/2$ and when the NIPA personal savings rate is used to compute z in our formula for λ [19]. The time series of the labor share is computed from the NIPA statistics and the capital-output ratio is fixed at the historically stable value, 2 (Maddison [16]). The predicted Pareto exponent is mostly too small to match the estimated exponent, possibly due to the underestimation of the contribution from the savings and due to the overestimation of the wealth growth variance. Also, the predicted exponent becomes extremely large or negative around 1970, due to the near-zero or negative values of the excess returns of S&P500 index during the periods. However, the movement of the predicted series is remarkably parallel to the actual series.

4 Conclusion

This paper demonstrated that the Bewley model with idiosyncratic investment shocks is able to generate the Pareto distribution as the stationary distribution of income at the balanced growth path. I argued that the Pareto distribution is generated by the investment shock that provides a multiplicative shock in the wealth accumulation process and by the borrowing constraints that induce precautionary savings and work as a reflective lower bound of the wealth accumulation. I showed that the capital tax reduces the Pareto exponent. The Pareto exponent was also analytically determined by fundamental parameters for a particular case.

I compared the distributions of income and wealth across different tax schemes that generate the same government expenditure-GDP ratio. Under calibrated parameter values, the simulation shows that the Pareto exponent is lower under capital tax than under consumption tax, while the exponent under consumption tax is similar to that under non distortionary tax. I also conducted a welfare comparison between the capital and consumption taxes. The result showed that a small fraction of the households with low shocks and high wealth prefers the stationary equilibrium under the consumption tax while others showed the opposite preference.

Appendix

A Computation

In this section I explain the numerical computation of the model in Section 2.2. A household's cash-at-hand W_i is discretized by 100 grids that are equally spaced in logarithmic scale. The pseudo code for the stationary equilibrium is as follows.

1. Outer loop: Set $n_t = 0$ and guess values for $\theta_0(W_i, \epsilon_i)$

   (a) Inner loop 1: Set $n_k = 0$ and guess values for $x_0(W_i, \epsilon_i)$ and aggregate capital K_0

      i. Compute w_{n_k} and r_{n_k}

      ii. Solve for the value function V_{n_k} and the savings function x_{n_k} by the endogenous gridpoints method, under fixed θ_{n_t}

      iii. Construct a transition matrix Γ_{n_k} for (W_i, ϵ_i) by x_{n_k}, θ_{n_t}, and the transition matrix for ϵ_i

      iv. Compute a stationary distribution $\Lambda_{n_k}(W_i, \epsilon_i)$ by iterating the transition matrix Γ_{n_k}

      v. Compute aggregate capital K_{n_k+1} by the stationary distribution Λ_{n_k} and θ_{n_t}.

      vi. If $|K_{n_k+1} - K_{n_k}| > \epsilon_K$, reset n_k to $n_k + 1$ and go to (i). Otherwise, go out of the loop.

   (b) Inner loop 2: Set $n_b = 0$ and guess a value for risk-free rate R_0

      i. Compute the portfolio function θ_{n_t+1} by solving the first order condition, under fixed x_{n_k}

      ii. Compute aggregate bond demand B_{n_b} by Λ_{n_k} and θ_{n_b}

      iii. If $|B_{n_b}| > \epsilon_B$, reset n_b to $n_b + 1$, update R_{n_b} and go to (i). Otherwise, go out of the loop.

2. If $\sup |\theta_{n_t+1}(W_i, \epsilon_i) - \theta_{n_t}(W_i, \epsilon)| > \epsilon_\theta$, reset n_t to $n_t + 1$ and go to 1(a). Otherwise, exit the algorithm.

To compute a transition path from $\tau_k = 0.5$ to $\tau_k = 0.28$, first I compute the stationary equilibria for both taxes. I set the transition period T sufficiently long (1,000 in this computation). I set the policy functions at T to be equal to the stationary policy functions under $\tau_k = 0.28$. Then, I guess a transition path of $(K_t, R_t, \theta_t(W_i, \epsilon_i))$ for which K_0 and K_T are set to the aggregate capital at the stationary equilibrium under $\tau_k = 0.5$ and $\tau_k = 0.28$, respectively. By using the value function at the end point and the guessed path of (K_t, R_t, θ_t), I solve backward for the savings function x_t for $t = 0, 1, \ldots, T-1$. Then, applying the policy functions successively forward to the initial distribution Λ_0 which is equal to the stationary distribution under $\tau_k = 0.5$, I obtain the transition path of Λ_t. Using the path of Λ_t and θ_t, I compute the path of K_t and compare it to the old path of K_t. If the two paths deviate sufficiently, I go back and solve backward the savings functions. If the two paths converge, then I compute the optimal portfolio θ_t and the market-clearing risk free rate R_t for each t. Then, I update the path of (R_t, θ_t) and go back to the beginning of the outer loop. I exit the outer loop when the paths (R_t, θ_t) converge.

In order to compute the stationary equilibrium when a tax rate is determined so that it can finance a fixed government expenditure-GDP ratio, I make a guess for the tax rate τ_0 along with the risk-free rate R_0 at 1(b): Inner loop 2. Then, at 1(b)(iii), I add a convergence criterion for the government budget constraint along with the market clearing condition for bonds.

B Derivation of the policy function in the Ramsey model

First, I guess and verify the policy functions (17,18,19) at the balanced growth path along with a guess on the value function $V(W) = BW^{1-\sigma}/(1-\sigma)$. The guessed policy functions for c, k', b' are consistent with the budget constraint (15).

The first-order conditions and the envelope condition for the Bellman equation (14) are:

$$c^{-\sigma} = \tilde{\beta}\mathrm{E}[r'V'(W')], \tag{29}$$

$$c^{-\sigma} = \tilde{\beta}R\mathrm{E}[V'(W')], \tag{30}$$

$$V'(W) = c^{-\sigma}. \tag{31}$$

Note that we used the condition $(1-\mu)(1+p) = 1$ from (3). By imposing the guess on these conditions, and by using $W' = (\phi r' + (1-\phi)R)(1+p)sW$ from (20), I obtain the equations that determine the constants:

$$0 = E[(r'-R)(\phi r' + (1-\phi)R)^{-\sigma}], \tag{32}$$

$$s/(1-s) = (1-\mu)\left(\beta E[r'(\phi r' + (1-\phi)R)^{-\sigma}]\right)^{1/\sigma}, \tag{33}$$

$$B = (1-s)^{-\sigma}. \tag{34}$$

Thus the guess is verified.

C Proof of Proposition 1

In this section, I solve the Ramsey model and show the existence of the balanced growth path. Then the proposition obtains directly by applying Manrubia and Zanette [17].

To be compatible with the notation in [19], I define $X_t = K_t/\gamma^t$ as detrended aggregate physical capital. At the steady state $\bar{X}$, the return to physical capital (6) is written as:

$$r_{i,t} = \alpha\epsilon_{i,t}^{(1-\alpha)/\alpha}\mathrm{E}(\epsilon_{i,t}^{(1-\alpha)/\alpha})^{\alpha-1}\bar{X}^{\alpha-1} + 1 - \delta, \tag{35}$$

which is a stationary process. The average return is:

$$\bar{r} \equiv \mathrm{E}(r) = \alpha\eta\bar{X}^{\alpha-1} + 1 - \delta. \tag{36}$$

The lending market must clear in each period, which requires $\int b_{i,t}di = 0$ for any t. By aggregating the non-human wealth and using the market clearing condition for lending, I obtain: $\int F_{i,t}di = \bar{r}K_t$. Thus the aggregate total wealth satisfies $\int W_{i,t}di = (1-\mu)^{-1}\bar{r}K_t + H_t$. At the balanced growth path, aggregate total wealth, non-human wealth and human wealth grow at rate γ. Let $\bar{W}$, $\bar{H}$, and $\bar{w}$ denote the aggregate total wealth, the human capital and the wage rate detrended by γ^t at the balanced growth path, respectively. Then I have:

$$\bar{W} = (1-\mu)^{-1}\bar{r}\bar{X} + \bar{H}. \tag{37}$$

Combining the market clearing condition for lending with the policy function for lending (19), I obtain the equilibrium risk-free rate:

$$R = \frac{\gamma(1-\mu)}{s(1-\phi)}\frac{\bar{H}}{\bar{W}}. \tag{38}$$

By using the conditions above and substituting the policy function (17), the budget constraint (15) becomes in aggregation:

$$(\gamma - s(1-\mu)^{-1}\bar{r})\bar{X} = (s - (1-\mu)R^{-1}\gamma)\bar{H}. \tag{39}$$

Plugging into (38), I obtain the relation:

$$R = \frac{\gamma(1-\mu)}{s(1-\phi)} - \frac{\phi}{1-\phi}\bar{r}. \tag{40}$$

Thus, the mean return to the risky asset and the risk-free rate are determined by $\bar{X}$ from (36,40). The expected excess return is solved as:

$$\bar{r} - R = \frac{1}{1-\phi}\left(\alpha\eta\bar{X}^{\alpha-1} + 1 - \delta - (1-\mu)\gamma/s\right). \tag{41}$$

If $\log\epsilon \sim N(-\sigma^2/2, \sigma^2)$, then I have $\eta = e^{\frac{\sigma^2}{2}(1-\alpha)(1/\alpha-2)}$. This shows a relation between the expected excess return and the shock variance σ^2.

Then, the human wealth is written as:

$$\bar{H} = \gamma^{-t}\left(\sum_{\tau=t}^{\infty}\bar{w}\gamma^{\tau}(1-\mu)^{\tau-t}\prod_{s=t+1}^{\tau}R_s^{-1}\right) = \frac{\bar{w}}{1-(1-\mu)\gamma R^{-1}} = \frac{(1-\alpha)\eta\bar{X}^{\alpha}}{1-(1-\mu)\gamma R^{-1}}. \tag{42}$$

Equations (36,39,40,42) determine $\bar{X}, \bar{H}, R, \bar{r}$. In what follows, I show the existence of the balanced growth path in the situation when the parameters of the optimal policy s, ϕ reside in the interior of $(0,1)$. By using (36,40,42), I have:

$$\frac{\bar{X}}{\bar{H}} = \frac{1 - \frac{(1-\mu)\gamma s(1-\phi)}{\gamma(1-\mu) - s\phi(1-\delta) - s\phi\alpha\eta\bar{X}^{\alpha-1}}}{(1-\alpha)\eta\bar{X}^{\alpha-1}}. \tag{43}$$

The right hand side function is continuous and strictly increasing in $\bar{X}$, and travels from 0 to $+\infty$ as $\bar{X}$ increases from 0 to $+\infty$.

Now, the right hand side of (39) is transformed as follows:

$$\begin{aligned}\bar{H}(s-(1-\mu)\gamma R^{-1}) &= \bar{H}\left(s - s(1-\phi)\frac{\bar{W}}{\bar{H}}\right) = \bar{H}s\left(1-(1-\phi)\left((1-\mu)^{-1}\frac{\bar{r}\bar{X}}{\bar{H}}+1\right)\right) \\ &= \bar{H}s\left(\phi-(1-\phi)(1-\mu)^{-1}\frac{\bar{r}\bar{X}}{\bar{H}}\right). \end{aligned} \tag{44}$$

Then I rearrange (39) as:

$$\frac{\gamma}{s\phi}\frac{\bar{X}}{\bar{H}} = 1 + (1-\mu)^{-1}\frac{\bar{r}\bar{X}}{\bar{H}}. \tag{45}$$

By (36), $\bar{r}$ is strictly decreasing in $\bar{X}$, and R is strictly increasing by (40). Thus $\bar{W}/\bar{H}$ is strictly decreasing by (38), and so is $\bar{r}\bar{X}/\bar{H}$ by (37). Thus, the right hand side of (45) is positive and strictly decreasing in $\bar{X}$. The left hand side is monotonically increasing from 0 to $+\infty$. Hence there exists the steady-state solution $\bar{X}$ uniquely. This verifies the unique existence of the balanced growth path.

The law of motion (20) for the detrended individual total wealth $x_{i,t}$ is now completely specified at the balanced growth path:

$$x_{i,t+1} = \begin{cases} \tilde{g}_{i,t+1} x_{i,t} & \text{with prob. } 1-\mu \\ \bar{H} & \text{with prob. } \mu, \end{cases} \tag{46}$$

where,

$$\tilde{g}_{i,t+1} \equiv (\phi r_{i,t+1} + (1-\phi)R)s/((1-\mu)\gamma). \tag{47}$$

This is the stochastic multiplicative process with reset events studied by Manrubia and Zanette [17]. By applying their result, I obtain the proposition.

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