A Stochastic Model of Labor Productivity and Employment

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Abstract

We investigate the productivity dispersion, i.e., allocation of workers among different levels of productivity and output, by employing the largest database for small and medium-sized companies, Credit Risk Database (CRD). Focusing on the manufacturing sector and small and medium levels of productivity, where more workers are distributed among higher levels of productivity, we have new empirical findings in a pivotal role of workers' allocation among different levels of output as a key to understand their allocation among varying levels of productivity. We also propose a stochastic process, mathematically a jump Markov process, in which workers are allocated to firms of differing output and productivity, interrupted by transitions to unemployment, where transitions are coupled with growth and contraction of firms' output that relate to fluctuations of demand.

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I. Introduction

The coming decade will see serious problems caused by ageing population and low birth-rate, namely the problems of slow growth and low productivity, rising public spending and labor shortages in our nation. As more people retire and fewer younger ones take their place, the labor force will shrink, which results in a drop of output growth, if productivity does not increases faster. Since the remaining workers will be older, they will be less productive. Ageing-related spending is estimated to be much larger in the long-term than even the fiscal burden of the financial crisis that we experience today (see IMF (2009)).

Labor and productivity also play central roles at the core of the problems of ageing population with low birth-rate in any developed country. Policy making has to involve utilization of bounded labor forces and maintaining productivity at a certain level. Good understanding of labor allocation among different productivity levels is required for this purpose; specifically the dynamics how workers are allocated and reallocated to industrial sectors with varying levels of productivity under temporally and also spatially, i.e. from one sector to another, changing demand.

Empirical studies have shown that productivity has dispersion among firms and sectors (see Shinohara (1955); Salter (1960); Yoshikawa (2000a) for example). Since the productivity dispersion relates to the notion of equilibrium in economics, it concerns the foundation of economic theory as pointed out lucidly in the works (Yoshikawa, 2000a,b; Aoki and Yoshikawa, 2007). Recent availability of abundant data at firm-levels can provide one to uncover empirical findings related to this important problem, and to obtain insight into the understanding of labor and productivity dynamics at a macroscopic level in the following sense.

The study of productivity dispersion is aided by a large body of empirical studies in labor economics (see Mortensen (2003) for example and references therein). Notably, Mortensen (2003) concluded as saying that the problem of wage dispersion, namely why are similar workers paid differently, reflects differences in employer productivity. The factors of exogenous labor turnover, job destruction and search friction prevent the labor market from converging into a state of equilibrium in which all workers are employed by the firms of highest productivity. Instead of such a static equilibrium, a steady-state allocation of labor across firms of differing productivity is generated stochastically by a process of reallocation of workers from less to more productive firms, including transitions to unemployment by job destruction and other reasons for labor turnover. This is further supported by a growing literature on the analysis of job flow (e.g. Davis et al. (1996); Albaek and Sorensen (1998)).

It should be remarked that the dynamical processes of allocation of labor across firms of differing productivity are essentially of stochastic nature. Since the system has many degrees of freedom —- millions of firms and workers —- the exact configuration of all degrees of freedom is usually not known. Instead one can resort to statistical description in which each state is assigned with a probability and stochastic transitions are given between those states. Additionally, the stochastic states are subject to a few macroscopic variables, such as the total number of workers and firms, and the total demand as given exogenously.

Following this reasoning, our recent papers with collaborators (Aoyama et al., 2008, 2009) performed a series of studies on productivity dispersion by using large databases for Japanese firms showing that

- productivity distribution has a heavy-tail, well characterized by a power-law for large productivity,

- Pareto index for the heavy tail decreases in its value as the level of aggregation goes

from employees to firms, and from firms to sectors,

- theoretical explanation can be given by extending the argument of maximum entropy and Boltzmann distribution advocated by Aoki-Yoshikawa (Aoki and Yoshikawa, 2007) to assimilate fluctuation of aggregate demand by using the concept of superstatistics in statistical physics (a weighted average of the Boltzmann distributions to take into account of fluctuating demand or temperature in the Boltzmann factor,

- analysis of non-manufacturing (service) sector shows different characteristics from the one of manufacturing sector, which seems to necessitate a further extension of the concept of superstatistics,

among other things.

In this paper, we investigate how workers are allocated among different levels of productivity by employing the largest database for small and medium-sized companies, Credit Risk Database (CRD), covering a million small-business firms and fifteen million workers in Japan. Since the different characteristics are present in non-manufacturing sector, we shall focus on manufacturing sector for which we established the empirical findings stated above, in the power-law regime basically for large productivity, where less workers are allocated to firms with higher productivity. We will focus on the small and medium levels of productivity, where more workers are distributed among higher levels of productivity, i.e. the opposite behavior, and have new findings in the pivotal role of workers' allocation among different levels of output. We also propose a stochastic process, mathematically a jump Markov process, in which workers are allocated to firms of differing output and productivity, interrupted by transitions to unemployment. We argue that the stochastic process generates a steady-state allocation of labor across firms of differing output and productivity resulting in the distributions of workers, productivity and output.

In Section II, we investigate by using database explained in Section A to show distributions of productivity and workers first, then our new findings about the allocation of workers among different productivity in Section B, and also about distributions of output and workers and their scaling relation. Then we propose a stochastic process and provide theoretical argument to explain the empirical findings briefly in Section III as a preliminary proposal. We summary in Section IV with discussions.

II.   Empirical Findings

A.   *Database*

Following the previous work (Aoyama et al., 2009), we employ the Credit Risk Database (CRD) as the largest database, which covers small and medium firms in Japan, in order to calculate productivity, number of workers and other information at the level of firms. The database includes a million small-business firms and fifteen million workers in the nation. Since our concern is the distribution of workers among different levels of productivity at a macroscopic scale, the coverage would be satisfactory for our purpose.

We select the firms in the manufacturing sector (defined by the primary business sector of each firm according to the Japan Standard Industrial Classification) comprising of 0.2 million firms and 5 million workers in gross. Productivity c of each firm is then calculated by

$$c = \frac{Y}{L} \ , \tag{1}$$

where Y is the value added and L is the labor measured in number of workers. In the previous work (Aoyama et al., 2009), Y was calculated from the financial information in the CRD (combined with a Nikkei database for large firms) by the method stated there. We shall use, as a proximity of value added, output of firms measured simply by business sales/profits. In a preliminary analysis, we can check the validity of our results here also for the value-added based calculation, so shall demonstrate that the results are quite robust and independent of a particular choice of proximate variables.

B.   *Distributions of Productivity and Workers*

The productivity c has a unimodal and heavy-tailed distribution. Fig. 1 shows the probability distribution function (PDF) denoted by $P_c(c)$. The distribution has a heavy tail which can be well characterized by a power-law for large c, namely

$$P_c(c) \propto c^{-\mu_c - 1} \ , \tag{2}$$

where μ_c is a constant called Pareto index. The power-law region can be represented more appropriately by the cumulative PDF defined by

$$P_>(c) = \int_c^\infty P_c(c') \, dc' \ , \tag{3}$$

which is shown in Fig. 2. The standard MLE (maximum likelihood estimate) gives $\mu_c = 1.88(\pm 0.10)$ for the power-law regime $c > 3 \times 10^4$ for the data shown (here and hereafter the data refers to the year 2006 unless otherwise stated). The power-law fit is depicted by dotted lines in Fig. 1 and Fig. 2. From Fig. 2, one can read that roughly 20% of the data points fall in the power-law regime. The previous papers (Aoyama et al., 2008, 2009) studied the power-law regime extending the work by Yoshikawa (2000b); Aoki and Yoshikawa (2007), and showed that the superstatistics can successfully describe several properties in the regime. In particular, we note that the power-law region extends for nearly two orders of magnitude accounting for roughly 20% of the entire data , and that the Pareto index takes the value $\mu_c \simeq 2$, as previously shown ($\mu_c \simeq 2 \sim 3$ depending on years).

Let us now examine how workers are distributed among different levels of productivity. To do so, we consider the joint PDF for the two variables of productivity c and the number of workers L, $P_{cL}(c, L)$, whose marginal distribution is $P_c(c)$:

$$P_c(c) = \int_0^\infty P_{cL}(c, L) \, dL \ . \tag{4}$$

We assume here and hereafter that the discrete variable L can be treated as a continuous variable as a good approximation at least for the regime of sufficiently large L. Fig. 3 is the scatter plot for $P_{cL}(c, L)$ depicted by gray points. To quantify how workers are allocated among different levels of productivity *in average*, one can use the conditional average for the number of workers who are present in a given level of c. This conditional average can be calculated by

$$\mathrm{E}[L|c] = \int_0^\infty L \, P_{L|c}(L|c) \, dL \ , \tag{5}$$

where $P_{L|c}(L|c)$ is the conditional probability defined by

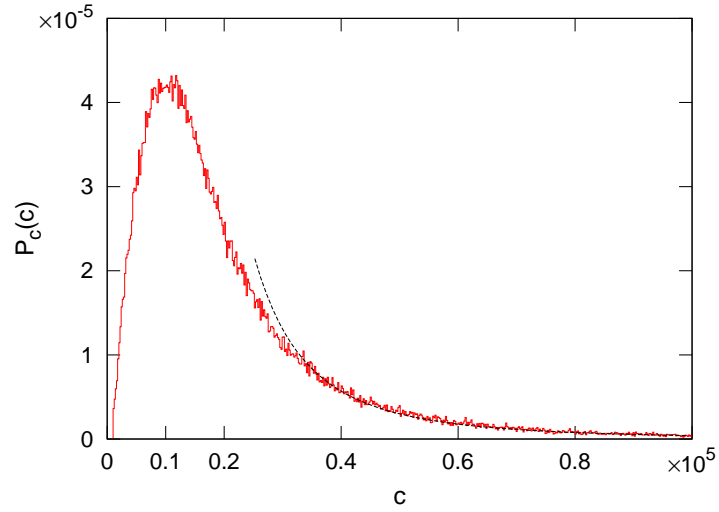$$P_{L|c}(L|c) = \frac{P_{cL}(c, L)}{P_c(c)} \ . \tag{6}$$

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Figure 1: Probability distribution function (PDF) $P_c(c)$ for productivity $c = Y/L$, where Y is output (sales/profit in thousand yen) and L is labor input (number of workers). The dotted line is a power-law fit $P_c(c) \propto c^{-\mu_c - 1}$ where $\mu_c = 1.88(\pm 0.10)$ (maximum likelihood estimation).
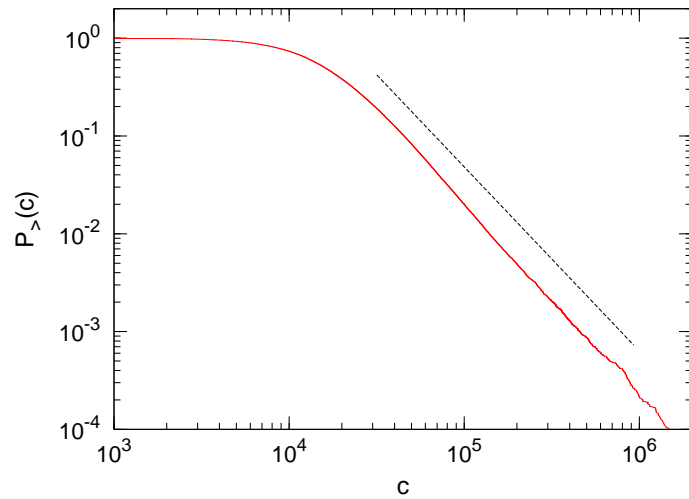

Figure 2: Cumulative PDF $P_>(c)$ for productivity. The dotted line is the same power-law fit as in Fig. 1 (shifted upward for visibility).
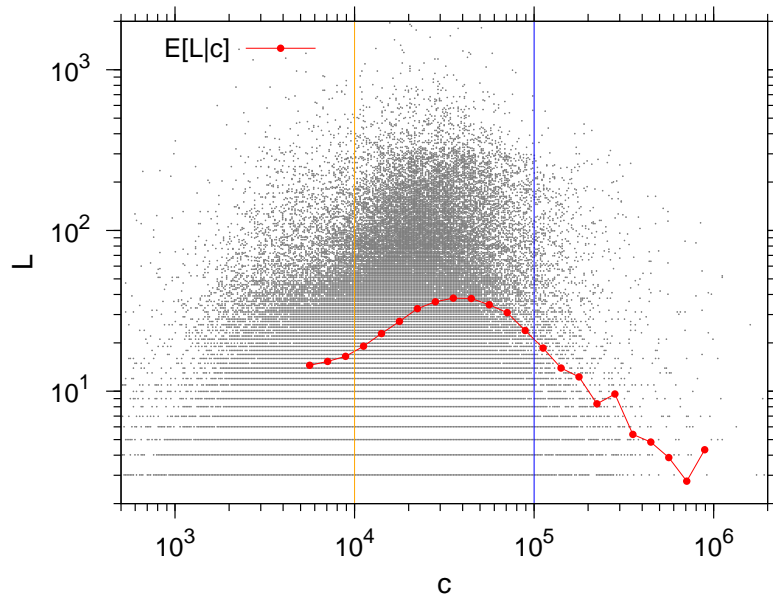
Figure 3: Scatter plot for the joint PDF of productivity c and number of workers L, $P_{cL}(c, L)$ (gray points). The red curve shows the distribution of L among different values of c in terms of conditional average $E[L|c]$ as a function of c. The range $10^4 < c < 10^5$ includes the region where $E[L|c]$ increases and then decreases in its value.

Fig. 3 also shows the conditional average $E[L|c]$ as a function of c depicted by a red curve. In the power-law regime stated above (compare with the horizontal axis in Fig. 2) for large c, approximately $c > 3 \times 10^4$, $E[L|c]$ is a decreasing function of c. This regime was extensively studied in Aoyama et al. (2008, 2009), and can be understood quite naturally. Namely, the framework of the standard Boltzmann distribution proposed by Yoshikawa (2000b); Aoki and Yoshikawa (2007) implies that for the higher the productivity the less workers are employed. Since the superstatistics proposed by Aoyama et al. (2008, 2009) is the weighted average of Boltzmann distributions, no matter how the weight is chosen, higher productivity means less workers, provided that the temperature appearing in Boltzmann distributions, which is proportional to macroeconomic demand, is positive.

Interestingly, in the other side of smaller value of productivity in the region $c < 3 \times 10^4$ approximately, $E[L|c]$ is an *increasing* function of c as clearly shown in Fig. 3.

This could be interpreted naturally as stating that more and more workers tend to move from firms with lower level of productivity to those with higher level. However, this seems beyond the scope of Boltzmann distribution or its extension to superstatistics.

One way to proceed is to consider superstatistics with *negative* temperature, which might be worth pursuing in opening up a new direction of statistical framework. In this paper, we proceed to take a different viewpoint in order to understand this novel empirical finding by examining a joint distribution of workers and *output*, i.e. how workers are allocated among different levels of output of firms, in what follows.

C. *Distributions of Workers and Output*

Since c is defined in terms of Y and L by Eq.(1), one could use the variables (Y, L) instead of (c, L), and vice versa, simply by a transformation of variables.

This leads us to consider the joint PDF $P_{YL}(Y, L)$, which is depicted in Fig. 4 by a scatter plot for the pair of variables (Y, L). To quantify how workers are allocated among different levels of output, one can use the conditional average for number of workers who are present in a given level of Y:

$$\mathrm{E}[L|Y] = \int_0^\infty L \, P_{L|Y}(L|Y) \, dL \; , \tag{7}$$

where $P_{L|Y}(L|Y)$ is the conditional probability defined by

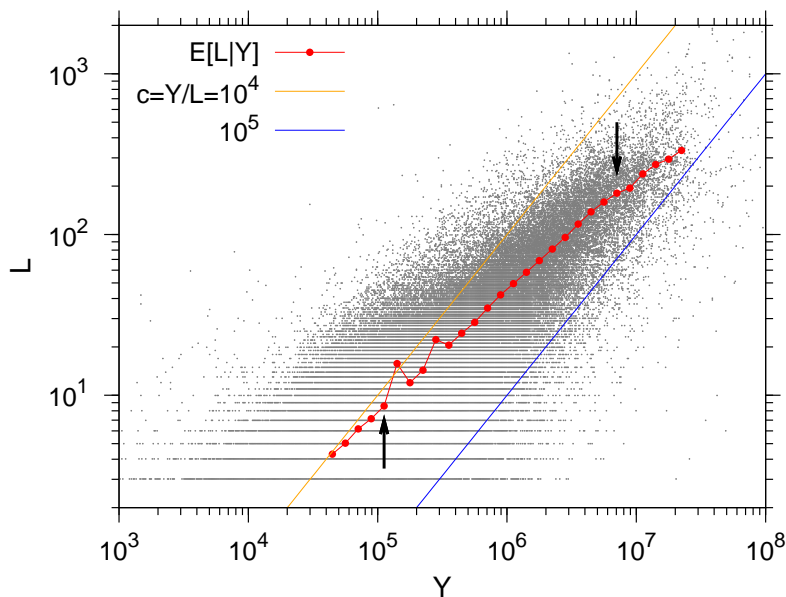$$P_{L|Y}(L|Y) = \frac{P_{YL}(Y, L)}{P_Y(Y)} \; . \tag{8}$$

Figure 4: Scatter plot for the PDF of output Y and number of workers L, $P_{YL}(Y, L)$ (gray points). The red curve shows the conditional average $\mathrm{E}[L|Y]$ as a function of Y. The diagonal band $10^4 < c < 10^5$ refers to the corresponding range depicted in Fig. 3. Left-upward and right-downward arrows correspond to $Y \simeq 10^5$ and $Y \simeq 10^7$ respectively.

The conditional average $\mathrm{E}[L|Y]$ is drawn by a red curve in Fig. 4. One can observe that $\mathrm{E}[L|Y]$ is a linear function of Y to a good accuracy in the most range of variables. Actually, the standard OLS (ordinary least square) linear regression reveals the validity of the equation:

$$\log \mathrm{E}[L|Y] = \alpha \log Y + \mathrm{const.} \; , \tag{9}$$

where the parameter α can be estimated as

$$\alpha = 0.70 \pm 0.02 \quad (\text{with } R^2 = 0.992) \; , \tag{10}$$

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Thus a larger number of workers are employed in firms with higher levels of output, in the *average* sense, in nearly the entire range of variables.

In addition, a stronger property holds for the conditional distribution $P_{L|Y}(L|Y)$. To establish this finding, we examine a set of distributions, $P_{L|Y}(L|Y)$, for different values of Y as shown in Fig. 5. The range of the conditioning variable Y is chosen approximately as $10^5 < y < 10^7$ such that it covers the corresponding range of $10^4 < c < 10^5$ on the curve of $E[L|Y]$ (see the arrows in Fig. 4). One can expect from Eq.(9), and also from the set of distributions $P_{L|Y}(L|Y)$ in the figure, that a *scaling relation* holds:

$$P_{L|Y}(L|Y) = Y^{-\alpha} \, \Phi(Y^{-\alpha} \, L) \; , \tag{11}$$

where $\Phi(\cdot)$ is a scaling function.

The normalization of $P_{L|Y}(L|Y)$ reads

$$1 = \int_0^\infty P_{L|Y}(L|Y) \, dL = \int_0^\infty \Phi(\eta) \, d\eta \; , \tag{12}$$

so the factor of $Y^{-\alpha}$ in front of the scaling function Phi in Eq.(11) is simply for the normalization.

It is easy to see that Eq.(9) is a consequence of the scaling relation (11). Indeed, Eqs.(7), (11) lead to

$$E[L|Y] = Y^\alpha \int_0^\infty \eta \, \Phi(\eta) \, d\eta \; , \tag{13}$$

which is equivalent to Eq.(9).

The scaling relation of Eq.(11) can be directly verified by choosing an appropriate value of α and by scaling the values of Y and the height of PDFs to check if the different curves of $P_{L|Y}(L|Y)$ fall onto a single curve. Actually, by using the value of α obtained in Eq.(10), one has Fig. 6 which evidently shows the collapse of curves onto a scaling function[1].

D.   *Consequences from the scaling relation*

We have immediate consequences from the scaling relation, Eq.(11). To see them, we note first that under the change of variables from (c, L) to (Y, L) one has

$$P_{cL}(c, L) = L \, P_{YL}(c \, L, L) \; . \tag{14}$$

It is then straightforward to see that

$$
\begin{aligned}
P_c(c) &= \int_0^\infty P_{cL}(c, L) \, dL = \int_0^\infty L \, P_{YL}(c \, L, L) \, dL \\
&= \int_0^\infty L \, P_Y(c \, L) \, P_{L|Y}(L|c \, L) \, dL \; ,
\end{aligned}
\tag{15}
$$

by using Eqs.(8), (11) and (14).

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[1]This graphical method can be augmented by a non-parametric statistical method such as a Kolmogorov-Smirnov test.
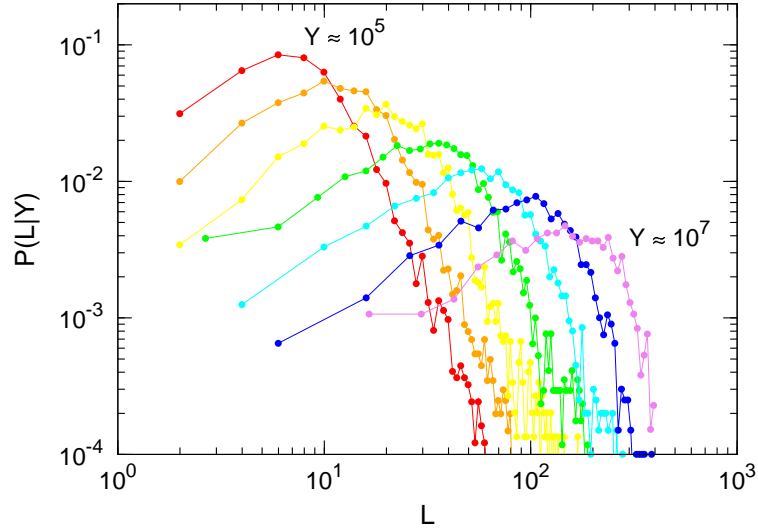
Figure 5: The conditional PDF $P_{L|Y}(L|Y)$ for L under the condition that $Y = $ constant. The range of Y covers roughly from $Y \simeq 10^5$ (red leftmost curve) to $Y \simeq 10^7$ (violet rightmost curve), which correspond to left-upward arrow and to right-downward arrows depicted in Fig. 5 respectively.
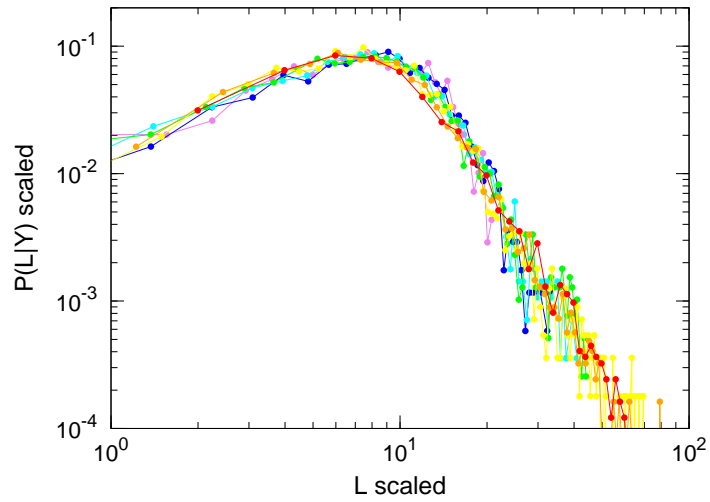

Figure 6: The curves of conditional PDFs $P_{L|Y}(L|Y)$ for different values of Y collapse on a single curve after scaling the values of Y and the height of PDFs, which shows the scaling relation in Eq.(11). The parameter α was determined from the curve $E[L|Y]$ in Fig. 3 by using Eq.(9).

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The conditional average $E[L|c]$ can be expressed similarly as

$$
\begin{aligned}
E[L|c] &= \int_0^\infty L\, P_{L|c}(L|c)\, dL = \frac{1}{P_c(c)} \int_0^\infty L\, P_{cL}(c,L)\, dL \\
&= \frac{1}{P_c(c)} \int_0^\infty L\, P_{YL}(c\,L, L)\, dL \\
&= \frac{1}{P_c(c)} \int_0^\infty L^2\, P_Y(c\,L) P_{L|Y}(L|c\,L)\, dL \;,
\end{aligned}
\tag{16}
$$

due to Eqs.(6), (11) and (14).

If one can assume that the scaling relation, Eq.(8), is valid for wide ranges of variables, Eq.(15) would be approximated by the following expression:

$$
P_c(c) = c^{-\alpha} \int_0^\infty L^{1-\alpha}\, P_Y(c\,L)\, \Phi(c^{-\alpha}\, L^{1-\alpha})\, dL \;,
\tag{17}
$$

and similarly for Eq.(16) as

$$
E[L|c] = \frac{c^{-\alpha}}{P_c(c)} \int_0^\infty L^{2-\alpha}\, P_Y(c\,L)\, \Phi(c^{-\alpha}\, L^{1-\alpha})\, dL \;.
\tag{18}
$$

After numerically parametrizing functional forms for P_Y and Φ from the empirical data by using a log-normal distribution and a polynomial approximation for $\log \Phi$ of its argument in log scale, one can calculate $P_c(c)$ and $E[L|c]$ numerically.

Thus we have two consequences from the scaling relation (11) for the allocation of workers among different levels of output.

1. The fact that the power-law in the PDF $P_c(c)$ holds for a wide range of productivity; in addition, the origin of the value $\mu_c \simeq 2 \sim 3$, which is larger than the Zipf law $\mu = 1$ for firm-size, i.e. for the number of workers and the output).

2. $E[L|c]$ is an *increasing* function of c in a region corresponding to small productivity as was shown in Fig. 3.

III.   Stochastic model

Let us propose a stochastic process, mathematically a jump Markov process, in which workers are allocated to firms of differing output, interrupted by transitions to unemployment. The firms are subject to relative demand which are continually shocked by events such as changes in tastes of consumers among many others, whose temporal variation is of relatively short time-scale. The purpose of this section is merely a sketch of the model.

Suppose that there are K firms and L_i workers in firm i. An individual can be either an employee of a firm or not employed by any one of the firms, i.e. in unemployment at a given moment of time. In a unit time interval dt, an individual is newly employed in the firm i with a probability $t_+(L_i)\, dt$ so that $L_i \to L_i + 1$. Similarly, with a probability $t_-(L)\, dt$, one of the L_i employees leaves so that $L_i \to L_i - 1$. In addition, we suppose that a new firm is born having a single worker with a probability $p\, dt$. On the other hand, a firm with $L = 1$ will exit with the probability $t_-(L = 1)\, dt$.

Note that when $L_i \to L_i - 1$, the leaving individual will not necessarily retain a job at another firm right away. The worker can be in the state of unemployment. Similarly

when $L_i \to L_i + 1$, the recruited worker may be employed either from another firm or from the state of unemployment. Thus the entire population consists of employees in the firms and a "reservoir" of unemployed individuals, and the number of employees is not necessarily conserved.

This model generically describes the stochastic process of (i) increase and decrease of labor, (ii) entry of a new firm having a unit of labor, and (iii) exit of a firm who lost the last unit of labor force.

A set of transition rates and entry probability specifies the jump Markov process. Let us denote by $N_{L,t}$ the average number of firms having labor L at time t. It obeys the master equation:

$$\frac{\partial_t N(L,t)}{\partial t} = t_+(L-1)\,N(L-1,t) + t_-(L+1)\,N(L+1,t)$$
$$-t_+(L)\,N(L,t) - t_-(L)\,N(L,t) + p\,\delta_{L,1}\,, \tag{19}$$

where $\delta_{L,1}$ is 1 if $L = 1$ and 0 otherwise.

The total number of firms is $K_t \equiv \sum_{L=1}^{\infty} N_{L,t}$, and the total number of workers in firms is $W_t \equiv \sum_{L=1}^{\infty} L\,N_{L,t}$. It follows from Eq.(19) that

$$\partial_t K_t = p - L_-(1)\,N_{1,t} \tag{20}$$

$$\partial_t W_t = p - \sum_{L=1}^{\infty} (t_-(L) - t_+(L))\,N_{L,t} \tag{21}$$

We are interested in the stationary solution of Eq.(19) such that $\partial_t N_{L,t} = 0$. The solution can be readily obtained by standard methods. Noting the boundary condition that $t_-(1)\,N(L=1) = p$ from Eq.(20) equated to zero, it is easy to show that

$$N(L) = N(1) \prod_{\ell=1}^{L-1} \frac{t_+(L-\ell)}{t_-(L-\ell+1)}\,. \tag{22}$$

Let us make a simple assumption on the transition rates. The employment (or unemployment) of worker would depend on the number of workers retained at the moment. If the number is larger, the larger is the chance of recruiting another worker. We argue shortly how this assumption is related to the output of firms. A simple assumption would be that the transition rates are of the form that $t_+(L) = a_+ L^\beta$ and $t_-(L) = a_- L^\beta$, where β is assumed to be greater than 1 for a moment. The stationary solution Eq.(22) becomes

$$N(L) = \frac{N(1)}{1 - N(1)/W_{(\beta)}} \frac{(1 - N(1)/W_{(\beta)})^L}{L^\beta} \simeq L^{-\beta}\,e^{-L/L^*}\,. \tag{23}$$

Here $L^* \equiv (N(1)/W_{(\beta)})^{-1}$, and $W_{(\beta)} \equiv \sum_{L=1}^{\infty} L^\beta N(L)$. We have used the relation $a_+/a_- = 1 - N(1)/W_{(\beta)}$, which follows from (20) and (21), both equated to zero. Since it is reasonable to suppose that $N(1)/W_{(\beta)} \ll 1$, we have made approximation in the second equality of (23). The exponential cut-off works as x approaches to x^*, but the value of x^* is practically quite huge. Therefore, one has the power law distribution $N(L) \propto L^{-\beta}$ for a wide range of L in spite of the cut-off.

The present model can be understood easily with the help of an analogy of the formation of "cities". Imagine that $N(L,t)$ is the number of cities with population L at time t. $t_+(L)$

corresponds to a birth in a city with population L, or an inflow into the city from another city. Similarly, $t_-(L)$ represents a death or an exit of a person moving to another city. The rates are the instantaneous probabilities that population of city with the current population L either increases or decreases by one. They are, therefore, the entry and exit rates of *one person* times population L, respectively. And a drifter forms his own one-person city with the instantaneous probability p. In this model, dynamics of $N(L,t)$, namely the average number of cities with population L is given by Eq.(23). In the case of population dynamics, one might assume that the entry (or birth) and exit (or death) rates of a person, a_+ and a_- are independent of the size of population of the city in which the person lives. Then, $t_+(L)$ and $t_-(L)$ become linear functions of L, namely, a_+L and a_-L. Even in population dynamics, though, one might assume that the entry rate of a person into big city is higher than its counterpart in small city because of the better job opportunity or the attractiveness of "city life." The same may hold for the exit and death rates because of congestion or epidemics.

This analogy captures our essential assumption; workers employed in the same firm are subject to the same aggregate shocks, which tend to increase or decrease the number of workers. Here the aggregate shocks means the demand shocks which vary from one sector to another, i.e. spatially, and also temporally. Thus our model can be interpreted as postulating that the transition $L \to L+1$ at a moment is driven by the increase of output at a previous time corresponding to the growth of demand, $Y \to Y + \Delta Y$ ($\Delta Y > 0$). And similarly $L \to L - 1$ by the decrease of output or the shrink of demand, $Y \to Y - \Delta Y$ at a previous time. Then, for the simplest case of $beta = 2$ for the transition rates, since there are more or less L^2 times of leaving and recruiting of workers, the net increase (or decrease) of the number of workers δL of the workers in a firm during an interval, would be linearly proportional to L. This is the multiplicative process which gives rise to power-law behavior.

Moreover, the empirical findings obtained in Section II.C show that the allocation of workers among different levels of productivity can be understood from their allocation among varying levels of output. The latter can be characterized by the linear relation in Eq.(9) which follows from the conditional distribution in Eq.(11) whose scaling function is given in Fig. 6. One can assume a set of coupled stochastic dynamics, one for L as described so far and the other for Y as a dynamics for the growth of firm-size reflecting the demand shocks. Comparison of such a simulation with the empirical findings is to be given elsewhere.

IV.   Summary and Discussions

By employing the largest database for small and medium-sized companies, Credit Risk Database (CRD), covering a million small-business firms and fifteen million workers in Japan. Since the different characteristics are present in non-manufacturing sector, we shall focus on manufacturing sector for which we established empirical findings previously (Aoyama et al., 2008, 2009) in the power-law regime basically for large productivity.

Focusing on the small and medium levels of productivity, we found that more workers are distributed among higher levels of productivity, i.e. the opposite to our previous results, and have new findings in the pivotal role of workers' allocation among different levels of output. Especially, the scaling relation related to the distribution of workers among varying levels of output leads that (1) the fact that the power-law in the PDF $P_c(c)$ holds for a wide range of productivity; in addition, the origin of the value $\mu_c \simeq 2 \sim 3$, which is larger than the Zipf law $\mu = 1$ for firm-size including for the number of workers and the output), and that (2) $\mathrm{E}[w|c]$ is an *increasing* function of c in a region corresponding to small productivity as was

shown in Fig. 3.

We also sketch a stochastic process, mathematically a jump Markov process, in which workers are allocated to firms of differing output and productivity, interrupted by transitions to unemployment. We propose that the stochastic process generates a steady-state allocation of labor across firms of differing output and productivity resulting in the distributions of workers, productivity and output, although a full comparison of the model with the obtained empirical facts should be done in a future work.

Our empirical finding of the scaling relation in Eq.(11) might be considered as a generalization of *production function* in the following sense. The concept of production function is a functional relation between output Y, labor input L and capital input K, where one usually assumes homogeneous production function, i.e. assumption of how output scales under a scaling of K and L. Our discovery is a *distributional* generalization of production function, which takes into account conditional distributions rather than conditional averages. This insight can potentially lead to a new perspective to look at distributions of productivity, labor, capital and output, as we have recently found (Aoyama and Fujiwara, 2010).

It should be also remarked that our argument along the conditional and joint distributions has similarity in a recent paper by Toda (2009) in its application to labor demand (in its Section 6). The argument of maximum entropy employed there as an analogous application of thermodynamics and statistical mechanics might have relation to our analysis here. We consider that this point is worth pursuing as a future work.

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