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Abstract

Business cycle accounting rests on the insight that the prototype neoclassical growth model with time-varying wedges can achieve the same allocation generated by a large class of frictional models: equivalence results. Equivalence results are shown under general conditions about the process of wedges while it is often specified to be the first order vector autoregressive when one applies business cycle accounting to actual data. In this paper, we characterize the class of models covered by the prototype model under the conventional first order vector autoregressive specification of wedges and find that it is much smaller than that believed in previous literature. We also apply business cycle accounting to an artificial economy where the equivalence does not hold and provide a numerical example that business cycle accounting works well even in such an economy.

Keywords: Equivalence results; business cycle accounting

JEL Classification: E32

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On Equivalence Results in Business Cycle Accounting[*]

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Abstract

Business cycle accounting rests on the insight that the prototype neoclassical growth model with time-varying wedges can achieve the same allocation generated by a large class of frictional models: equivalence results. Equivalence results are shown under general conditions about the process of wedges while it is often specified to be the first order vector autoregressive when one applies business cycle accounting to actual data. In this paper, we characterize the class of models covered by the prototype model under the conventional first order vector autoregressive specification of wedges and find that it is much smaller than that believed in previous literature. We also apply business cycle accounting to an artificial economy where the equivalence does not hold and provide a numerical example that business cycle accounting works well even in such an economy.

Keywords: Business cycle accounting; equivalence results

JEL classification: C68, E10, E32

1 Introduction

Business cycle accounting (hereafter BCA) is a method that is used (i) to measure distortions by the prototype model with time-varying *wedges*, which resemble aggregate productivity, labor and investment taxes, and government consumption, such that the prototype model perfectly accounts for observed data, and (ii) to investigate the importance of each wedge in business cycles by counterfactual simulations. In order to substantiate their prototype model, Chari, Kehoe and McGrattan (2007a) (hereafter CKM) claim *equivalence results* – namely, their prototype model covers a large class of frictional detailed business cycle models. Equivalence results suggest that the prototype model is equivalent to a detailed model if and only if the first order conditions of the prototype model are satisfied given any allocation of consumption, investment, labor, and output generated by the detailed model through the adjustment of wedges. CKM show equivalence results under general conditions about evolutions of wedges. However, in practice, they impose that wedges evolve according to the first order vector autoregressive, VAR(1), process and it is not clear whether CKM's VAR(1) specification of wedges is consistent with conditions in terms of equivalence results. Many papers that apply BCA also employ CKM's VAR(1) specification of wedges. Therefore, it is important to investigate the class of models covered by the prototype model with the VAR(1) specification of wedges. And if the class of models covered by the prototype model is small, it is also important to investigate whether or not BCA can measure true wedges in an economy where equivalence result does not hold. Non equivalence might lead a mismeasurement of wedges. In the literature of BCA, there are two opposite results on the importance of investment wedge for business cycles and these results depend on minor difference in the procedure of BCA. The diversion of results might be accounted for by the non equivalence between the prototype model and the true model of the real world.

In this paper, we examine the equivalence results by focusing on the VAR(1) representation of wedges. We characterize the class of frictional models covered by the prototype model with the conventional VAR(1) specification of wedges. We find that the prototype model covers a detailed model if and only if wedges have sufficient information about the endogenous and exogenous states of the detailed model. Intuitively, the number of independent wedges should be larger than that of endogenous and exogenous states variables in the detailed model. We also find that the class covered by the prototype model is much smaller than that is shown in CKM under general conditions of wedges. Some examples of the equivalence in CKM are not covered by the prototype model with the VAR(1) specification of wedges. Therefore, the condition for equivalence results is highly restrictive if we employ the VAR(1) representation of wedges. We extend our analysis to an alternative specification which has the VAR(1) specification as a special case and find that the class of models covered by the prototype model is not so large even in such case. We also apply BCA to an artificial medium-scale dynamic stochastic general equilibrium (hereafter DSGE) model, which is not covered by the prototype model, in order to assess the empirical usefulness of BCA. We provide an example that measured wedges capture the properties of the true wedges almost correctly even in such an economy. Therefore, our result tells us that even if the prototype model is not equivalent to a detailed model, BCA might work well since the prototype model is as a good approximation of the detailed model.

In the following, we describe the related literature. CKM propose BCA and claim that their prototype model covers a large class of frictional business cycle models.[1] CKM also conclude that investment wedge is not promising for business cycle research. Christiano and Davis (2006) critique the procedure of BCA and claim that results of BCA are fragile if they employ an alternative procedure.[2] This paper is closely related to Bäeurle and Burren (2007). They investigate the class of frictional models covered by the prototype model and their study was conducted at the same time as this paper. We show the class of models which are equivalent to the prototype model while they consider only the class of models where there exists the VAR(1) specification of wedges, which is a necessary

[1]Chari, Kehoe, and McGrattan (2002) apply BCA using the deterministic prototype model. In the case of the deterministic prototype model, there is no theoretical problem of equivalence results. However, when we implement BCA in the deterministic economy, we have to assume the future path of wedges and such assumption might cause mismeasurement of wedges.

[2]Chari, Kehoe, and McGrattan (2007b) critique the alternative procedure of Christiano and Davis (2006).

condition for the equivalence.

The rest of this paper is as follows. Section 2 introduces the basic framework for the analysis: the prototype model, a detailed model, and the definition of the equivalence. Section 3 presents our main results. We characterize the class of models covered by the prototype model with the VAR(1) specification of wedges and discuss implications of our results. We extend our analysis to more general class of the process of wedges in Section 4. In Section 5, we apply BCA to an artificial economy where the equivalence result does not hold and provide a numerical example that BCA can measure wedges almost correctly even in such an economy. Section 6 draws certain concluding remarks.

2 Basic Setting

2.1 Prototype Model

The prototype economy of the equivalence result is as follows.[3] The representative household problem is given by the following:

$$\max_{c_t,\ell_t,i_t} E_0 \sum_{t=0}^{\infty} \beta^t \Big[\log(c_t) + \nu \log(1-\ell_t) \Big], \tag{1}$$

$$\text{s.t. } c_t + (1+\tau_{x,t}) i_t \leq (1-\tau_{\ell,t}) w_t \ell_t + r_t k_{t-1} - T_t, \tag{2}$$

$$k_t = (1-\delta) k_{t-1} + i_t, \tag{3}$$

where c_t denotes consumption; ℓ_t, labor supply; $\tau_{x,t}$, imaginary investment tax; $1/(1+\tau_{x,t})$, *investment wedge*; $\tau_{\ell,t}$, imaginary labor income tax; $(1-\tau_{\ell,t})$, *labor wedge*; i_t, investment; w_t, wage rate; r_t, rental rate of capital; k_{t-1}, capital stock at the end of period t; and T_t, the lump-sum tax.

The production function of competitive firms is

$$y_t = A_t k_{t-1}^{\alpha} \ell_t^{1-\alpha}, \tag{4}$$

[3]While CKM consider a general model with histroty dependence, we restrict a model which has the recursive structure in the present paper.

where y_t denotes output, and A_t denotes *efficiency wedge*.[4] Finally, the market clearing condition is

$$c_t + i_t + g_t = y_t, \tag{5}$$

where g_t denotes *government wedge.*

The equilibrium system of the prototype model is summarized as

$$\nu \frac{c_t}{1-\ell_t} = (1-\tau_{\ell,t})(1-\alpha) \cdot \frac{y_t}{\ell_t}, \tag{6}$$

$$\frac{1+\tau_{x,t}}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left\{ (1+\tau_{x,t+1})(1-\delta) + \alpha \frac{y_{t+1}}{k_t} \right\} \right], \tag{7}$$

$$y_t = A_t k_{t-1}^{\alpha} \ell_t^{1-\alpha}, \tag{8}$$

$$k_t = (1-\delta)k_{t-1} + i_t, \tag{9}$$

$$c_t + i_t + g_t = y_t, \tag{10}$$

where (6) denotes the consumption-leisure-choice optimization condition; (7), the Euler equation; (8), the aggregate production function; (9), the evolution of aggregate capital stock; (10), the resource constraint. To close the model, we have to specify the evolution of wedges. CKM specify that the vector of wedges $\boldsymbol{s}_t \equiv [\log(A_t), \tau_{\ell,t}, \tau_{x,t}, \log(g_t)]'$ evolves according to the first order vector autoregressive, VAR(1), process as

$$\boldsymbol{s}_{t+1} = \boldsymbol{P}_0 + \boldsymbol{P}\boldsymbol{s}_t + \boldsymbol{\varepsilon}_{t+1}, \tag{11}$$

where $\boldsymbol{\varepsilon}_t$ is i.i.d. over time and is normally distributed with mean zero.

CKM also consider the prototype model with *capital wedge*, which resembles to capital income tax, instead of investment wedge. In this case, the budget constraint (2) becomes

$$c_t + i_t \leq (1-\tau_{\ell,t})w_t\ell_t + (1-\tau_{k,t})r_t k_{t-1} - T_t, \tag{12}$$

[4] We consider a stationary economy for simplicity.

where $(1-\tau_{k,t})$ denotes capital wedge. The analogue of (7) is

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left\{ (1-\delta) + (1-\tau_{k,t+1})\alpha \frac{y_{t+1}}{k_t} \right\} \right], \tag{13}$$

and $\boldsymbol{s}_t \equiv [\log(A_t), \tau_{\ell,t}, \tau_{k,t}, \log(g_t)]'$.[5] We consider the case of the prototype model with investment wedge in this paper, but the analogues of results in Section 3 are applicable to the prototype model with capital wedge.

2.2 Detailed and Extended Detailed Models

Here, we define two models: a detailed model and an extended detailed model for the analysis.

Let $\boldsymbol{x}_t$ be a vector of the endogenous state variables; $\boldsymbol{y}_t$, a vector of endogenous jump variables; and $\boldsymbol{z}_t$, a vector of exogenous state variables. let n and q is the numbers of variables contained in $[\boldsymbol{x}'_{t-1}, \boldsymbol{z}'_t]'$ and $\boldsymbol{y}_t$, respectively. We consider these variables to be defined as deviations from the deterministic steady-state.

As in Uhlig (1999), generally, a linearized DSGE detailed model is described as

$$\boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{x}_{t-1} + \boldsymbol{C}\boldsymbol{y}_t + \boldsymbol{D}\boldsymbol{z}_t = \boldsymbol{0}, \tag{14}$$

$$E_t\Big[\boldsymbol{F}\boldsymbol{x}_{t+1} + \boldsymbol{G}\boldsymbol{x}_t + \boldsymbol{H}\boldsymbol{x}_{t-1} + \boldsymbol{J}\boldsymbol{y}_{t+1} + \boldsymbol{K}\boldsymbol{y}_t + \boldsymbol{L}\boldsymbol{z}_{t+1} + \boldsymbol{M}\boldsymbol{z}_t\Big] = \boldsymbol{0}, \tag{15}$$

$$\boldsymbol{z}_{t+1} = \boldsymbol{N}\boldsymbol{z}_t + \boldsymbol{u}_{t+1}, \quad E_t\Big[\boldsymbol{u}_{t+1}\Big] = \boldsymbol{0}, \tag{16}$$

where $\boldsymbol{u}_{t+1}$ is i.i.d. over time and is normally distributed with mean zero. We assume that consumption c_t, investment i_t, labor ℓ_t, capital stock k_t, and output y_t are definable in this detailed model and that capital stock at the beginning of period k_{t-1} is included

[5]CKM employ some types of capital wedge. This version is employed for equivalence result of the input-financing friction model as we will show in Section 3.

in $\boldsymbol{x}_{t-1}$. The state-space form solution of this detailed model is

$$\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{z}_{t+1} \end{bmatrix} = \boldsymbol{\Psi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix}, \tag{17}$$

$$\boldsymbol{y}_t = \boldsymbol{\Omega} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}, \tag{18}$$

where $\boldsymbol{\Psi}$ is an $n \times n$ matrix and $\boldsymbol{\Omega}$ is a $q \times n$ matrix. The aggregate decision rule of consumption, investment, labor, output, and capital stock at the end of period is

$$\begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{\ell}_t \\ \hat{y}_t \\ \hat{k}_t \end{bmatrix} = \boldsymbol{\Theta} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}, \tag{19}$$

where $\boldsymbol{\Theta}$ is a $5 \times n$ matrix and variables with hat $\hat{}$ denote (log) deviations from the deterministic steady-state.

Here, we introduce *the extended detailed model* in order to consider wedges which are consistent with the detailed model as follows.

Definition 1. *An extended detailed model consists of (i) a equilibrium system of a detailed model (14), (15), and (16), and (ii) the linearized equations of the first order conditions of the prototype model (6), (7), (8), (9), and (10).*

The extended detailed model implies that any realized sequences of consumption, investment, labor, and output generated by the detailed model are consistent with the first order conditions of the prototype model (6), (7), (8), (9), and (10). Since $\hat{\boldsymbol{s}}_t$ is the endogenous variables in the extended detailed model, we can calculate the aggregate decision rule of $\hat{\boldsymbol{s}}_t$ as

$$\hat{\boldsymbol{s}}_t = \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}, \tag{20}$$

where $\boldsymbol{\Phi}$ is an $m \times n$ matrix; m, the number of wedges $\boldsymbol{s}_t$; and n, the number of endogenous and exogenous state variables $[\boldsymbol{x}'_{t-1}, \boldsymbol{z}'_t]'$.[6] We are also able to calculate the aggregate decision rule of endogenous and exogenous state variables as

$$\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{z}_{t+1} \end{bmatrix} = \boldsymbol{\Psi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix}, \tag{21}$$

where $\boldsymbol{\Psi}$ is an $n \times n$ matrix.

2.3 Equivalence Results

The definition of the equivalence is as follows.

Definition 2. *The prototype model is equivalent to (or covers) a detailed model if the prototype model can achieve any realized sequences of consumption, investment, labor, output, and capital stock generated by the detailed model.*

In other words, the prototype model is equivalent to a detailed model if the intratemporal condition (6), the Euler equation (7), the aggregate production function (8), and the resource constraint (10) are satisfied given any realized sequences of $\{c_t, i_t, \ell_t, y_t, k_t\}$ generated by the detailed model. These four equations are satisfied by suitable adjustments of wedges. CKM claim equivalence results by comparing the first order conditions of two models and derivate conditions where the first order conditions of the detailed model are identical to (6), (7), (8), (9), and (10) in some examples. Their equivalence results are true under general conditions about process of wedges. However, in practice, CKM impose that wedges evolve according to VAR(1) and it is not clear whether or not VAR(1) specification of wedges is consistent with such conditions. It is also unclear that

[6] We assume that this extended detailed model satisfies the suitable conditions in order to solve the aggregate decision rule, that is, the Blanchard-Kahn condition.

even if such VAR(1) specification exists, the prototype model is equivalent to a detailed model in general.

Generally, the necessary and sufficient condition of the equivalence is summarized as follows.

Proposition 1. *The prototype model is equivalent to a detailed model if and only if (i) a process of wedges to close the prototype model is consistent with the detailed model, and (ii) there exists, in the extended detailed model, a mapping from the states of the prototype model given the process of wedges to variables of the prototype model.*

The necessity of (i) is obvious since it is needed to close the prototype model. The necessity of (ii) is as follow. There exists, in the prototype model, a mapping from the states to variables and if the condition (ii) does not hold, realized sequences of variables generated by the detailed model are not achieved in the prototype model. The sufficiency is also easily shown as follows. The mapping (ii) generated in the extended detailed model must satisfy the first order conditions of the prototype model since they are also included in the extended detailed model.[7] Note that the condition (ii) requires that the state-space form solution of the prototype model must exists in the extended detailed model.

We investigate these problems through a formal discussion of the linearized economy in the following two sections.[8] In Section 3, we consider the case of the VAR(1) specification of wedges. We extend our analysis to an alternative specification which has the VAR(1) specification as a special case in Section 4.

[7]Bäeurle and Burren (2007) consider only the problem about (i). One of contributions of the present paper is to consider the condition (ii).

[8]We only consider a linearized economy. However, results in this paper hold in a non-linear economy in the neighborhood of a steady-state.

3 VAR(1) Specification of Wedges

3.1 Conditions for Equivalence

In the case of CKM's VAR(1) specification (11), the condition (i) is the existence of

$$\hat{\boldsymbol{s}}_{t+1} = \boldsymbol{P}\hat{\boldsymbol{s}}_t + \varepsilon_{t+1}, \tag{22}$$

in the extended detailed model. Since, in the case of VAR(1) specification of wedges, $[\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ are the state variables in the prototype model, the condition (ii) is the existence of the mapping from $[\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ to consumption, investment, labor, output and capital stock at the end of period:

$$\begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{\ell}_t \\ \hat{y}_t \\ \hat{k}_t \end{bmatrix} = \boldsymbol{\Lambda} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{\boldsymbol{s}}_t \end{bmatrix}, \tag{23}$$

where $\boldsymbol{\Lambda}$ is a $5 \times (n+1)$ matrix, *in the extended detailed model.* Note that we do not need to consider the existence of the mapping from $[\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ to $\hat{\boldsymbol{s}}_{t+1}$ if the VAR(1) specification of wedges (22) exists.

First, we consider the existence of the VAR(1) specification of wedges. The necessary and sufficient condition for the existence of the VAR(1) representation is as follows.

Lemma 1. *Assume that (20) and (21) hold in the extended detailed model. There exists* $\boldsymbol{P}$ *that satisfies CKM's specification (22) if and only if*

$$rank(\boldsymbol{\Phi}) = rank\left(\begin{bmatrix} \boldsymbol{\Phi}' \vdots \boldsymbol{\Psi}'\boldsymbol{\Phi}' \end{bmatrix} \right). \tag{24}$$

Proof. See Appendix A. □

If $rank(\boldsymbol{\Phi}) = n$, the following Lemma 1 holds and provides us with some intuitions with regard to the existence of $\boldsymbol{P}$.

Lemma 2. *Assume that (20) and (21) hold in the extended detailed model. If the rank of* $\boldsymbol{\Phi}$ *equals the number of endogenous and exogenous states in the detailed model,* n*, there exists* $\boldsymbol{P}$ *that satisfies CKM's specification (22).*

Proof. Since $rank(\boldsymbol{\Phi}) = n$, there exists $(\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}$. Then, (20) becomes

$$(\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'\hat{\boldsymbol{s}}_t = \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}. \tag{25}$$

By (25) and (21), we obtain

$$\hat{\boldsymbol{s}}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\Psi}(\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'\hat{\boldsymbol{s}}_t + \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix}. \tag{26}$$

Therefore, there exists a VAR(1) representation of wedges (22) where $P = \boldsymbol{\Phi}\boldsymbol{\Psi}(\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'$ and $\boldsymbol{\varepsilon}_{t+1} = \boldsymbol{\Phi}[\boldsymbol{0},\ \boldsymbol{u}'_{t+1}]'$. □

Lemma 2 can be interpreted as follows. As seen in (25), $rank(\boldsymbol{\Phi}) = n$ implies that $[\boldsymbol{x}'_{t-1}\ \boldsymbol{z}'_t]'$ are identified by $\hat{\boldsymbol{s}}_t$ if we know $\boldsymbol{\Phi}$. In other words, wedges $\hat{\boldsymbol{s}}_t$ has sufficient information with regard to the endogenous and exogenous states $[\boldsymbol{x}'_{t-1}\ \boldsymbol{z}'_t]'$, and such wedges can be written as a VAR(1) process. Intuitively, the number of independent wedges should be that of endogenous and exogenous state variables in the detailed model at least in order to let the prototype model cover the detailed model. It is easily shown that, in general, $\boldsymbol{P}$ does not exist if $rank(\boldsymbol{\Phi}) < n$. Generally, if the number of wedges is strictly smaller than that of the endogenous and exogenous states in the detailed model or $n > m$, wedges cannot contain adequate information regarding the endogenous and exogenous states in general. Then, there is no VAR(1) representation of wedges. Even if the number of wedges are larger than that of the endogenous and exogenous states in the detailed model or $m \geq n$, and if $rank(\Phi) < n$, it implies that wedges do not have sufficient information about the endogenous and exogenous states in the detailed model, and there is no VAR(1) representation in general.

Next, we consider the existence of (23). Let a $(m+1)\times n$ matrix $\tilde{\mathbf{\Phi}}$ to be

$$\tilde{\mathbf{\Phi}} = \begin{bmatrix} \boldsymbol{e} \\ \mathbf{\Phi} \end{bmatrix}, \tag{27}$$

where $\boldsymbol{e} = [1, 0, 0, \cdots, 0]$ is a $1 \times n$ vector and it satisfies

$$\begin{bmatrix} \hat{k}_{t-1} \\ \hat{\boldsymbol{s}}_t \end{bmatrix} = \tilde{\mathbf{\Phi}} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}. \tag{28}$$

The necessary and sufficient condition for the existence of (23) is as follows.

Lemma 3. *Assume that (20) and (28) hold in the extended detailed model. There exists* $\mathbf{\Lambda}$ *that satisfies (23) if and only if*

$$rank(\tilde{\mathbf{\Phi}}) = rank\left(\left[\tilde{\mathbf{\Phi}}' \vdots \mathbf{\Theta}' \right] \right). \tag{29}$$

Proof. See Appendix B. □

If $rank(\tilde{\mathbf{\Phi}}) = n$, the following Lemma 4 holds and provides us with some intuitions with regard to the existence of $\mathbf{\Lambda}$.

Lemma 4. *Assume that (19) and (20) hold in the extended detailed model. If the rank of* $\tilde{\mathbf{\Phi}}$ *equals the number of endogenous and exogenous states in the detailed model,* n*, there exists* $\mathbf{\Lambda}$ *that satisfies (23).*

Proof. Since $rank(\tilde{\mathbf{\Phi}}) = n$, there exists $(\tilde{\mathbf{\Phi}}'\tilde{\mathbf{\Phi}})^{-1}$ and

$$\begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} = \left(\tilde{\mathbf{\Phi}}'\tilde{\mathbf{\Phi}}\right)^{-1} \tilde{\mathbf{\Phi}}' \begin{bmatrix} k_{t-1} \\ \boldsymbol{s}_t \end{bmatrix}. \tag{30}$$

By (19) and (30), it is easily shown that $\mathbf{\Lambda} = \mathbf{\Theta}\left(\tilde{\mathbf{\Phi}}'\tilde{\mathbf{\Phi}}\right)^{-1} \tilde{\mathbf{\Phi}}'$ is a solution. □

It is easily shown that, in general, $\mathbf{\Lambda}$ does not exist if $rank(\tilde{\mathbf{\Phi}}) < n$ as in the case of $\boldsymbol{P}$. Roughly speaking, if the number of wedges plus one is strictly smaller than that of the endogenous and exogenous states in the detailed model or $n > m+1$, wedges cannot contain adequate information regarding the endogenous and exogenous states in general. Then, there is no mapping from $[\hat{k}_{t-1},\ \hat{\boldsymbol{s}}_t']'$ to consumption, investment, labor, output, and capital stock.

Finally, the necessary and sufficient condition for the equivalence is summarized as follow.

Theorem 1. *Assume that (19), (20), (21), and (28) hold in the extended detailed model. The prototype model with the VAR(1) specification of wedges is equivalent to the detailed model if and only if*

$$rank(\boldsymbol{\Phi}) = rank\left(\begin{bmatrix} \boldsymbol{\Phi}' \vdots \boldsymbol{\Psi}'\boldsymbol{\Phi}' \end{bmatrix} \right), \tag{31}$$

$$rank(\tilde{\boldsymbol{\Phi}}) = rank\left(\begin{bmatrix} \tilde{\boldsymbol{\Phi}}' \vdots \boldsymbol{\Theta}' \end{bmatrix} \right). \tag{32}$$

Proof. It is obvious by Lemma 1 and 3. □

The following sufficient condition is useful to understand the intuition.

Theorem 2. *Assume that (19), (20), (21), and (28) hold in the extended detailed model. The prototype model with the VAR(1) specification of wedges is equivalent to the detailed model if the rank of $\boldsymbol{\Phi}$ equals the number of endogenous and exogenous states in the detailed model, n.*

Proof. It is obvious by Lemma 2, 4 and that $rank(\boldsymbol{\Phi}) = n$ implies $rank(\tilde{\boldsymbol{\Phi}}) = n$. □

Theorem 2 implies the sufficient condition of the existence of the VAR(1) representation of wedges $rank(\boldsymbol{\Phi}) = n$ is the sufficient conditions of the existence of the mapping from the states of the prototype model to consumption, investment, labor, capital, and output in the extended detailed model. The prototype model is equivalent to a detailed model if the VAR(1) representation of wedges exists in many cases. This tells us that, in the case of VAR(1) of wedges, the condition (ii) is less restrictive. We will show that the condition (ii) is restrictive when we introduce an alternative specification of the process of wedges in Section 4.

3.2 Implications of Our Results

We showed that the VAR(1) representation of wedges exists if and only if wedges have sufficient information about the endogenous and exogenous states in the detailed model and that equivalence holds in many cases if the VAR(1) representation exists. We can roughly verify the condition for the existence of $\boldsymbol{P}$ by comparing the number of wedges m and that of endogenous and exogenous states n of the detailed model. If $rank(\boldsymbol{\Phi}) < n$, there is no $\boldsymbol{P}$ in general. If the detailed model is a medium-scale DSGE model of the current generation, which has many exogenous shocks and endogenous states, the equivalence results do not hold in many cases since the maximum number of wedges is at most four.

Even if the number of endogenous and exogenous states is small, a VAR(1) representation might not exist. Here, we show that the class of frictional models covered by the prototype model with the VAR(1) specification of wedges is much smaller than that is shown in CKM under general conditions of wedges.

The sticky-wage model in CKM is not covered by the associated prototype model. In their sticky-wage model, the endogenous state is aggregate capital k_{t-1} and the exogenous state is money supply M_t. In the associated prototype model, there is only one wedge – labor wedge, or $rank(\boldsymbol{\Phi}) = 1$. Then, according to Lemma 1, a VAR(1) representation does not exist in general since the number of wedges is strict smaller than that of endogenous and exogenous variables: $m < n$.

The input-financing friction model in CKM is also not covered by the associated prototype model. The endogenous state is aggregate capital k_{t-1}, and the exogenous states are the interest rate spreads of sector 1 and 2: $\tau_{1,t}$ and $\tau_{2,t}$ in the model with input-financing friction.[9] In the associated prototype model, there are three wedges – efficiency, labor, and capital wedges. Subsequently, in this case, $n = m$ in this case;

[9] CKM do not specify what are exogenous shocks. Here, we consider two interest rate spreads are exogenous shocks to close the model.

however CKM's Proposition 1 shows that $\tau_{\ell,t}$ equals $\tau_{k,t}$. This implies that

$$
\begin{aligned}
\boldsymbol{s}_t = \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} \\
\iff \begin{bmatrix} \log(A_t) \\ \tau_{\ell,t} \\ \tau_{k,t} \end{bmatrix} = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} \end{bmatrix} \begin{bmatrix} k_{t-1} \\ \tau_{1,t} \\ \tau_{2,t} \end{bmatrix},
\end{aligned} \tag{33}
$$

where $\phi_{2,i} = \phi_{3,i}$ for $i = 1, 2$, and 3. It is obvious that $rank(\boldsymbol{\Phi}) = 2 < 3 = n$ in general, and then there is no VAR(1) representation.[10] These results imply that equivalence result is highly restrictive under the VAR(1) specification of wedges.

Note that the equivalence depends on the whole structure of a detailed model, not only frictions. We show that CKM's sticky-wage model is not covered by the prototype model with the VAR(1) specification of wedges, however, this does not mean that all models with sticky-wage are not covered by the prototype model. For example, if the sticky-price model with only technology shocks, not money supply, the prototype model is equivalent to it since the number of endogenous and exogenous variables are two: capital stock and technology, and the number of wedges are two: efficiency and labor wedges.

4 Alternative Specification of Wedges

In Section 3, we consider the case of the VAR(1) specification of wedges. Here, we consider an alternative specification of wedges, which lets the prototype model cover larger class of frictional models than that in the case of the VAR(1) representation. We will show that the prototype model with our alternative specification can cover larger class of models but such class is not so large.

Let $\boldsymbol{\pi}_t$ is a $r \times 1$ vector of variables included in both the prototype model and the

[10] This non equivalence arises since the prototype model is described by capital wedge as in (13). The prototype model with investment wedge is equivalent to this input-financing friction model.

detailed model such that

$$\boldsymbol{\pi}_t = \boldsymbol{\Xi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix}, \tag{34}$$

where $\boldsymbol{\Xi}$ is an $r \times n$ matrix. Then, a candidate of variables included in $\boldsymbol{\pi}_t$ is jump or state variables at period t in the extended detailed model. Consider the following alternative specification:

$$\boldsymbol{s}_{t+1} = \boldsymbol{R}_0 + \boldsymbol{R}\boldsymbol{\pi}_t + \boldsymbol{\eta}_{t+1}, \tag{35}$$

where $\boldsymbol{R}_0$ is an $m \times 1$ vector, $\boldsymbol{R}$ is an $m \times r$ matrix, and $\boldsymbol{\eta}_{t+1}$ is i.i.d. over time and is normally distributed with mean zero. Note that the VAR(1) specification of wedges (11) is a special case of (35) where $\boldsymbol{\pi}_t = \boldsymbol{s}_t$. The linearized version of (35) is

$$\hat{\boldsymbol{s}}_{t+1} = \boldsymbol{R}\hat{\boldsymbol{\pi}}_t + \boldsymbol{\eta}_{t+1}. \tag{36}$$

Even if we do not specify variables in $\boldsymbol{\pi}_t$, we can discuss about the existence of (36) by (34). The necessary and sufficient condition is as follows.

Lemma 5. *Assume that (20), (21) and (34) hold in the extended detailed model. There exists $\boldsymbol{R}$ that satisfies our specification (36) if and only if*

$$rank(\boldsymbol{\Xi}) = rank\left(\begin{bmatrix} \boldsymbol{\Xi}' \vdots \boldsymbol{\Psi}'\boldsymbol{\Phi}' \end{bmatrix} \right). \tag{37}$$

Proof. It is easily shown that there exists $\boldsymbol{R}$ that satisfies $\boldsymbol{R\Xi} = \boldsymbol{\Psi\Phi}$ if and only if (37) holds. The rest of the proof is almost similar to that in the proof of Lemma 1. Necessity and sufficiency are easily shown. □

Lemma 5 is the analogue of Lemma 1. The analogue of Lemma 2 is as follows.

Lemma 6. *Assume that (20), (21), and (34) hold in the extended detailed model. If the rank of $\boldsymbol{\Xi}$ equals the number of endogenous and exogenous states in the detailed model,* *n, there exists a unique $\boldsymbol{R}$ that satisfies the alternative specification (36).*

Proof. It is easily shown by the same logic in the proof of Lemma 2. □

These lemmas imply that our specification (35) holds if and only if $\boldsymbol{\pi}_t$ has sufficient information about the endogenous and exogenous states in the detailed model. Roghly speaking, if the number of π_t is larger than that of the endogenous and exogenous states in the detailed model, there exists (35).

The rest is the condition (ii) in Proposition 2. To consider it, we have to specify $\boldsymbol{\pi}_t$.

Specification of Bäurle and Burren (2007): A candidate of the specification of $\boldsymbol{\pi}_t$ is as follows.

$$\boldsymbol{\pi}_t \equiv \begin{bmatrix} \overline{k}_{t-1} \\ \boldsymbol{s}_t \end{bmatrix}. \tag{38}$$

where $\overline{k}_{t-1}$ is the "aggregate" capital stock and $\overline{k}_{t-1} = k_{t-1}$ in equilibrium. This "aggregate" trick is employed in order to let the first order conditions of the prototype model to be same (6), (7), (8), (9), and (10). This specification of $\boldsymbol{\pi}_t$ is the same as that proposed in Bäurle and Burren (2007).

Since, under the specification (36), the state variables of the prototype model is $[\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ and the condition (ii) in Proposition 1 is the same as (23). Therefore, Lemma 3 and 4 is applicable. In the case of this alternative specification, the necessary and sufficient condition for the equivalence is summarized as follow.

Theorem 3. *Assume that (19), (20), (21), and (28) hold in the extended detailed model. The prototype model with the specification of wedges :*

$$\boldsymbol{s}_{t+1} = \boldsymbol{R}_0 + \boldsymbol{R}\boldsymbol{\pi}_t + \boldsymbol{\eta}_{t+1},$$

where

$$\boldsymbol{\pi}_t \equiv \begin{bmatrix} \bar{k}_{t-1} \\ \boldsymbol{s}_t \end{bmatrix}$$

is equivalent to the detailed model if and only if

$$rank(\boldsymbol{\Xi}) = rank\left(\left[\boldsymbol{\Xi}' \vdots \boldsymbol{\Psi}'\boldsymbol{\Phi}' \right] \right), \tag{39}$$

$$rank(\tilde{\boldsymbol{\Phi}}) = rank\left(\left[\tilde{\boldsymbol{\Phi}}' \vdots \boldsymbol{\Theta}' \right] \right). \tag{40}$$

Proof. It is obvious by Lemma 3 and 5. □

In this case of the specification, $\boldsymbol{\Xi} = \tilde{\boldsymbol{\Phi}}$ and the sufficient condition for the existence of the alternative specification (36), $\boldsymbol{\Xi}$, implies the condition (23): the mapping from the states of the prototype model to variables. Thus, the equivalence holds if the existence of the alternative specification (36) exists in many cases.

Since there are $m + 1$ variables in $\boldsymbol{\pi}_t$, this alternative specification exists in the case where the number of endogenous and exogenous states is smaller than $m + 1$ in the detailed model. It is easily verified that the prototype model with this specification covers CKM's sticky-wage model and the input-financing friction model shown in Section 3.

Note that the equivalence depends on the whole structure of a detailed model, not only frictions as explained in Section 3. CKM's sticky-wage model is not covered by the prototype model with alternative specification of wedges (36) and (38), however, it does not mean that all sticky-wage models are covered. The equivalence depends on the number of independent state variables in the detailed model. Note also that the prototype model with this specification, generally, can only cover a detailed model in which the number of the state variables is less than five. Medium-scale DSGE models in the current generation, like Christiano, Eichenbaum, and Evans (2005), cannot be covered since there are many endogenous and exogenous variables.

Even if we augment $\boldsymbol{\pi}_t$... Consider the case where there are more variables in $\boldsymbol{\pi}_t$. For example,

$$\boldsymbol{\pi}_t \equiv \begin{bmatrix} \overline{k}_{t-1} \\ \boldsymbol{s}_t \\ \overline{c}_t \end{bmatrix}, \tag{41}$$

where $\overline{c}_t$ is aggregate consumption and $c_t = \overline{c}_t$ in equilibrium. We augment the previous specification of $\boldsymbol{\pi}_t$ by $\overline{c}_t$.

The alternative specification exists even if the number of the endogenous and exogenous wedges is six, which is the specification of Bäurle and Burren (2007) does not exist. However, the class of detailed models covered by the prototype model with this specification is not larger than that covered by the prototype model with the specification of Bäurle and Burren (2007) since the condition (ii) in Proposition 1 is restrictive. In this specification, the condition (ii) in Proposition 1 is the same as (23). It is described as in Proposition 2.

Proposition 2. *If a detailed model is covered by the prototype model with the alternative specification (35) and $\boldsymbol{\pi}_t \equiv [\overline{k}_{t-1}, \boldsymbol{s}_t', \overline{c}_t]'$, it is also covered by the prototype model with (35) and $\boldsymbol{\pi}_t \equiv [\overline{k}_{t-1}, \boldsymbol{s}_t']'$.*

Proof. Assume that a detailed model is covered by the prototype model with (36) and (41). Then, (23) and

$$\hat{\boldsymbol{s}}_{t+1} = \boldsymbol{R} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{\boldsymbol{s}}_t \\ \hat{c}_t \end{bmatrix} + \boldsymbol{\eta}_t \tag{42}$$

hold. Since there is a mapping from $[\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ to $\hat{c}_t$, (42) is rewritten as

$$\hat{\boldsymbol{s}}_{t+1} = \tilde{\boldsymbol{R}} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{\boldsymbol{s}}_t \end{bmatrix} + \boldsymbol{\eta}_t. \tag{43}$$

(23) and (43) imply that this detailed model is covered by the prototype model with (36) and (41). □

The analogue of Proposition 2 holds if we augment $\boldsymbol{\pi}_t$ by other variables in the prototype model. Therefore, generally, the number independent endogenous and exogenous variables in detailed models covered by the prototype model with (35) is less than six and the class of detailed models is not so large.[11]

5 Business Cycle Accounting without Equivalence

We showed that equivalence results is highly restrictive and the class of models covered by the prototype model with the VAR(1) specification is small in previous two sections. However, there is a possibility that the prototype model works well as an approximation of the detailed model. In this section, we investigate what happens if we apply BCA to an economy where the equivalence result does not hold in order to assess the usefulness of BCA.[12] We provide an example that measured wedges capture the properties of the true wedges almost correctly even if the equivalence does not hold.

5.1 An artificial economy

We employ a kind of medium-scale DSGE model as in Christiano, Eichenbaum, and Evans (2005).[13] The reason why we employ medium-scale DSGE model is that such model has rich enough dynamics to produce the actual tendency found in the data and we already know the empirically plausible range of parameters in this sort of models. So, using this model, we can investigate the empirical usefulness of BCA under plausible parameter values in a realistic economy.

[11] It is well-known that the VARMA representation exists under more general conditions than VAR(1) by the VAR literature as in Ravenna (2007). One of our future tasks is to investigate the case of the prototype model with the VARMA representation of wedges.

[12] In the literature of the structural VARs, Ravenna (2007) and Christiano, Eichenbaum, and Vigfusson (2006) investigate the empirical usefulness of VARs by applying VARs to artificial economies.

[13] The details about our model and our parameter values are described in Appendix C.

Our model has the following properties: (i) sticky price with backward price indexation, (ii) sticky wage with backward price indexation, (iii) habit persistence, and (iv) forward-looking Taylor rule. In our model, there are seven independent endogenous and exogenous state variables and both prototype models with VAR(1) specification and alternative specification with $\boldsymbol{\pi}_t \equiv [\hat{k}_{t-1}, \hat{\boldsymbol{s}}_t']'$ cannot cover this model. We eliminate adjustment cost of investment since there is no adjustment cost in the prototype model for BCA in Section 2. [14] The parameter values of our medium-scale DSGE model are described in Table 1. We specify the model to be quarterly. We employ the log-linear approximation to calculate the aggregate decision rule and generate 200 periods (50 years) artificial data.

5.2 Business cycle accounting

Our procedure of BCA follows the standard way. We set the parameter values of the prototype model as follows: discount factor β, depreciation rate of capital δ, and share of capital in production α to be the same as in our detailed model, and the weight of leisure in utility ν to be 2.[15] We employ the Bayesian method based on the Kalman filter to estimate the process of wedge $\boldsymbol{P}$.[16] Table 2 shows prior distributions of the parameters and Table 3 shows posterior distributions. Given the evolution of wedges $\boldsymbol{P}$, wedges are measured by

$$\begin{bmatrix} \hat{i}_t^d \\ \hat{\ell}_t^d \\ \hat{y}_t^d \end{bmatrix} = \tilde{\Lambda} \begin{bmatrix} \hat{k}_{t-1}^d \\ \hat{\boldsymbol{s}}_t \end{bmatrix}, \tag{44}$$

$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{i}_t^d$, $\hat{k}_{-1} = \hat{k}_{-1}^d$, and $\hat{g}_t = \hat{g}_t^d$, where variables with superscript d denote actual data and $\tilde{\Lambda}$ is 3×5 matrix. There are three equations and three unknown wedges

[14] While the adjustment costs of investment is one of important feature in the medium-scale DSGE models, we eliminate it. To investigate the case of the prototype model with adjustment costs is our future task.

[15] We do so since the utility function in our detailed model is different from that of the prototype model.

[16] Estimations are performed using Dynare (http://www.cepremap.cnrs.fr/dynare/). The detail about the estimation is described in Appendix D.

– A_t, $\tau_{\ell,t}$, and $\tau_{x,t}$, in (44).

We apply BCA to our medium-scale DSGE economy using the prototype model with the VAR(1) specification of wedges. Figure 6 shows the true and measured investment wedges $1/(1+\tau_{x,t})$. The true wedges are generated by the extended detailed model. We do not show other wedges since they are measured correctly by the intratemporal conditions of the prototype model. The estimated process of wedge $\boldsymbol{P}$ affects only the measurement of the investment wedge. The measured investment wedge looks to be close to the true one. Table 4 reports the cyclical behavior of the true and measured investment wedges: means, standard deviations, autocorrelations, correlations with current output, and correlations the true current investment wedge. Measured investment wedge is more volatile and persistent than the true one and the correlation between measured investment wedge and output is smaller than that between the true and output. However, differences are small. Therefore, the bias of measured investment is not so large in this case and measured wedges capture the properties of the true wedges almost correctly[17]

6 Conclusion

Business cycle accounting rests on the insight that the prototype neoclassical growth model with time-varying wedges can achieve the same allocation generated by a large class of frictional models: equivalence results. Equivalence results are shown under general conditions about the process of wedges while it is often specified to be the first order vector autoregressive when one applies business cycle accounting to actual data.

In this paper, we characterized the class of frictional models covered by the prototype model under the conventional specification and found that it is much smaller than that believed in previous literature. We also show that even if we employ alternative specification of the process of wedges, the class covered by the prototype model is not so large. We also applied BCA to an medium-scale DSGE economy where the equivalence does not hold and provided an example that measured wedges capture the properties

[17] Of course, this is just an example. To investigate the statistical properties of the true and measured investment wedges, we have to apply BCA to sufficiently many sample paths. It is our future task.

of the true wedges almost correctly even in such an economy. This result tells us that equivalence results is highly restrictive from the theoretical view, however BCA might work well empirically since the prototype model is a good approximation of the detailed model.

Appendix A: Proof of Lemma 1

Proof. There exist $\boldsymbol{P}$ that satisfies $\boldsymbol{\Phi\Psi} = \boldsymbol{P\Phi}$ if and only if (24) holds according to the standard knowledge of linear algebra. In the rest of the proof, we elaborate on sufficiency and then subsequently, on necessity.

(Sufficiency) Assume that there exists $\boldsymbol{P}$ that satisfies $\boldsymbol{\Phi\Psi} = \boldsymbol{P\Phi}$. By (20), at period $t+1$,

$$\begin{aligned}
\boldsymbol{s}_{t+1} &= \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{z}_{t+1} \end{bmatrix} \\
&= \boldsymbol{\Phi\Psi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix} \\
&= \boldsymbol{P\Phi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix} \\
&= \boldsymbol{P}\boldsymbol{s}_t + \varepsilon_{t+1},
\end{aligned} \tag{45}$$

where

$$\varepsilon_{t+1} \equiv \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix}. \tag{46}$$

(Necessity) Assume that there exists $\boldsymbol{P}$ that satisfies CKM's specification (22). (22) and (20) imply that

$$\boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{z}_{t+1} \end{bmatrix} = \boldsymbol{P\Phi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \varepsilon_{t+1}. \tag{47}$$

(21) and (47) imply that

$$\boldsymbol{\Phi\Psi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \boldsymbol{\Phi} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{u}_{t+1} \end{bmatrix} = \boldsymbol{P\Phi} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} + \varepsilon_{t+1}. \tag{48}$$

Since $E_t[\boldsymbol{\varepsilon}_{t+1}] = \mathbf{0}$ and (48) must hold for any $\boldsymbol{x}_{t-1}$, $\boldsymbol{z}_t$ and $\boldsymbol{u}_{t+1}$, (46) and

$$\boldsymbol{\Phi\Psi} = \boldsymbol{P\Phi} \tag{49}$$

hold. □

Appendix B: Proof of Lemma 3

Proof. There exists $\boldsymbol{\Lambda}$ that satisfies $\boldsymbol{\Lambda}\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Theta}$ if and only if (37) holds. In the rest of the proof, we elaborate on sufficiency and then subsequently, on necessity.

(Sufficiency) Assume that there exists $\boldsymbol{\Lambda}$ that satisfies $\boldsymbol{\Lambda}\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Theta}$. (19) becomes

$$\begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{\ell}_t \\ \hat{y}_t \\ \hat{k}_t \end{bmatrix} = \boldsymbol{\Lambda}\tilde{\boldsymbol{\Phi}} \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} \tag{50}$$

$$= \boldsymbol{\Lambda} \begin{bmatrix} \hat{k}_{t-1} \\ \hat{\boldsymbol{s}}_t \end{bmatrix}. \tag{51}$$

(Necessity) Assume that (23). By (19) and (28),

$$\left(\boldsymbol{\Lambda}\tilde{\boldsymbol{\Phi}} - \boldsymbol{\Theta}\right) \begin{bmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{z}_t \end{bmatrix} = \mathbf{0}. \tag{52}$$

Therefore, $\boldsymbol{\Lambda}\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Theta}$ must hold. □

Appendix C: Our Medium-Scale DSGE Model

In this appendix, we present our medium-scale DSGE model used in the investigation of BCA. Our model has the following properties as in Christiano, Eichenbaum, and Evans (2005): (i) sticky price with backward price indexation, (ii) sticky wage with backward price indexation, (iii) habit persistence, and (iv) forward-looking Taylor rule. We employ the Rotemberg adjustment costs for sticky price and wage, not Calvo-pricing. However, this does not matter since both specification is approximately equivalent in a linearized economy.[18] There are seven independent endogenous and exogenous state variables.[19] Therefore, Theorem 1 does not hold and the prototype model cannot cover it.

C.1 Firm

There are competitive final-goods firms and monopolistic competitive intermediate-goods firms. The production function of final-goods firms is as follows:

$$y_t = \left[\int_0^1 Y_t(z)^{\frac{1-\theta_p}{\theta_p}} dz \right]^{\frac{\theta_p}{1-\theta_p}}. \tag{53}$$

The profit maximization of final-goods firms implies demand function of intermediate goods indexed by z:

$$Y_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\theta_p} y_t. \tag{54}$$

The production function of intermediate-goods firm indexed by z is

$$Y_t(z) = A_t K_t(z)^{\alpha} L_t(z)^{1-\alpha}, \tag{55}$$

and the productivity A_t evolves according to

$$\log(A_{t+1}) = \rho_A \log(A_t) + (1 - \rho_A) \log(A) + \sigma^A \varepsilon_{t+1}^A. \tag{56}$$

where A denotes the steady-state of A_t; σ^A, the standard deviation of technology shocks; and ε_{t+1}^A, i.i.d. shock with standard normal distribution. The cost minimization of

[18] We employ the log-linear approximation to calculate the aggregate decision rule.

[19] The five endogenous states are consumption c, capital k, nominal interest rate R, inflation rate π, and wage rate w and the three exogenous states are technology A, government consumption g, and monetary policy shock ε_t^R. However, nominal interest rate is not independent from monetary policy shock because of the Taylor rule.

intermediate-goods firms implies

$$r_t = \alpha \cdot mc_t \cdot \frac{Y_t(z)}{K_t(z)}, \tag{57}$$

$$w_t = (1-\alpha) \cdot mc_t \cdot \frac{Y_t(z)}{L_t(z)}. \tag{58}$$

The Rotemberg adjustment costs which is approximately equivalent to the Calvo-pricing with price indexation is

$$\frac{\phi_p}{2}\left[\frac{P_t(z)}{P_{t-1}(z)}\Big/\left(\pi^{1-\xi_p}\pi_{t-1}^{\xi_p}\right)-1\right]^2 y_t, \tag{59}$$

where $\pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. Finally, intermediate-goods firm indexed by z sets price such that it is a solution to

$$\max_{P_t(z)} \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0}\left\{\left(\frac{P_t(z)}{P_t}\right)^{1-\theta_p} - mc_t\left(\frac{P_t(z)}{P_t}\right)^{-\theta_p} - \frac{\phi_p}{2}\left[\frac{P_t(z)}{P_{t-1}(z)}\Big/\left(\pi^{1-\xi_p}\pi_{t-1}^{\xi_p}\right)-1\right]^2 y_t\right\}. \tag{60}$$

C.2 Households

The utility function with habit persistence of household indexed by i is

$$U = \sum_{t=0}^{\infty} \beta^t \left[\log\left(c_t(i) - bc_{t-1}(i)\right) - \psi_\ell \frac{\ell_t(i)^{\sigma_\ell+1}}{\sigma_\ell+1}\right], \tag{61}$$

where $b > 0$. The evolution of capital stock is

$$k_t(i) = (1-\delta)k_{t-1}(i) + x_t(i). \tag{62}$$

Households have differentiated labor as endowment and they have a power to offer nominal wage. The labor demand is

$$\ell_t(i) = \left(\frac{W_t(i)}{W_t}\right)^{-\theta_w} L_t, \tag{63}$$

where W_t denotes nominal wage rate. The Rotemberg adjustment costs of nominal wage which is approximately equivalent to the Calvo-pricing with price indexation is

$$\frac{\phi_w}{2}\left[\frac{W_t(i)}{W_{t-1}(i)}\Big/\left(\pi^{1-\xi_w}\pi_{t-1}^{\xi_w}\right)-1\right]^2\frac{W_t}{P_t}\ell_t. \tag{64}$$

Finally, the budget constraint is

$$\begin{aligned}c_t(i)+x_t(i)+\frac{B_t(i)}{P_t}&=R_{t-1}\frac{B_{t-1}(i)}{P_t}+r_tk_{t-1}(i)\\&+\frac{W_t(i)}{P_t}\ell_t(i)\left\{1-\frac{\phi_w}{2}\left[\frac{W_t(i)}{W_{t-1}(i)}\Big/\left(\pi^{1-\xi_w}\pi_{t-1}^{\xi_w}\right)-1\right]^2\right\}-T_t+\Omega_t.\end{aligned} \tag{65}$$

C.3 Policy

The government consumption g_t is AR(1):

$$\log(g_{t+1})=\rho_g\log(g_t)+(1-\rho_g)\log(g)+\sigma^g\varepsilon_{t+1}^g, \tag{66}$$

where g denotes the steady-state of g_t; σ^g, standard deviation of fiscal policy shock; and ε_{t+1}^g, i.i.d. shock with standard normal distribution. The monetary authority follows the forward looking Taylor rule:

$$\hat{R}_t=\rho_R\hat{R}_{t-1}+(1-\rho_R)\left[\rho_\pi E_t\left\{\hat{\pi}_{t+1}\right\}+\rho_y\hat{y}_t\right]+\sigma^R\varepsilon_t^R, \tag{67}$$

where variables with the hat ˆ denotes the log-deviations from the steady-state.

C.4 Resource constraint

The resource constraint is

$$\begin{aligned}c_t+x_t+g_t=y_t&\left\{1-\frac{\phi_p}{2}\left[\pi_t\Big/\left(\pi^{1-\xi_p}\pi_{t-1}^{\xi_p}\right)-1\right]^2\right\}\\&-\frac{\phi_w}{2}\left[\pi_{w,t}\Big/\left(\pi^{1-\xi_w}\pi_{t-1}^{\xi_w}\right)-1\right]^2\frac{W_t}{P_t}\ell_t,\end{aligned} \tag{68}$$

where $\pi_{w,t}=W_t/W_{t-1}$.

Appendix D: Estimation of the Process of Wedges

Here, we explain how to estimate the process of wedges:

$$\hat{\boldsymbol{s}}_{t+1} = \boldsymbol{P}\hat{\boldsymbol{s}}_t + \boldsymbol{\varepsilon}_{t+1}.$$

We estimate the parameters of process of wedges $\boldsymbol{P}$ and the lower triangular matrix $\boldsymbol{Q}$ to ensure that our estimate of $\boldsymbol{\Sigma}$ is positive semidefinite, where $\boldsymbol{\Sigma} = \boldsymbol{Q}\boldsymbol{Q}'$ is a variance covariance matrix of error term. The first step of the estimation is to describe the log-linearized solution of prototype model as a state-space form and define the maximum likelihood function based on the Kalman filter for the estimation. Then, the parameters of wedges process are estimated by the maximum likelihood estimation to fit observations in the log-linearized prototype model to the observations in the actual economy.

Estimations are performed using Dynare. The parameters of the wedge process in the prototype model are estimated using the Bayesian method. We employ the Metropolis-Hastings algorithm in Chib and Greenberg (1995) to draw 100,000 sample draws from the posterior distribution[20]. The first half draws are discarded. The prior distributions of parameters are as follows. The prior for the (i, i) elements of the coefficient matrix of the process of wedges follows a beta distribution. The prior for the (i, j) elements, $i \neq j$, of the coefficient matrix of the process of wedges follows a normal distribution. The prior for diagonal terms of matrix $\boldsymbol{Q}$ follows an uniform distribution. The prior for non-diagonal terms of matrix $\boldsymbol{Q}$ follows an normal distribution. The prior for the steady state values of wedges follows a normal distribution . The prior distributions are summarized in Table 2 and the posterior distributions are summarized in Table 3.

[20] Otsu (2007) and Saijo (2008) apply the Bayesian estimation method to applying BCA. The first half draws are discarded. An and Schorfheide (2007) review Bayesian methods to estimate and evaluate dynamic stochastic general equilibrium models.

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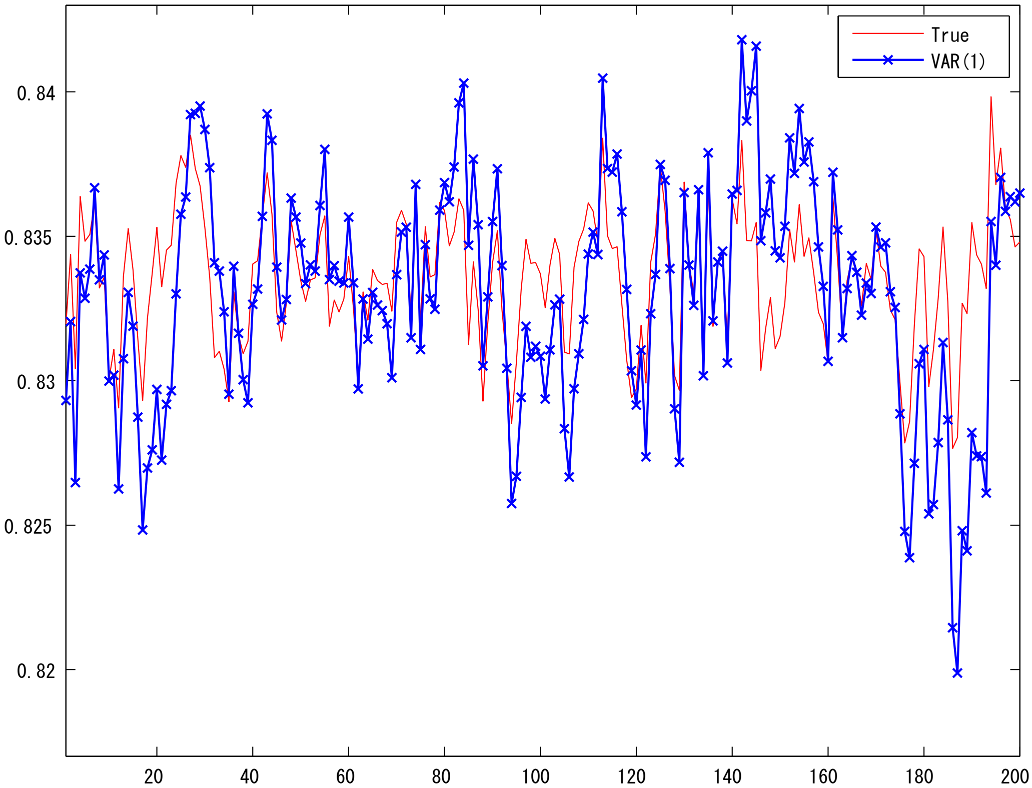
Figure 1: True and measured investment wedges



Notes: The red solid line is the true wedge generated by the extended detailed model. The blue line with x-mark is the measured wedge by the prototype model with the VAR(1) specification of wedges.

parameter	symbol	value
discount factor of households	β	.99
Frish elasticity of labor substitution	σ_ℓ	1
habit persistence	b	.63
steady-state labor supply	ℓ	.3
depreciation rate of capital	δ	.025
Rotemberg adjustment cost of price	ϕ_p	30
Rotemberg adjustment cost of wage	ϕ_w	30
indexation of price	ξ_p	1
indexation of wage	ξ_w	1
persistence of technology level	ρ_A	.95
steady-state technology level	A	1
standard deviation of technology shock	σ^A	.01/4
persistence of government consumption	ρ_g	.95
standard deviation of g shock	σ^g	.01/4
steady-state ratio of government consumption	g/y	.1
steady-state gross inflation	π	1
steady-state markup of price	$\frac{\theta_p}{\theta_p-1}$	1.2
steady-state markup of wage	$\frac{\theta_w}{\theta_w-1}$	1.2
persistence of nominal interest rate	ρ_R	.8
weight of inflation in Taylor rule	ρ_π	2
weight of output in Taylor rule	ρ_y	.2
standard deviation of monetary policy shock	σ^R	.01/4

Table 1: Parameter values of our medium-scale DSGE model

Name	Domain	Prior density	Parameter (1)	Parameter (2)
parameters				
$\boldsymbol{P}(1,1)$	[0,1)	Beta	.8	.1
$\boldsymbol{P}(2,2)$	[0,1)	Beta	.8	.1
$\boldsymbol{P}(3,3)$	[0,1)	Beta	.8	.1
$\boldsymbol{P}(4,4)$	[0,1)	Beta	.8	.1
$\boldsymbol{P}(1,2)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(1,3)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(1,4)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(2,1)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(2,3)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(2,4)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(3,1)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(3,2)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(3,4)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(4,1)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(4,2)$	$\mathbb{R}$	Normal	0	10
$\boldsymbol{P}(4,3)$	$\mathbb{R}$	Normal	0	10
$\log(A)$	$\mathbb{R}$	Normal	0	.0069
$\log(1-\tau_\ell)$	$\mathbb{R}$	Normal	-.082	.1496
$\log\left(\frac{1}{1-\tau_x}\right)$	$\mathbb{R}$	Normal	.189	.0027
$\log(g)$	$\mathbb{R}$	Normal	-2.298	.0078
standard deviations of shocks				
ε_A	$\mathbb{R}^+$	Uniform	0	.2
ε_ℓ	$\mathbb{R}^+$	Uniform	0	.2
ε_x	$\mathbb{R}^+$	Uniform	0	.2
ε_g	$\mathbb{R}^+$	Uniform	0	.2
correlations of shocks				
$(\varepsilon_A, \varepsilon_\ell)$	[-1,1]	Normal	0	.3
$(\varepsilon_A, \varepsilon_x)$	[-1,1]	Normal	0	.3
$(\varepsilon_A, \varepsilon_g)$	[-1,1]	Normal	0	.3
$(\varepsilon_\ell, \varepsilon_x)$	[-1,1]	Normal	0	.3
$(\varepsilon_\ell, \varepsilon_g)$	[-1,1]	Normal	0	.3
$(\varepsilon_x, \varepsilon_g)$	[-1,1]	Normal	0	.3

Notes: Parameter (1) and (2) list the means and the standard deviations for beta and normal distributions and the upper and lower bounds of the support for the uniform distributions. $\boldsymbol{P}(i,j)$ is the (i,j) component of $\boldsymbol{P}$ and $\boldsymbol{\varepsilon} \equiv [\varepsilon_A, \varepsilon_\ell, \varepsilon_x, \varepsilon_g]'$ denotes the error term.

Table 2: Prior distributions of the prototype model with the VAR(1) specification of wedges

Name	Posterior means	90 % confidence intervals
parameters		
$\boldsymbol{P}(1,1)$	.9137	[.9106 , .9149]
$\boldsymbol{P}(2,2)$	.205	[.2004 , .2103]
$\boldsymbol{P}(3,3)$	.7448	[.7448 , .7472]
$\boldsymbol{P}(4,4)$	.9299	[.9239 , .9386]
$\boldsymbol{P}(1,2)$	-.0033	[-.0035 , -.0027]
$\boldsymbol{P}(1,3)$	-.1024	[-.1027 , -.1017]
$\boldsymbol{P}(1,4)$	-.0014	[-.0044 , -.0003]
$\boldsymbol{P}(2,1)$	6.8071	[6.813 , 6.8196]
$\boldsymbol{P}(2,3)$	17.859	[17.8471, 17.8576]
$\boldsymbol{P}(2,4)$	-2.5094	[-2.5229, -2.4901]
$\boldsymbol{P}(3,1)$	-.0881	[-.0881 , -.0881]
$\boldsymbol{P}(3,2)$	-.0019	[-.0019 , -.0018]
$\boldsymbol{P}(3,4)$	.0552	[.0548 , .0552]
$\boldsymbol{P}(4,1)$	.007	[.0072 , .0075]
$\boldsymbol{P}(4,2)$	.0016	[.0015 , .0017]
$\boldsymbol{P}(4,3)$	.0391	[.0391 , .0393]
$\log(A)$	.0036	[.0036, .0037]
$\log(1-\tau_\ell)$	-.094	[-.0942, -.094]
$\log\left(\frac{1}{1-\tau_x}\right)$	.1833	[.1833, .1833]
$\log(g)$	-2.2988	[-2.2988, -2.2988]
standard deviations of shocks		
ε_A	.0027	[.0027, .0029]
ε_ℓ	.1263	[.1182, .1338]
ε_x	.0033	[.003, .0035]
ε_g	.0026	[.0027, .0027]
correlations of shocks		
$(\varepsilon_A, \varepsilon_\ell)$	-.0728	[-.0849 ,-.0632]
$(\varepsilon_A, \varepsilon_x)$	-.2222	[-.2341, -.207]
$(\varepsilon_A, \varepsilon_g)$	-.0833	[-.0891, -.0413]
$(\varepsilon_\ell, \varepsilon_x)$	-.9491	[-.9493, -.9488]
$(\varepsilon_\ell, \varepsilon_g)$	.1512	[.153, .1812]
$(\varepsilon_x, \varepsilon_g)$	-.0979	[-.1391, -.0922]

Notes: $\boldsymbol{P}(i,j)$ is the (i,j) component of $\boldsymbol{P}$ and $\boldsymbol{\varepsilon} \equiv [\varepsilon_A, \varepsilon_\ell, \varepsilon_x, \varepsilon_g]'$ denotes the error term.

Table 3: Posterior means and confidence intervals of the prototype model with the VAR(1) specification of wedges

	mean	std	autocorr.	corr w/ y_t	corr w/ true
true	.8335	.0023	.4857	.7254	1
VAR(1)	.8030	.0039	.7170	.3649	.7091

Notes: Means, standard deviations, autocorrelations, correlations with current output, and correlations with the true investment wedge are reported.

Table 4: Cyclical behavior of true and measured investment wedges