RIETI Discussion Paper Series 06-E-010

Transaction Services and Asset-price Bubbles (Revised)

KOBAYASHI Keiichiro
RIETI

Research Institute of Economy, Trade & Industry, IAA
The Research Institute of Economy, Trade and Industry
http://www.rieti.go.jp/en/

RIETI Discussion Paper Series 06-E-010

Transaction services and asset-price bubbles*

(Revised)

Keiichiro Kobayashi†

November 10, 2005 (First draft: June 20, 2004)

Abstract

This paper examines asset-price bubbles in an economy where a nondepletable asset (e.g., land) can provide transaction services, using a variant of the cash-in-advance model.

When a landowner can borrow money immediately using land as collateral, one can say that land essentially provides a transaction service.

The transaction services that such an asset can provide increase as its price rises, since the asset owner can borrow more money against the asset's increased value.

Thus an asset-price bubble can emerge due to the externality of *self-reference*, wherein the asset price reflects the transaction services that it can provide, while the amount of the transaction services reflects the asset price.

If the collateral ratio of the asset (θ) and money supply (m) are not very large, a steady state equilibrium exists where the asset price has a bubble component and resource allocation is inefficient; if θ and/or m become large, the bubble component of the asset price vanishes and the equilibrium allocation becomes efficient.

The paper shows that in the case where the equilibrium concept is relaxed to allow for sticky prices and a temporary supply-demand gap, an equilibrium exists where a bubble develops temporarily and eventually bursts.

---

* I am grateful to Toni Braun and anonymous referees for helpful comments. All remaining errors are mine.

† Research Institute of Economy, Trade and Industry,
e-mail: kobayashi-keiichiro@rieti.go.jp

Transaction services and asset-price bubbles*

(Incomplete and preliminary)

Keiichiro Kobayashi

Fellow, Research Institute of Economy, Trade and Industry.

1-3-1 Kasumigaseki, Chiyoda-ku, Tokyo 100-8901, JAPAN

+81-3-3501-8308 (phone), +81-3-55101-3927 (fax)

E-mail: kobayashi-keiichiro@rieti.go.jp

November 10, 2005 (First draft: June 20, 2004)

Abstract

This paper examines asset-price bubbles in an economy where a nondepletable asset (e.g., land) can provide transaction services, using a variant of the cash-in-advance model. When a landowner can borrow money immediately using land as collateral, one can say that land essentially provides a transaction service. The transaction services that such an asset can provide increase as its price rises, since the asset owner can borrow more money against the asset's increased value. Thus an asset-price bubble can emerge due to the externality of *self-reference*, wherein the asset price reflects the transaction services that it can provide, while the amount of the transaction services reflects the asset price. If the collateral ratio of the asset (θ) and money supply (m) are not very large, a steady state equilibrium exists where the asset price has a bubble component and resource allocation is inefficient; if θ and/or m become large, the bubble component of the asset price vanishes and the equilibrium allocation becomes efficient. The paper shows that in the case where the equilibrium

*I am grateful to Toni Braun and anonymous referees for helpful comments. All remaining errors are mine.

concept is relaxed to allow for sticky prices and a temporary supply-demand gap, an equilibrium exists where a bubble develops temporarily and eventually bursts.

1 Introduction

An asset that is easily exchanged for money can be said to work as a de facto medium of exchange, just like money itself. In other words, the asset can provide transaction services. This paper is a theoretical study of the deviation of an asset price from its fundamental value when the asset can provide transaction services as a medium of exchange. The basic idea can be roughly described as follows: Suppose that there exists a nondepletable asset (land) and that the landowner can obtain money immediately by borrowing from banks using the land as collateral. If the price of the asset is Q_t, it can be plausibly assumed that the amount of money the owner of one unit of land can borrow from a bank is weakly increasing in $Q_t\theta_t$, where θ_t $(0 \leq \theta_t < 1)$ is a parameter representing the collateral ratio of the asset, which may be exogenously given or may be an equilibrium outcome determined by the inefficiencies of the real estate market. Therefore, the amount of transaction services (L_t) that the asset can provide can be expressed as

$$L_t = M(q_t\theta_t),$$

where $q_t \equiv \frac{Q_t}{P_t}$ is the real asset price, P_t is the general price level, and $M(\cdot)$ is a weakly increasing function. At the same time, the real price of the asset is determined as a discounted sum of the flow of dividends that the land yields and the flow of the value of the transaction services that it provides. The land price can be expressed as

$$q_t = \sum_{s=t}^{\infty} \{y_s + g_s(L_s)\},$$

where y_s is the present value of the dividend at date s as of date t and $g_s(L_s)$ is the present value of the transaction services L_s at date s as of date t. For simplicity, let us focus on the steady state where we can omit time subscripts. In the steady state, the transaction services L and the real asset price q are determined by

$$L = M(q\theta) \text{ and } q = Q(L), \tag{1}$$

where $Q(L)$ is an increasing function of L. As Figure 1 shows, the L^* that solves (1) may be positive.

Figure 1. Land prices and the transaction services

Thus, in the equilibrium, the asset may provide a positive amount of transaction services L^* and its price may become $q^* = Q(L^*)$, which is higher than the fundamental price of the asset $Q(0)$. The difference $Q(L^*) - Q(0)$ can be regarded as the "bubble" component of the asset price.[1] The bubble is generated by a particular type of externality, or a *self-reference* in the transaction services that the asset can provide: An increase in the asset price results in an increase in the transaction services that the asset can provide, since the asset is exchangeable for more money; and the increase in transaction services enhances the value of the asset, causing a further increase in the asset price. Thus the amount of the transaction services that the asset can provide reflects the asset price, which reflects, in turn, the transaction services.

There is a considerable amount of literature on asset-price bubbles (see Camerer [1989] for a survey of rational growing bubbles, fads, and information bubbles). Examples of recent theoretical developments are Allen and Gale (2000), in which information asymmetry and limited liability cause risk-shifting from investors to banks, which leads to asset-price bubbles; and Allen, Morris, and Shin (2003), in which higher order beliefs under noisy public information generate distortions in asset pricing. But few authors have addressed the problem of the transaction services that the asset can provide. Among these few authors are Kiyotaki and Wright (1989) and Bansal and Coleman (1996).

Kiyotaki and Wright show that a bubble equilibrium exists in which an intrinsically useless asset (cash) has positive value, since it provides transaction services. The difference in their model from the present paper is that the amount of transaction services

[1] To use the word "bubble" in this context may be somewhat misleading, since the difference $Q(L^*) - Q(0)$ reflects the fact that the asset provides transaction services in addition to the dividends. Thus we may be able to say that the fundamental price of an asset when it provides transaction services ($Q(L^*)$) is higher than its fundamental price when it does not provide transaction services ($Q(0)$). Nevertheless, I call the difference $Q(L^*) - Q(0)$ a bubble throughout in this paper, since the fundamental price of an asset usually refers to the value derived from dividends, not from transaction services.

that the cash can provide in their model is physically limited by the assumption that an exogenously fixed amount of cash is exchangeable for one unit of goods. Since I assume that the amount of transaction services that the asset can provide increases as the real price of the asset increases, the asset price can follow a complicated path, as discussed in Section 4. Bansal and Coleman analyze a one-period bond as an asset that provides transaction services. Because their asset is a fixed-payment security with a short maturity, the bubble component generated by the transaction services is small, while in the present paper the asset is infinitely long-lived and allows the emergence of large and unstable bubbles.

My model is quite similar to the model in Kiyotaki and Moore (2001) in which a borrowing constraint plays a crucial role in determining the asset price. The differences are that the labor-supply decision is explicitly introduced[2] and that, in Section 4, equilibrium paths with sticky prices and supply-demand gaps are analyzed.

Introducing the labor supply, I show that if the collateral ratio θ and money supply m are too small, there exists no equilibrium; that for the middle range of θ and m, there exists an equilibrium where the asset price has a bubble component and the resource allocation is inefficient; and that for a large θ and/or a large m, there exists only the optimal equilibrium, where the asset price does not have a bubble component. Introducing sticky prices and supply-demand gaps, I show that an equilibrium path exists in which an asset-price bubble develops temporarily and eventually bursts.

The organization of the paper is as follows: In the next section, I present the basic structure of the model, specify the conditions for the existence of steady state equilibria, and characterize the asset price and efficiency of the equilibria. In Section 3, I demondtrate that there is no other equilibrium than the steady states described in Section 2. Section 4 examines an equilibrium path under the assumption that prices are sticky and a temporary supply-demand gap can exist. Under sticky prices, there exist equilibrium paths in which an asset-price bubble temporarily develops and eventually bursts. Section

[2]Kiyotaki and Moore (2001) also introduce the labor supply. But in their model where entrepreneurs and workers are different, the agents who supply labor do not invest in capital, while in my model the representative agent both invest in land and supply labor.

5 provides some concluding remarks.

2 The model

The model is a general equilibrium model with a variant of the cash-in-advance constraint, which is composed of an infinite number of consumers and banks and one government. The economy is populated with a continuum of consumers with identical preferences, whose measure is normalized to one. There is also a continuum of banks with measure one. At date 0, a representative consumer maximizes the following utility:

$$\sum_{t=0}^{\infty} \beta^t U(c_t, l_t), \tag{2}$$

where β is the discount factor ($0 < \beta < 1$), c_t is the consumption at date t, and l_t is the labor supply at date t. For expositional convenience, the functional form is assumed to be $U(c, l) = \ln c + \gamma \ln(1 - l)$.

At each date t, the consumer is endowed with one unit of time, which can be divided into labor supply (l_t) and leisure ($1 - l_t$). There is a nonperishable asset (land) in this economy, which has a fixed total supply of 1. Initially each consumer owns 1 unit of land at the beginning of date 0. I assume that one unit of land yields y units of consumer goods at each date without any cost, and that one unit of labor yields A units of consumer goods. Thus the total supply of consumer goods is $y + Al_t$ at each date.

At each date t, the government provides M^s_{t+1} units of cash to this economy. The difference $X_t \equiv M^s_{t+1} - M^s_t$ is a lump-sum transfer to (from) the consumer from (to) the government at date t. (The initial amount M^s_0 is given to consumers at, say, date -1 as a lump-sum subsidy.)

At each date t, the consumer chooses the amount of consumption c_t, the labor supply l_t, cash holdings M_{t+1}, and land holdings a_{t+1}, given that he owns M_t units of cash and a_t units of land at the beginning of date t. If we denote the price of cash in terms of consumer goods by p_t and the real land price by q_t, the budget constraint for the consumer at date t is written as

$$c_t + p_t M_{t+1} + q_t a_{t+1} \leq y a_t + A l_t + q_t a_t + p_t (M_t + X_t). \tag{3}$$

Note that $\frac{1}{p_t}$ is the nominal price of the consumer goods. I assume as in the ordinary cash-in-advance model that a consumer cannot consume his own products $ya_t + Al_t$ and needs to buy c_t in the goods market from other consumers.

Consumers can buy the goods using cash and intraperiod bank borrowing b_t. Thus, consumers must choose c_t under the following liquidity constraint:

$$c_t \leq p_t M_t + b_t.$$

Banks lend b_t to consumers competitively at the beginning of date t, and consumers repay $R_t b_t$ to the banks at the end of date t. As a result of competition among banks, the rate of return on bank borrowing within one date must be one: $R_t = 1$. I assume that b_t works as a medium of exchange exactly like cash. In other words, I assume that b_t is given in the form of a bank deposit and that banks can create and provide transaction services to depositors without cost.

I assume, as in Kiyotaki and Moore (1997), that borrowers can freely abscond, leaving a part of their land (θa_t, where $0 < \theta < 1$), and that there is no way for banks to penalize such borrowers. Therefore, consumers cannot precommit to repay b_t to banks, and the only thing that banks can do when the borrowers abscond is to seize the remaining land θa_t. Following the arguments by Kiyotaki and Moore, this assumption implies that a consumer is subject to the borrowing constraint:

$$b_t \leq q_t \theta a_t,$$

where a_t is the land held by the consumer and θ ($0 \leq \theta < 1$) is the collateral ratio.[3] Under this borrowing constraint, a consumer who borrows b_t will never abscond and will repay b_t at the end of date t, since otherwise the bank will seize a part of his land, the value of which is $q_t \theta a_t$ ($\geq b_t$).

[3] As described below in the timing of events, q_t is realized after bank credit is given to the borrowers. If q_t is a random variable at the time of loan contracting, the borrowing constraint must be much more complicated in form, since it must be derived as the solution of the optimal contracting problem between agents who have rational expectations toward q_t. For simplicity, it is assumed as in Kiyotaki and Moore (1997) that there is no (aggregate) risk in this economy, so that the future path of the asset price $\{q_t\}_{t=0}^{\infty}$ is perfectly foreseen by the agents.

The above arguments imply that the reduced form of the liquidity constraint for the consumer is

$$c_t \leq p_t M_t + q_t \theta a_t. \tag{4}$$

Therefore, the representative consumer's problem is to maximize (2) subject to (3) and (4).

Timing of events It is useful to clarify the timing of events. The representative consumer enters date t with cash holdings M_t and land holdings a_t. At the beginning of date t, he produces Al_t by supplying labor (l_t), he is given yields on the land (ya_t), and he borrows $b_t (= q_t \theta a_t)$ from a bank. The goods market opens first, and the consumer sells goods $ya_t + Al_t$ and buys c_t under constraint (4). After consumption takes place, the consumer repays b_t to the bank. After repayment, the asset market then opens, and the consumer buys M_{t+1} and a_{t+1} by selling the remaining assets, the real value of which is $ya_t + Al_t + p_t M_t + q_t a_t - c_t$.

The equilibrium conditions for cash, land, and consumer goods are

$$M_t = M_t^s, \tag{5}$$

$$a_t = 1, \tag{6}$$

$$c_t = ya_t + Al_t. \tag{7}$$

The competitive equilibrium is defined as follows:

Definition 1 *The competitive equilibrium is a set of prices $\{p_t, q_t\}_{t=0}^{\infty}$ and allocations $\{c_t, l_t, a_t, M_t\}_{t=0}^{\infty}$ that satisfies the following conditions: (a) The prices are nonnegative and finite for all t: $0 \leq p_t, q_t < \infty$; (b) given the prices, the allocations solve the consumer's problem (i.e., maximization of [2] subject to [3] and [4]); (c) the allocations satisfy the equilibrium conditions (5)–(7); and (d) the transversality conditions for cash and land are satisfied.*

Denoting the Lagrange multipliers for (3) and (4) by λ_t and μ_t, respectively, we find that the first order conditions (FOCs) for the consumer's problem are

$$\beta^t U_c(t) = \lambda_t + \mu_t, \tag{8}$$

$$-\beta^t U_l(t) = A\lambda_t, \tag{9}$$

$$q_t\lambda_t = \lambda_{t+1}(y + q_{t+1}) + \mu_{t+1}q_{t+1}\theta, \tag{10}$$

$$p_t\lambda_t = p_{t+1}(\lambda_{t+1} + \mu_{t+1}), \tag{11}$$

where $U_c(t)$ and $U_l(t)$ are the derivative of $U(c_t, l_t)$ with respect to c_t and l_t, respectively. The transversality conditions can be written as

$$\lim_{t\to\infty} q_t a_{t+1}\lambda_t = 0, \tag{12}$$

$$\lim_{t\to\infty} p_t M_t\{\lambda_t + \mu_t\} = 0. \tag{13}$$

Since $a_t = 1$ in the equilibrium, the transversality condition for land (12) is satisfied if q_t is bounded from above and c_t is bounded from below by a positive number. This transversality condition excludes the rational bubble as an equilibrium price path. It is easily confirmed that (12) is satisfied in the equilibrium paths that are examined in what follows. In this paper I focus on the case where the government sets the real money supply (p_tM_t) at a constant for all t:

$$p_tM_t = m, \tag{14}$$

where $m < A$.[4] In this case, the transversality condition for money (13) is satisfied if c_t is bounded from below by a positive number. It is easily confirmed that (13) is satisfied in the equilibrium paths examined in this paper.

The FOCs imply that the inflation rate is determined by

$$\frac{p_t}{p_{t+1}} = \frac{\beta A}{\gamma}\frac{1-l_t}{c_{t+1}}. \tag{15}$$

Note that the gross rate of inflation is defined as $\frac{p_t}{p_{t+1}}$, since p_t is the inverse of the nominal price of consumer goods. Note also that since the fundamental price of the asset (q_t^F) is defined as the real asset price where the asset does not provide transaction services, q_t^F satisfies

$$q_t^F\lambda_t = \lambda_{t+1}(y + q_{t+1}^F). \tag{16}$$

[4]Though I assume (14) mainly for analytical simplicity, this assumption seems realistic since the monetary authorities in reality seem to target stabilizing the real money balance in the nonfinancial sector.

Since $1 = \frac{p_{t+1}}{p_t}\left(\frac{\lambda_{t+1}}{\lambda_t} + \frac{\mu_{t+1}}{\lambda_t}\right)$ from (11), it can be interpreted that $\frac{p_{t+1}}{p_t}\frac{\lambda_{t+1}}{\lambda_t}$ is the real present value at t of one unit of cash at $t+1$ as a store of value, and $\frac{p_{t+1}}{p_t}\frac{\mu_{t+1}}{\lambda_t}$ is the real present value at t of the transaction services that one unit of cash can provide at $t+1$. The second term of the right-hand side of (10) is the real present value at t of the transaction services that one unit of land can provide at date $t+1$.

In what follows in this section, I analyze whether a steady-state equilibrium exists where resource allocations are constant over time. I characterize the conditions of parameters θ and m for the existence (or nonexistence) of a type of equilibrium.

2.1 Equilibrium in which the liquidity constraint is nonbinding

First, I characterize the condition for the existence of a steady-state equilibrium where (4) is not binding, i.e., $\mu_t = 0$ for all t. I hereafter call this equilibrium a "nonbinding equilibrium." Setting $\mu_t = 0$ for all t, the FOCs imply that

$$l_t = \bar{l} \equiv \frac{A - \gamma y}{(1+\gamma)A},$$

and

$$q_t = \frac{\beta}{1-\beta}y.$$

Since in the nonbinding equilibrium (4) must hold with strict inequality, the condition for the existence of the nonbinding equilibrium is that $A\bar{l} + y = \frac{A-\gamma y}{1+\gamma} + y$ is less than $\frac{\beta}{1-\beta}y\theta + m$, which can be rewritten as:

$$\frac{m}{y} > \frac{A+y}{(1+\gamma)y} - \frac{\beta}{1-\beta}\theta. \tag{17}$$

If and only if parameters m and θ satisfy (17), there exists the unique nonbinding equilibrium.

In the nonbinding equilibrium, consumption c_t is determined by $c_t = A\bar{l} + y = \frac{A+y}{1+\gamma}$, and (15) implies that the inflation rate is determined by $\frac{p_t}{p_{t+1}} = \beta$. Thus if the real money supply is held constant, the price of consumer goods, i.e., $\frac{1}{p_t}$, must fall at the rate of time discount factor in the nonbinding equilibrium. This deflation is necessary for people to hold cash in the equilibrium where $\mu_t = 0$: Since the transaction services that cash can supply are valued at zero, cash needs to have a gross return of no less than $\frac{1}{\beta}$, since otherwise the representative agents will not hold cash as their asset.

2.2 Equilibrium in which the liquidity constraint is binding

Now I characterize the conditions for the existence of a steady-state equilibrium where (4) is binding, i.e., $\mu_t > 0$ for all t. I hereafter call this equilibrium a "binding equilibrium."

The FOCs (8) and (9) imply that $\mu_t > 0$ is equivalent to

$$l_t < \bar{l}. \tag{18}$$

In the binding equilibrium, $q_t = \frac{1}{\theta}\{Al_t + y - m\}$, since the liquidity condition (4) is binding. Thus l_t in the binding equilibrium must satisfy the following condition too, since $q_t > 0$.

$$Al_t + y - m > 0. \tag{19}$$

Given that l_t satisfies (18) and (19), the FOC (10) implies that $\{l_t\}_{t=0}^{\infty}$ in the binding equilibrium must evolve by $L(l_t) = R(l_{t+1})$, where

$$L(l_t) = \frac{Al_t + y - m}{1 - l_t}\frac{\gamma}{A\theta}, \tag{20}$$

$$R(l_t) = \frac{(1-\theta)(Al_t + y - m)}{1 - l_t}\frac{\beta\gamma}{A\theta} + \frac{\beta\gamma y}{(1 - l_t)A} + \frac{\beta(Al_t + y - m)}{Al_t + y}. \tag{21}$$

The condition for the existence of the binding equilibrium where $l_t = l_{t+1} = l$ is that l satisfies (18), (19), and $L(l) = R(l)$. As shown later in this subsection, the existence of the binding equilibrium is determined by the combinations of $L(0) \lesseqgtr R(0)$ and $L(\bar{l}) \lesseqgtr R(\bar{l})$. Thus I first characterize the conditions of parameters for $L(0) > R(0)$ (or $L(0) < R(0)$) and $L(\bar{l}) > R(\bar{l})$ (or $L(\bar{l}) < R(\bar{l})$). The inequality $L(0) > R(0)$ is rewritten as $\left(\frac{1}{\beta} - 1\right)\frac{\gamma y}{A\theta} - 1 > \left[\left(\frac{1}{\beta} - 1 + \theta\right)\frac{\gamma y}{A\theta} - 1\right]\frac{m}{y}$. Since $\left[\left(\frac{1}{\beta} - 1 + \theta\right)\frac{\gamma y}{A\theta} - 1\right] > 0$ is equivalent to $\theta < \frac{\left(\frac{1}{\beta}-1\right)\gamma y}{A - \gamma y}$, the condition for $L(0) > R(0)$ is written as

$$\theta < \frac{\left(\frac{1}{\beta} - 1\right)\gamma y}{A - \gamma y} \quad \text{and} \quad 0 < \frac{m}{y} < \frac{\left(\frac{1}{\beta} - 1\right)\frac{\gamma y}{A\theta} - 1}{\left(\frac{1}{\beta} - 1 + \theta\right)\frac{\gamma y}{A\theta} - 1}, \tag{22}$$

or

$$\theta > \frac{\left(\frac{1}{\beta}-1\right)\gamma y}{A-\gamma y} \quad \text{and} \quad \frac{m}{y} > \frac{\left(\frac{1}{\beta}-1\right)\frac{\gamma y}{A\theta}-1}{\left(\frac{1}{\beta}-1+\theta\right)\frac{\gamma y}{A\theta}-1}. \tag{23}$$

Note that the first condition in (23) implies that the numerator of the left-hand side of the second condition is less than $-\frac{\gamma y}{A}$. Since the denominator is negative, the conditions (23) are satisfied by sufficiently large m and θ.

Noting that $\frac{1}{A\bar{l}+y} = \frac{\gamma}{A(1-\bar{l})}$, the condition for $L(\bar{l}) < R(\bar{l})$ is rewritten as

$$\frac{m}{y} > \frac{A+y}{(1+\gamma)y} - \frac{\beta}{1-\beta}\theta. \tag{24}$$

Note that (24) is exactly the same as (17).

Assuming that the values of A, y, β, and γ are fixed, the conditions (22), (23), and (24) divide the first quadrant of $(\theta,\ \frac{m}{y})$-space into the following five regions: Region I, where $L(\bar{l}) > R(\bar{l})$ and $L(0) < R(0)$; Region II, where $L(\bar{l}) > R(\bar{l})$, $L(0) > R(0)$, and $m < y$; Region III, where $L(\bar{l}) < R(\bar{l})$ and $L(0) < R(0)$; Region IV, where $L(\bar{l}) < R(\bar{l})$ and $L(0) > R(0)$; and Region V, where $L(\bar{l}) > R(\bar{l})$, $L(0) > R(0)$, and $m > y$. Note that (22) is satisfied in Region II, while (23) is satisfied in Region V.

This division of the $(\theta,\ \frac{m}{y})$-space is illustrated in Figure 2.

Figure 2. Division of parameter space

It is shown below that each region corresponds to the existence or nonexistence of the binding equilibrium. Before proceeding, the following facts must be kept in mind.

Fact 1 *Since $L(l) = R(l)$ is a quadratic equation, the number of real and nonnegative solutions to this equation must be at most two.*

Fact 2 *Since $A - m > 0$ and*

$$\lim_{l \to 1-} \frac{R(l)}{L(l)} = \left\{1 - \theta + \frac{\theta y}{A+y-m}\right\}\beta,$$

it is the case that $0 < \lim_{l\to 1-} \frac{R(l)}{L(l)} < 1$. In other words, $L(l) > R(l)$ for $l(< 1)$ that is sufficiently close to 1.

Now I turn to each region. In Region III, where $L(\bar{l}) < R(\bar{l})$ and $L(0) < R(0)$, the continuity of the functions $L(l)$ and $R(l)$ implies that the number of the solutions to $L(l) = R(l)$ that satisfy $0 < l < \bar{l}$ must be an even number: $2j$ $(j = 0, 1, 2, \cdots)$. Fact 2 and the continuity of the functions imply that $L(l) = R(l)$ has at least one solution which is larger than $\bar{l}$. Since the total number of the solutions must be at most two (Fact 1) and one solution is larger than $\bar{l}$, it must be the case that $j = 0$, i.e., there is no solution to $L(l) = R(l)$ that satisfies (18). Therefore, if the parameters are in Region III, the binding equilibrium does not exist.

In Region I, where $L(\bar{l}) > R(\bar{l})$ and $L(0) < R(0)$, the continuity of the functions $L(l)$ and $R(l)$ implies that the number of the solutions to $L(l) = R(l)$ that satisfy (18) must be an odd number: $2j + 1$ $(j = 0, 1, 2, \cdots)$. Fact 1 implies that the number of the solutions must be 1. Denoting this solution to $L(l) = R(l)$ by l^*, I can show as follows that l^* satisfies (19): Note that (19) is equivalent to $l > l'$ where l' is defined by $L(l') = 0$; it is easily shown that $R(l') = \frac{\beta\gamma}{(1-l')A} y > 0$; therefore, $L(l') < R(l')$; the same reasoning as for Region III implies that l^* cannot be less than l'; therefore, l^* satisfies condition (19), which means that the asset price at the steady state $q^* \equiv \frac{Al^*+y-m}{\theta}$ is positive. It is also shown as follows that $q^* > q^F = \frac{\beta y}{1-\beta}$ if (19) is satisfied.[5]

Lemma 1 *If the parameters are in Region I, it is the case that*

$$\frac{Al^* + y - m}{\theta} > \frac{\beta y}{1 - \beta}, \tag{25}$$

where l^* is the solution to $L(l) = R(l)$.

See the Appendix for the proof. This lemma implies that $q^* > q^F$ if the parameters are in Region I. Therefore, it has been shown that if the parameters are in Region I, the binding equilibrium uniquely exists. The inflation rate is constant in this equilibrium

[5]For l^* to be the binding equilibrium, it is necessary that the asset price satisfies $q^* > q^F$. This condition is necessary because if the economy is in this steady state the representative agent can obtain the present value of q^F by holding one unit of land forever, and she can also obtain the necessary transaction services. Therefore, if $q^* < q^F$, the agent never sells the land at q^*, implying that q^* can never be the equilibrium price of the asset.

and is determined by (15): The equilibrium rate of inflation may be positive or negative, depending on the values of the parameters.

In Region II, where $L(\bar{l}) > R(\bar{l})$, $L(0) > R(0)$, and $m < y$, the continuity of the functions $L(l)$ and $R(l)$ implies that the number of the solutions to $L(l) = R(l)$ that satisfy (18) must be an even number: $2j$ $(j = 0, 1, 2, \cdots)$. The equation $L(l) = R(l)$ can be rewritten as $G(l) = H(l)$, where

$$G(l) = \{1 - (1-\theta)\beta\}\frac{\gamma}{A\theta}(Al + y - m)(Al + y), \tag{26}$$

$$H(l) = \beta(Al + y - m)(1 - l) + \frac{\beta\gamma y}{A}(Al + y). \tag{27}$$

Since $m < y$, the two roots of $G(l) = 0$, i.e., $-\frac{y}{A}$ and $-\frac{y-m}{A}$, are both negative numbers. Since $G(l)$ is convex and $H(l)$ is concave, $G(-\frac{y-m}{A}) = 0 < H(-\frac{y-m}{A}) = \frac{\beta\gamma y m}{A}$ implies that $G(l) = H(l)$ has at most one positive solution. Meanwhile, the positive solutions to $G(l) = H(l)$ must be equal to the positive solutions to $L(l) = R(l)$, the number of which is an even number. Therefore, the number of the solutions to $L(l) = R(l)$ in the region $0 < l < \bar{l}$ is zero. This shows that if the parameters are in Region II, the binding equilibrium does not exist.

In Region IV, where $L(\bar{l}) < R(\bar{l})$ and $L(0) > R(0)$, the continuity of the functions $L(l)$ and $R(l)$ implies that $L(l) = R(l)$ has at least one solution that satisfies $0 < l < \bar{l}$. Fact 2 implies that $L(l) = R(l)$ has at least one solution that is larger than $\bar{l}$. Since the total number of the solutions must be at most two (Fact 1), it is the case that $L(l) = R(l)$ has one solution l_4^* in the region where $0 < l < \bar{l}$ and the other solution in the region where $\bar{l} < l < 1$. It is shown as follows that the solution l_4^* is not the equilibrium: As shown in the reasoning for Region I, l', the solution to $L(l) = 0$, satisfies $L(l') < R(l')$; the continuity of $L(l)$ and $R(l)$ implies that $l_4^* < l'$; therefore, l_4^* does not satisfy (19), meaning that l_4^* is not the equilibrium. This shows that if the parameters are in Region IV, the binding equilibrium does not exist.

Finally, in Region V, where $L(\bar{l}) > R(\bar{l})$, $L(0) > R(0)$, and $m > y$, the continuity of the functions $L(l)$ and $R(l)$ implies that the number of the solutions to $L(l) = R(l)$ that satisfy (18) must be an even number: $2j$ $(j = 0, 1, 2, \cdots)$. Since $m > y$, one root of $G(l) = 0$, i.e., $\frac{m-y}{A}$, is a positive number. Since $G(l)$ is convex and $H(l)$ is concave, $G(\frac{m-y}{A}) = 0 < H(\frac{m-y}{A}) = \frac{\beta\gamma y m}{A}$ implies that $G(l) = H(l)$ has at least one positive solution. Meanwhile, the positive solutions to $G(l) = H(l)$ must be equal to the positive solutions to $L(l) = R(l)$, the number of which is an even number. Therefore, the number of the solutions to $L(l) = R(l)$ in the region $0 < l < \bar{l}$ is two. Let us denote these solutions by l^*_{51} and l^*_{52}, where $0 < l^*_{51} < l^*_{52}$. A similar argument as for Region I implies that $0 = L(l') < R(l')$, where $l' = \frac{m-y}{A} > 0$. Therefore, the continuity of $L(l)$ and $R(l)$ implies that $0 < l^*_{51} < l' < l^*_{52}$, and thus, $L(l^*_{51}) < 0 < L(l^*_{52})$. Since this result means that the asset price is negative if $l = l^*_{51}$, the first solution l^*_{51} is not the equilibrium allocation of labor supply. On the other hand, since the same lemma as Lemma 1 holds for l^*_{52}, it is the equilibrium allocation. Therefore, if the parameters are in Region V, the binding equilibrium uniquely exists.

2.3 Discussion

Summing up the results in the previous subsections, we can say the following for the existence of the equilibria: If the parameters are in Region I or V, the binding equilibrium exists and the nonbinding equilibrium does not exist; In Region III or IV, the binding equilibrium does not exist and the nonbinding equilibrium exists; In Region II, there exists no equilibrium. This result is illustrated in Figure 3.

Figure 3. Regions for the existence of equilibria

Note that in the nonbinding equilibrium, the resource allocation is efficient: The efficient allocation is defined as the solution to $\max_{c_t,l_t} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$ subject to $c_t \leq Al_t + y$. This allocation is realized in the nonbinding equilibrium. It has been shown that if the parameters satisfy (24), i.e., m and/or θ are sufficiently large, there exists only the optimal (nonbinding) equilibrium. This leads to the following (trivial) intuition: If the amount of transaction services that cash (M_t) and land (a_t) can supply is sufficiently large so that the amount supplied exceeds that needed, the liquidity constraint (4) becomes nonbinding and the resource allocation becomes efficient.

In the binding equilibrium, the resource allocation is inefficient: If the parameters are in Region I or V, the labor supply in the binding equilibrium is strictly less than the first best value $\bar{l}$.

Note also that even in the case that $m = 0$, the nonbinding or binding equilibrium can exist depending on the value of θ: If $\frac{1-\beta}{\beta}\frac{\gamma y}{A} < \theta < \frac{1-\beta}{\beta}\frac{A+y}{(1+\gamma)y}$, there exists only the binding equilibrium, and if $\theta > \frac{1-\beta}{\beta}\frac{A+y}{(1+\gamma)y}$, there exists only the nonbinding equilibrium. Since in this economy land works as "commodity money" that can loosen the liquidity constraint (4), the fiat money m may not be necessary in an equilibrium. Figure 3 shows that this is exactly the case.

The same argument holds for fiat money. In the case that $\theta = 0$, the nonbinding or binding equilibrium can exist depending on the value of m: If $y < m < \frac{A+y}{1+\gamma}$, there exists only the binding equilibrium, and if $m > \frac{A+y}{1+\gamma}$, there exists only the nonbinding equilibrium. Therefore, if a sufficient amount of fiat money is supplied, the fixed asset (land) need not play the role of commodity money in the equilibrium.

Since in the analysis in this paper it is assumed that the agents hold both fiat and commodity monies, the equilibrium inflation rate is determined so that rational agents are willing to have both monies. The nominal supply of cash M_t is determined passively in the equilibrium of this model so that M_t is consistent with the equilibrium inflation rate. If we assume that the monetary authority fixes the inflation rate or the nominal amount of cash M instead of the real money balance m, we will find that for a large θ, the commodity money a_t dominates the fiat money M_t. This is one of the results in Kiyotaki and Moore (2001): They show that in their model, if θ is large and the (nominal) supply of fiat money M_t is fixed, fiat money is valued at zero, and only the physical asset (i.e., land in my model) plays the role of money in the equilibrium. Fiat money becomes useless paper in such an equilibrium.

In my model, it is demonstrated that the efficiency of the equilibrium (i.e., whether it is binding or nonbinding) is not directly determined by the existence or nonexistence of fiat money m. What is relevant to the efficiency is the total amount of transaction services that is provided by fiat and commodity monies ($m + q_t\theta a_t$). If this amount is small, the equilibrium becomes binding and inefficient, and if this amount is large, the equilibrium becomes nonbinding and efficient. In other words, if the value of the transaction services that the asset can provide, i.e., μ_t, is high, the equilibrium is inefficient, and if it is zero, the equilibrium is efficient. Figure 4 shows that the asset price q^* is higher than the fundamental price q^F in the binding equilibrium and that it converges to q^F as the equilibrium becomes nonbinding. In this figure, the parameters are set as follows: $\beta = 0.98$, $\gamma = 2$, $A = 1$, and $y = 0.02$. The upper panel shows the case where $\frac{m}{y}$ is fixed at 1.5, and the lower panel shows the case where θ is fixed at 0.2.

Figure 4. Equilibrium asset prices

Another point that my model shows is that if m and θ are too small so that the parameters fall in Region II, no equilibrium exists in this economy. We can intuitively grasp this as follows. In the case where the parameters are in Region II, the asset price that satisfies (4) today is too expensive for a buyer, compared to the return from the asset that he will get tomorrow if the economy stays at a steady state $l_t = l$, where $0 < l < \bar{l}$. Therefore, the labor supply tommorrow l_{t+1} must be larger than today in order to justify q_t. Thus, there is no steady state equilibrium. A caveat for this nonexistence result is that Region II may be negligibly small for plausible parameter values. Suppose, for example, that $\beta = 0.98$, $\gamma = 2$, $A = 1$, and $y = 0.02$. In this case, since (23) must be satisfied in Region II, θ in Region II is smaller than 0.0009. Since a realistic value of θ is probably in the range between 0.1 and 0.8, Region II is negligibly small. Therefore, the nonexistence of an equilibrium for Region II may not be a relevant result for actual economies.

Implications for monetary policy The above results imply that the real money supply m must be sufficiently large in order to realize the social optimum (the nonbinding equilibrium), while the inflation rate must be set at a negative value (i.e., $\frac{p_t}{p_{t+1}} = \beta$). Therefore, the optimal monetary policy in this model is the Friedman's rule, i.e., to set the level of the real balance at a sufficiently large value and to reduce the nominal money supply gradually. The results also indicate that it may be inappropriate to interpret the discrepancy $q^* - q^F$ in the binding equilibrium as the asset-price bubble. This is because the model shows that the discrepancy becomes large when the money supply m is small. In reality, the emergence of asset-price bubbles is usually associated with expansion of money and credit (see Allen and Gale [2000]). It may be necessary to consider a different factor, such as the risk-shifting effect in Allen and Gale (2000), in order to explain the bubbles associated with monetary expansion.

3 On deviation from the steady state equilibrium

If the parameters are in Regions III and IV, the economy always stays in the nonbinding equilibrium, in which the allocations of labor and consumption are constant over time. In this section I assume that the parameters are in Region I or V, and examine whether there is any equilibrium path that is not the steady state, i.e., the binding equilibrium.

The goal of this section is to prove the following lemma:

Lemma 2 *If the parameters are in Region I or V, no equilibrium path exists other than the binding equilibrium.*

This lemma confirms that in the model of this paper, in which there is no capital accumulation or productivity growth, an equilibrium path is always a steady state.

(Proof of Lemma 2)

The proof is by contradiction. Suppose that there is an equilibrium path other than the binding equilibrium. Since the nonbinding equilibrium in which $\mu_t = 0$ for all t does not exist for the parameters in Region I or V, one of the following three must be true in this equilibrium path: (i) $\mu_t > 0$ for all t; (ii) there exists some t, such that $\mu_t > 0$ and $\mu_{t+k} = 0$ for all $k \geq 1$; or (iii) there exists some t, such that $\mu_t = 0$ and $\mu_{t+1} > 0$. If (i) is true, the labor supply l_t in this equilibrium path must be determined by $L(l_t) = R(l_{t+1})$. Denote the solution to $L(l) = R(l)$ by l^*. Fact 2 implies that if $l_0 > l^*$, l_t eventually exceeds $\bar{l}$, and thus the condition $\mu_t > 0$ is violated. If $l_0 < l^*$, l_t or $L(l_t)$ eventually becomes less than zero. Therefore, (i) is not true in the equilibrium path. It is obvious that (ii) cannot be true, since otherwise this equilibrium is identical to the nonbinding equilibrium from $t+1$ onward, which cannot exist for the parameters in Region I or V. Therefore, (iii) must be true if this equilibrium path exists. Note that in this event (iii), the labor at $t+1$ must satisfy (18). Since $\mu_t = 0$, the FOCs imply that $l_t = \bar{l}$. The FOC (10) implies that $\frac{q_t}{A\bar{l}+y} = R(l_{t+1})$, since $\frac{\gamma}{(1-\bar{l})A} = \frac{1}{A\bar{l}+y}$. Meanwhile, since $\mu_t = 0$, the liquidity constraint (4) is not binding at t. Thus, $q_t > \frac{A\bar{l}+y-m}{\theta}$. These conditions imply that

$$R(l_{t+1}) = \frac{q_t}{A\bar{l}+y} > \frac{A\bar{l}+y-m}{(A\bar{l}+y)\theta} = L(\bar{l}). \tag{28}$$

Since $L(l)$ and $R(l)$ are both continuous and increasing functions and $L(\bar{l}) > R(\bar{l})$, (28) implies that $l_{t+1} > \bar{l}$. Therefore, l_{t+1} does not satisfy (18), meaning that the event (iii) cannot occur in the equilibrium. Since neither of (i), (ii), or (iii) can occur in an equilibrium, there is no equilibrium other than the binding equilibrium. (End of proof)

The results of this and the previous sections imply that in this model the economy always stays in a steady state, i.e., either the binding equilibrium or the nonbinding equilibrium, unless the parameters are in Region II, where neither of them exists.

4 Equilibrium with sticky prices and bursting bubbles

The arguments in the previous section imply that only the steady states are the equilibrium in the case where prices are flexible and there is no supply-demand gap.

If we introduce sticky prices and allow the existence of a supply-demand gap, there may be various equilibrium paths, in which the evolution of the asset price looks like the formation and bursting of a bubble. In this section, it is demonstrated that such an equilibrium path exists.

The parameters are assumed to be in Region I or V in what follows. It is shown that if the equilibrium condition $c_t = Al_t + y$ is relaxed to $c_t \leq Al_t + y$ and prices are sticky in the sense that p_t and q_t are predetermined at date $t-2$ and the prices at dates 0 and 1 (p_0, q_0, p_1, and q_1) are appropriately given, then for appropriate parameters there exist equilibria in which a bubble develops temporarily and bursts at some date.

First, the following fact is easily confirmed:

Fact 3 *Assume that the market clearing condition* $c_t = Al_t + y$ *is satisfied at dates 0 and 1. To prefix the prices at dates 0 and 1 (*p_0*,* q_0*,* p_1*, and* q_1*) at appropriate values is equivalent to prefixing the labor supply at date 0 (*l_0*).*

Since we are considering a equilibrium path in which $\mu_t > 0$, l_1 is determined by $L(l_0) = R(l_1)$, implying that l_1 and c_1 are functions of l_0. Thus, the right-hand side of equation (15) for $t = 0$ is a function of l_0. This equation implies that to prefix the inflation rate at date 0 ($\frac{p_0}{p_1}$) is equivalent to prefixing l_0. The asset prices q_0 and q_1 must be given at consistent values, i.e., $q_0 = \frac{Al_0+y-m}{\theta}$ and $q_1 = \frac{Al_1+y-m}{\theta}$.

In order to characterize such an equilibrium, it is necessary to relax the definition of a competitive equilibrium to allow for a temporary supply-demand gap:

Definition 2 *A sticky price equilibrium is the same as a competitive equilibrium defined by Definition 1, except for that (a) prices* p_t *and* q_t *are predetermined at date* $t-2$*; (b) instead of (7),* $c_t \leq Al_t + y$ *is satisfied; and (c) if* $c_t < Al_t + y$*, the supply-demand gap (*$Al_t + y - c_t$*) perishes without being consumed by anyone at date* t *and is borne as a lump-sum cost by the consumer (= seller), and the budget constraint for the consumer at date* t *becomes*

$$c_t + p_t M_{t+1} + q_t a_{t+1} \leq Al_t + ya_t + q_t a_t + p_t(M_t + X_t) - \delta_t, \tag{29}$$

where δ_t *is a lump-sum cost, which is exogenous for the consumer, and* $\delta_t = Al_t + y - c_t$ *holds in the equilibrium.*

The prices at date t are predetermined at $t-2$, and the consumer (=seller) cannot change the price p_t at date t even though he cannot sell all of his goods at p_t. Note that even under sticky prices, the FOCs: (8)–(11) must be satisfied, since the consumers solve their optimization problem taking the entire price path as given. Therefore, the FOCs are always satisfied in a sticky price equilibrium, while the market clearing conditions may not be. This implies that if the market clearing conditions are satisfied for t and $t+1$ in the sticky price equilibrium, the equation $L(l_t) = R(l_{t+1})$ must be satisfied.

The concept of sticky price equilibrium is useful for analyzing the situation where the initial value of the labor supply l_0 is not l^*. (If prices are not sticky, p_0, q_0, p_1, and q_1 adjust instantaneously at date 0 so that l_0 never deviates from l^*.) If $l_0 > l^*$ and prices are sticky, the equilibrium path, or l_t, is determined by the difference equation $L(l_t) = R(l_{t+1})$ for the time being, but the equation becomes impossible to solve eventually, since l_t must be less than $\bar{l}$ in the equilibrium where $\mu_t > 0$. It is shown as follows that if the parameters satisfy a certain condition, there exists a sticky price equilibrium where the bubble bursts at some date τ and there emerges a supply-demand gap: $c_\tau < Al_\tau + y$.

4.1 A bubble path with the binding liquidity constraint

Define c ($> m$) as a solution to

$$\frac{\theta L(l^*)c}{c-m} = \frac{m}{y} + \left[\frac{L(\bar{l})}{\beta} - (1-\theta)L(l^*) + 1\right]\frac{c}{y}. \tag{30}$$

It is graphically confirmed that equation (30) has only one solution that is larger than m. The parameter c is a function of θ and $\frac{m}{y}$. Using c, the following condition determines a region in (θ, $\frac{m}{y}$)-space:

$$\frac{\theta L(l^*)c}{c-m} < 1. \tag{31}$$

Similarly, c' ($> c$) can be defined as a solution to

$$\frac{\theta L(l^*)c'}{c'-m} = \frac{m}{y} + \left[\frac{R(\bar{l})}{\beta} - (1-\theta)L(l^*) + 1\right]\frac{c'}{y}. \tag{32}$$

The following condition determines another region in (θ, $\frac{m}{y}$)-space:

$$\frac{\gamma}{A} < \frac{\theta L(l^*)}{c'-m}, \tag{33}$$

and

$$c' < \frac{(A+y)\theta L(l^*) + \gamma m}{\gamma + \theta L(l^*)}. \tag{34}$$

Lemma 3 *Suppose that the parameters are in Region I or V and that they also satisfy (31), (33), and (34). Suppose also that the initial value of the labor supply l_0 is fixed and that it exceeds l^*. There exists a sticky price equilibrium where $a_t = 1$ and $\mu_t > 0$*

for all t, and $\exists\tau$ is such that $l^* < l_t < \bar{l}$ for $t < \tau$, $l_t = l^*$ for $t > \tau$, $c_\tau < Al_\tau + y$, and $c_t = Al_t + y$ for $t \neq \tau$.

The region of the parameters that satisfy the assumptions of this lemma is illustrated in Figure 5. The fixed parameters are set at the same values as those in Figure 4. In this case Region II is negligibly small. This figure implies that the equilibrium path described in this lemma exists if $\frac{m}{y}$ is very small.

Figure 5. Region for the existence of a bubble path with $\mu_\tau > 0$

(Proof) Proof is by construction. I construct an equilibrium where the economy jumps to the binding equilibrium (where $l_t = l^*$) at date $\tau + 1$. Define $\tilde{l}$ by $L(\tilde{l}) = R(\bar{l})$. The sequence $\{l_0, l_1, \cdots, l_{\tau-1}\}$ is constructed by $L(l_t) = R(l_{t+1})$ $(t = 0, 1, \cdots, \tau - 2)$ and $l_{\tau-2} < \tilde{l} \leq l_{\tau-1}$. It is easily confirmed graphically that $l_{\tau-1} < \bar{l}$. The sequence c_t $(t = 0, 1, \cdots, \tau - 1\ )$ is also defined by $c_t = Al_t + y$. Since it is assumed that $\mu_\tau > 0$, the asset price at τ must satisfy $q_\tau = \frac{c_\tau - m}{\theta}$. The FOC for a_τ implies that

$$L(l_{\tau-1}) = \frac{\beta\gamma}{(1 - l_\tau)A}\left\{y + (1-\theta)\frac{c_\tau - m}{\theta}\right\} + \beta\frac{c_\tau - m}{c_\tau}. \tag{35}$$

Since the economy is assumed to go to the binding equilibrium at $\tau + 1$, the FOC for $a_{\tau+1}$ implies that

$$\frac{c_\tau - m}{\theta}\frac{\gamma}{(1 - l_\tau)A} = R(l^*) = L(l^*). \tag{36}$$

Equations (35) and (36) are two equations for two unknowns: c_τ and l_τ. These equations imply that c_τ $(> m)$ is uniquely determined as the solution to

$$\frac{\theta L(l^*)c_\tau}{c_\tau - m} = \frac{m}{y} + \left[\frac{L(l_{\tau-1})}{\beta} - (1-\theta)L(l^*) + 1\right]\frac{c_\tau}{y}. \tag{37}$$

Since $\tilde{l} < l_{\tau-1} < \bar{l}$ and $L(l)$ is increasing in l, it is confirmed graphically that the solution to (37) satisfies $c < c_\tau < c'$. In order for c_τ and l_τ to be the equilibrium allocations, they must satisfy $\mu_\tau > 0$ and $0 < l_\tau < 1$. The condition $\mu_\tau > 0$ is equivalent to $\frac{1}{c_\tau} > \frac{\gamma}{(1-l_\tau)A}$, which is rewritten using (36) as $\frac{\theta L(l^*)c_\tau}{c_\tau - m} < 1$. Since $c_\tau > c$, this condition is satisfied if c satisfies (31). Since $c_\tau > m$, equation (36) implies that $l_\tau < 1$. The condition $l_\tau > 0$ is equivalent to $\frac{\theta L(l^*)}{c_\tau - m} > \frac{\gamma}{A}$. This condition is satisfied for $c_\tau < c'$ if c' satisfies (33). Finally, the condition $c_\tau < Al_\tau + y$ is equivalent to $c_\tau < \frac{(A+y)\theta L(l^*)+\gamma m}{\gamma+\theta L(l^*)}$. This condition is satisfied if the parameters satisfy (34), since $c < c_\tau < c'$. Therefore, c_τ and l_τ are the equilibrium allocations. For $t \geq \tau + 1$, the allocations are defined by $l_t = l^*$ and $c_t = c^* = Al^* + y$. The asset price is determined by $q_t = \frac{c_t - m}{\theta}$ for all t, and (the inverse of) the general price is determined by (15). Note that q_t in this equilibrium always exceeds the fundamental price q_t^F that is defined by (16) for the allocations of this equilibrium: This is because $\mu_t > 0$ for all t in this equilibrium path and q_t satisfies (10). (End of proof)

This lemma implies that there exists a sequence of prices $\{p_t, q_t\}_{t=0}^{\infty}$ that support the above sequence $\{c_t, l_t\}_{t=0}^{\infty}$ as the resource allocation in a sticky price equilibrium.

Note that this is not the unique sticky price equilibrium for a given initial value (l_0). The bubble may burst at some time that is less than or equal to τ defined above. There may be a sticky price equilibrium that corresponds to each timing of the bubble's puncturing. The timing τ in the above lemma is the upper limit for continuation of the bubble when the initial labor supply is given as $l_0 > l^*$.

One example of the equilibrium path described in Lemma 3 is shown in Figure 6. The equilibrium path corresponds to $\theta = 0.2$ and $\frac{m}{y} = 0.01$. Consumption and the asset price rise gradually, and they collapse at date τ. The labor supply also rises gradually, and it jumps up at date τ. This jump of labor at the bubble's collapse may be interpreted as a boom in the real sector that is stimulated by the last stage of the asset-price bubble of $\tau - 1$. An interesting feature of this simulation is the inflation rate. It goes *down* gradually during the period when the asset price and consumption continue rising. The inflation rate jumps up at $\tau - 1$ and down to severe deflation at τ. The slowing inflation during the period of the asset-price bubble seems consistent with the observations in Japan during the late 1980s: In that period, inflation did not accelerate, while stock prices and land prices skyrocketed; this steady inflation was one reason why the Bank of Japan decided not to respond preemptively.

Figure 6. A bubble path with $\mu_\tau > 0$

4.2 A bubble path with the nonbinding liquidity constraint

It is also shown in the following lemma that for appropriate parameter values there exists a sticky price equilibrium in which the bursting of the bubble occurs and $\mu_t = 0$ at the time of the bursting. The following three inequalities define a region in $(\theta, \frac{m}{y})$-space:

$$\frac{m}{y} > \frac{\{1 - L(l^*)\theta\}\beta}{L(\bar{l}) - \beta L(l^*)}, \tag{38}$$

$$1 > \frac{\beta\gamma y}{\{R(\bar{l}) - \beta L(l^*)\}A}, \tag{39}$$

and

$$\frac{\beta y}{\{R(\bar{l}) - \beta L(l^*)\}A} < \frac{A + y}{1 + \gamma}. \tag{40}$$

Lemma 4 *Suppose that the parameters are in Region I or V and that they also satisfy (38), (39), and (40). Suppose also that the initial value of labor supply l_0 is fixed and exceeds l^*. A sticky price equilibrium exists where $a_t = 1$, and $\exists\tau$ is such that $l^* < l_t < \bar{l}$ for $t < \tau$, $l_t = l^*$ for $t > \tau$, $c_\tau < Al_\tau + y$, $c_t = Al_t + y$ for $t \neq \tau$, $\mu_t > 0$ for $\forall t \neq \tau$, and $\mu_\tau = 0$.*

The region of the parameters that satisfy the assumptions of this lemma is illustrated in Figure 7. The fixed parameters are set at the same values as those in Figure 4. This figure implies that the equilibrium path described in this lemma exists for a wide range of θ and $\frac{m}{y}$.(Region II is negligibly small in this figure.) Thus, this lemma may hold for realistic parameter values.

Figure 7. Region for the existence of a bubble path with $\mu_\tau = 0$

(Proof) Proof is by construction. I construct an equilibrium where the economy jumps to the binding equilibrium at date $\tau+1$. The sequence $\{l_0, l_1, \cdots, l_{\tau-1}\}$ is constructed by $L(l_t) = R(l_{t+1})$ $(t = 0, 1, \cdots, \tau-2)$ and $l_{\tau-2} < \tilde{l} \leq l_{\tau-1}$. It is easily confirmed graphically that $l_{\tau-1} < \bar{l}$. The sequence c_t $(t = 0, 1, \cdots, \tau - 1$) is also defined by $c_t = Al_t + y$. Since it is assumed that $\mu_\tau = 0$, the FOCs for c_τ and l_τ imply that

$$\frac{1}{c_\tau} = \frac{\gamma}{(1 - l_\tau)A}. \tag{41}$$

The FOC for a_τ implies that

$$L(l_{\tau-1}) = \frac{\beta\gamma}{(1-l_\tau)A}\{y + (1-\theta)q_\tau\} + \frac{\beta q_\tau \theta}{c_\tau}. \tag{42}$$

Since the economy jumps to the binding equilibrium at $\tau+1$, the FOC for $a_{\tau+1}$ implies that

$$\frac{q_\tau \gamma}{(1-l_\tau)A} = R(l^*) = L(l^*). \tag{43}$$

Solving (41), (42), and (43), we get:

$$l_\tau = 1 - \frac{\beta\gamma y}{\{L(l_{\tau-1}) - \beta L(l^*)\}A}, \tag{44}$$

$$c_\tau = \frac{(1-l_\tau)A}{\gamma}, \tag{45}$$

$$q_\tau = c_\tau L(l^*). \tag{46}$$

The condition for $l_\tau > 0$ is that $1 > \frac{\beta\gamma y}{\{L(l_{\tau-1})-\beta L(l^*)\}A}$, which is satisfied if the parameters satisfy (39), since $l_{\tau-1} > \tilde{l}$ and $L(\tilde{l}) = R(\bar{l})$. The condition for $\mu_\tau = 0$ is $q_\tau\theta + m > c_\tau$, which is rewritten as $\frac{m}{y} > \frac{\{1-L(l^*)\theta\}\beta}{L(l_{\tau-1})-\beta L(l^*)}$. This condition is satisfied if the parameters satisfy (38), since $l_{\tau-1} < \bar{l}$. Since l_τ and c_τ are determined by the above equations, the condition that $c_\tau < Al_\tau + y$ is rewritten as $\frac{\beta y}{L(l_\tau)-\beta L(l^*)} < \frac{A+y}{1+\gamma}$. This condition is satisfied if (40) is satisfied, since $L(l_\tau) > R(\bar{l})$. Therefore, if the parameters satisfy the assumptions of this lemma, there exists a sticky price equilibrium in which the bubble bursts at τ and the economy jumps to the binding equilibrium at $\tau+1$. (End of proof)

One example of the equilibrium path described in Lemma 4 is shown in Figure 8. The equilibrium path corresponds to $\theta = 0.2$ and $\frac{m}{y} = 1.5$. The behaviors of variables are qualitatively the same as those in Figure 6.

Figure 8. A bubble path with $\mu_\tau = 0$

4.3 Discussion

In the previous subsections it is demonstrated that sticky price equilibrium exists in which the asset price rises temporarily and collapses eventually. Similarly, it can be shown that for appropriate parameters an equilibrium path exists in which the asset price falls temporarily following $L(l_t) = R(l_{t+1})$ with $l_0 < l^*$, and jumps up to the steady state value eventually. Similar lemmas as Lemmas 3 and 4 can be established for the negative bubble paths.

Note that in the previous subsections it is implicitly assumed that the agents' expectations are well coordinated, such that the economy eventually jumps to the steady state (i.e., the binding or nonbinding equilibrium). Depending on the expectations on the jump at date τ, the economy may follow a much more complicated path. For example, it is possible that the rise and collapse of the asset price will be repeated cyclically; or that the economy may follow a positive bubble path for several periods, jump to a negative bubble path for subsequent periods, and jump again to the steady state or a positive or negative bubble path.

What has been shown in this section is that if sticky prices are assumed and a supply-demand gap is allowed, the model can exhibit rich dynamics in which the asset price, labor supply, and consumption change over time and jump sometimes.

5 Concluding remarks

This paper has examined the equilibrium of an economy with a nondepletable asset (i.e., land) in the case where the asset can provide transaction services, using a variant of the cash-in-advance model. The transaction services the asset can provide increase as its (real) price becomes higher, since the owner of the asset can borrow more money by putting it up as collateral. Thus the asset price may exceed its fundamental price, since the transaction services that it can provide are an increasing function of the asset price, which reflects the value of the transaction services that it can provide.

Introducing a parameter that represents the collateral ratio of the asset (θ), I showed that if the total supply of transaction services ($q\theta a + m$) is small, the equilibrium is inefficient and the asset price exceeds its fundamental price, and that if $q\theta a + m$ is large, the equilibrium is efficient and the asset price equals its fundamental price. It was also shown that if the equilibrium concept is relaxed to allow for sticky prices and a temporary supply-demand gap, there exists an equilibrium in which a bubble develops temporarily and eventually bursts.

6 Appendix

In this appendix, I show a mathematical proof of Lemma 1 and describe the economic intuition behind it:

(Proof of Lemma 1)

Condition (25) is equivalent to $l^* > \underline{l}$, where

$$\underline{l} = \frac{1}{A}\left\{m - \frac{1-(1-\theta)\beta}{1-\beta}y\right\}. \tag{47}$$

Since the parameters are in Region I, the above condition is satisfied if and only if $L(\underline{l}) < R(\underline{l})$. Using the fact that $\frac{A\underline{l}+y-m}{\theta} = \frac{\beta y}{1-\beta}$, we can rewrite the condition $L(\underline{l}) < R(\underline{l})$ as

$$(1+\gamma)\underline{l} < 1 - \frac{\gamma y}{A}. \tag{48}$$

From (47) and (48), it can be said that (25) holds if and only if

$$\frac{m}{y} < \frac{A+y}{(1+\gamma)y} - \frac{\beta}{1-\beta}\theta. \tag{49}$$

Comparing this condition and (24), it is easily confirmed that this condition is satisfied if the parameters are in Region I. (End of proof)

The intuition of this lemma is as follows. Since (19) is satisfied, both $L(l_t)$ and $R(l_t)$ are positive. Since equation $L(l_t) = R(l_{t+1})$ is equivalent to (10), it can be said that q^* satisfies the FOC (10). On the other hand, q^F satisfies (16). Since $l^* < \bar{l}$, it is the case that $\mu_t > 0$ for $l_t = l^*$ and $c_t = Al^* + y$. Therefore, equations (10) and (16) imply that $q^* > q^F$. In other words, since the value of the transaction services that the asset can provide is positive, the actual asset price q^* is larger than its fundamental price q^F as long as (19) is satisfied.

7 References

Allen, F., and D. Gale (2000). "Bubbles and crises." *Economic Journal* 110:236–55.

Allen, F., S. Morris, and H. S. Shin (2003). "Beauty contests, bubbles and iterated expectations in asset markets." Wharton Financial Institutions Center Working Paper 03-06, University of Pennsylvania.

Bansal, R., and W. J. Coleman II (1996). "A monetary explanation of the equity premium, term premium, and risk-free rate puzzles." *Journal of Political Economy* 104 (6): 1135–71.

Camerer, C. (1989). "Bubbles and fads in asset prices: A review of theory and evidence." *Journal of Economic Surveys* 3:3–41.

Hoshi, T., and A. Kashyap (1999). "The Japanese banking crisis: Where did it come from and how will it end?" NBER Working Paper w7250.

Kiyotaki, N., and J. Moore (1997). "Credit cycles." *Journal of Political Economy* 105 (2): 211–48.

Kiyotaki, N., and J. Moore (2001). "Liquidity, business cycles, and monetary policy." mimeo.

Kiyotaki, N., and R. Wright (1989). "On money as a medium of exchange." *Journal of Political Economy* 97 (4): 927–54.

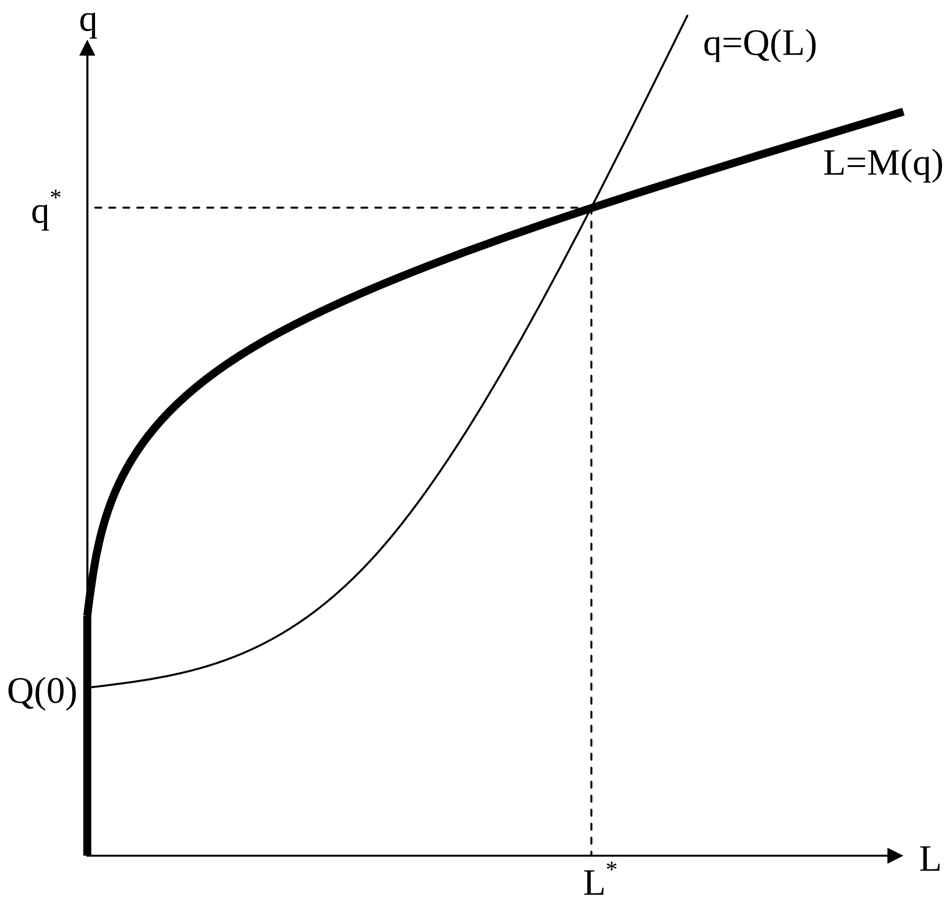


Figure 1. Land prices and the transaction services

Figure 2. Division of parameter space

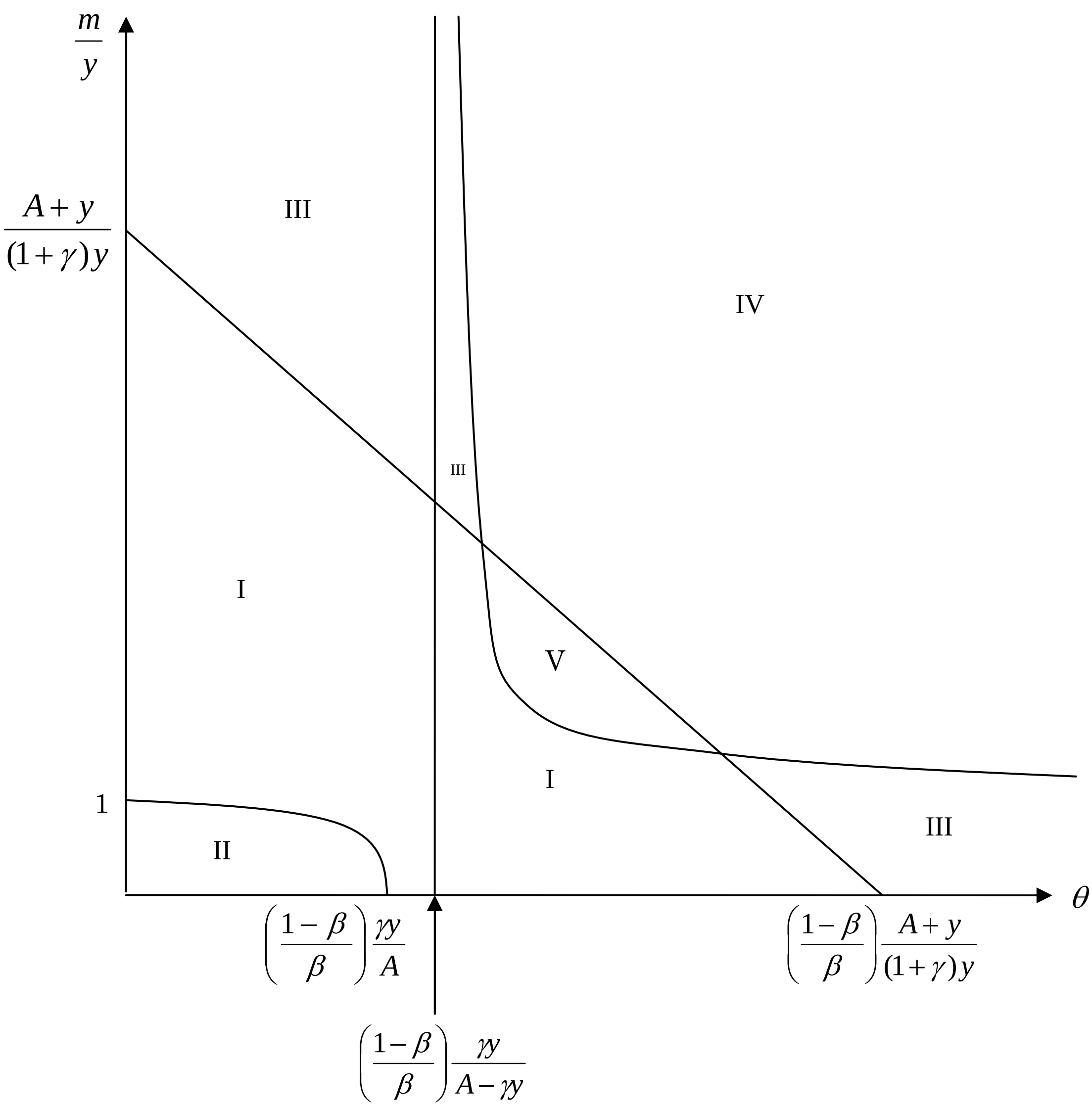


Figure 3. Regions for the existence of equilibria

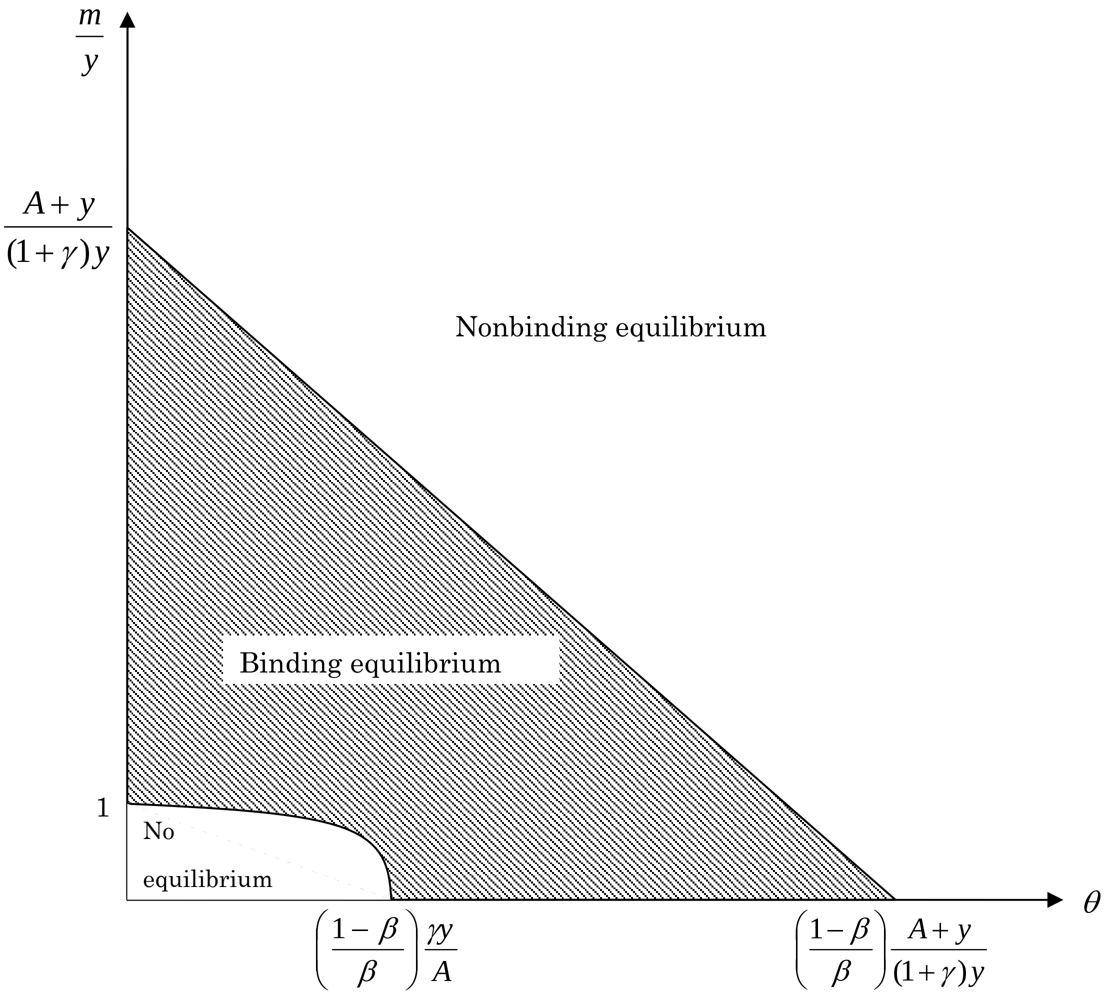
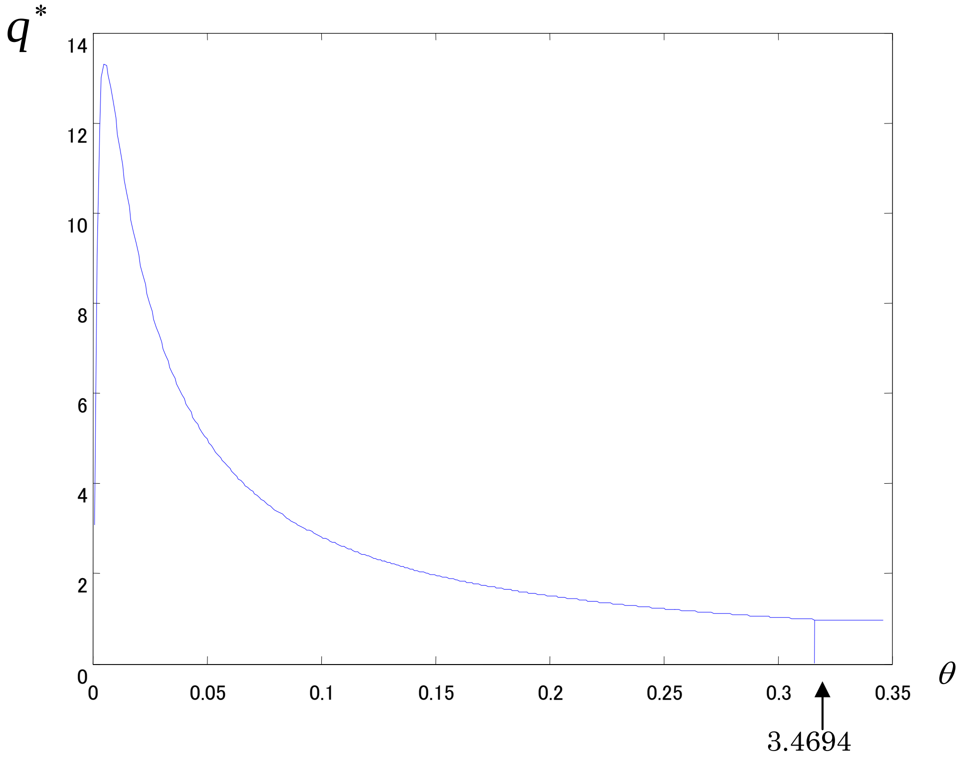


Figure 4. Equilibrium asset prices



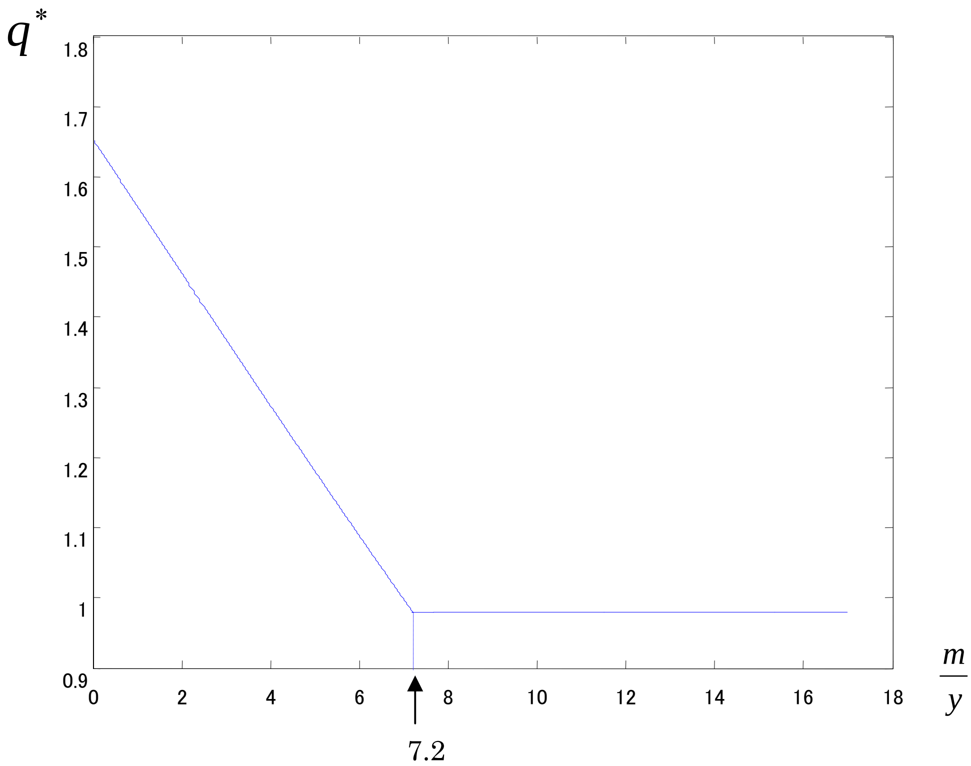


Figure 5. Region for the existence of a bubble path with $\mu_\tau > 0$

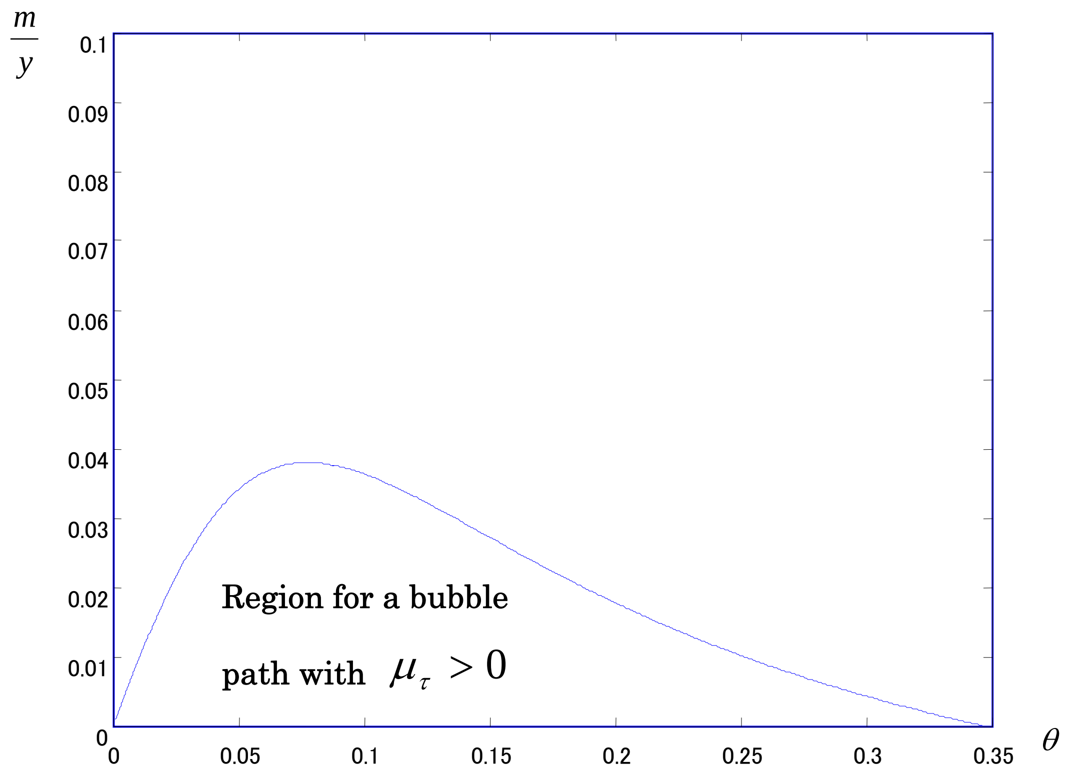
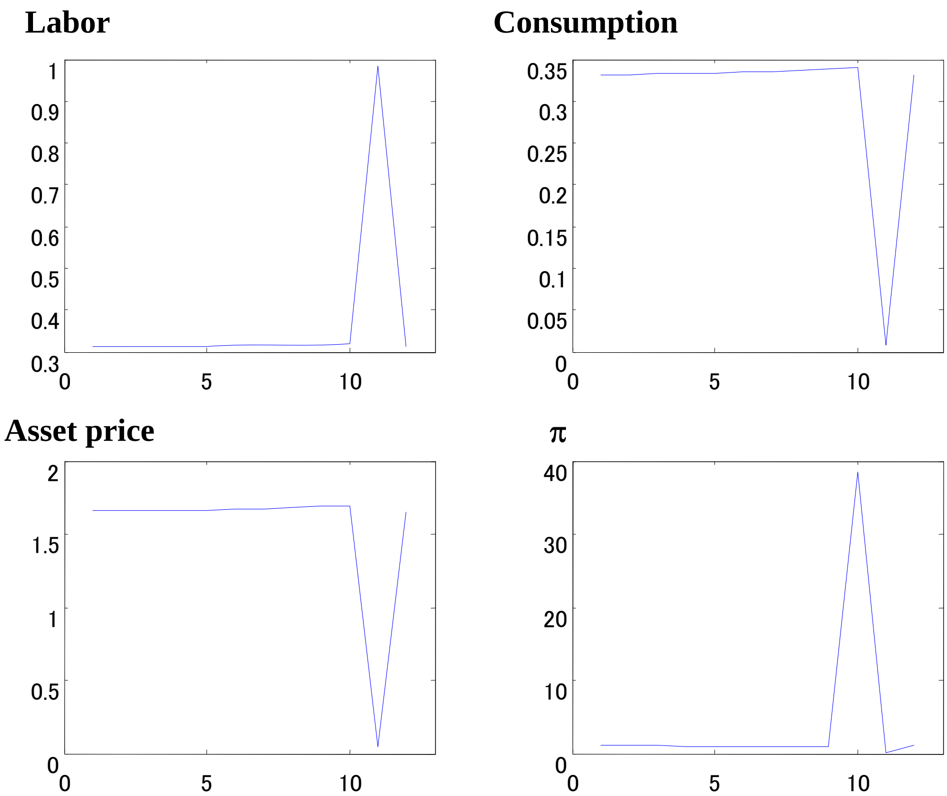


Figure 6. A bubble path with $\mu_\tau > 0$



time	Labor	Consumption	Asset price	π
1	0.31186	0.33186	1.6583	1.0152
2	0.31213	0.33213	1.6597	1.0138
3	0.31248	0.33248	1.6614	1.0119
4	0.31292	0.33292	1.6636	1.0096
5	0.31348	0.33348	1.6664	1.0066
6	0.31419	0.33419	1.67	1.0029
7	0.31509	0.33509	1.6744	0.99816
8	0.31622	0.33622	1.6801	0.99227
9	0.31766	0.33766	1.6873	0.98489
10	0.31947	0.33947	1.6964	38.57
11(τ)	0.9824	0.008646	0.042228	0.029659
12(τ+1)	0.31084	0.33084	1.6532	1.0207

Figure 7. Region for the existence of a bubble path with $\mu_\tau = 0$

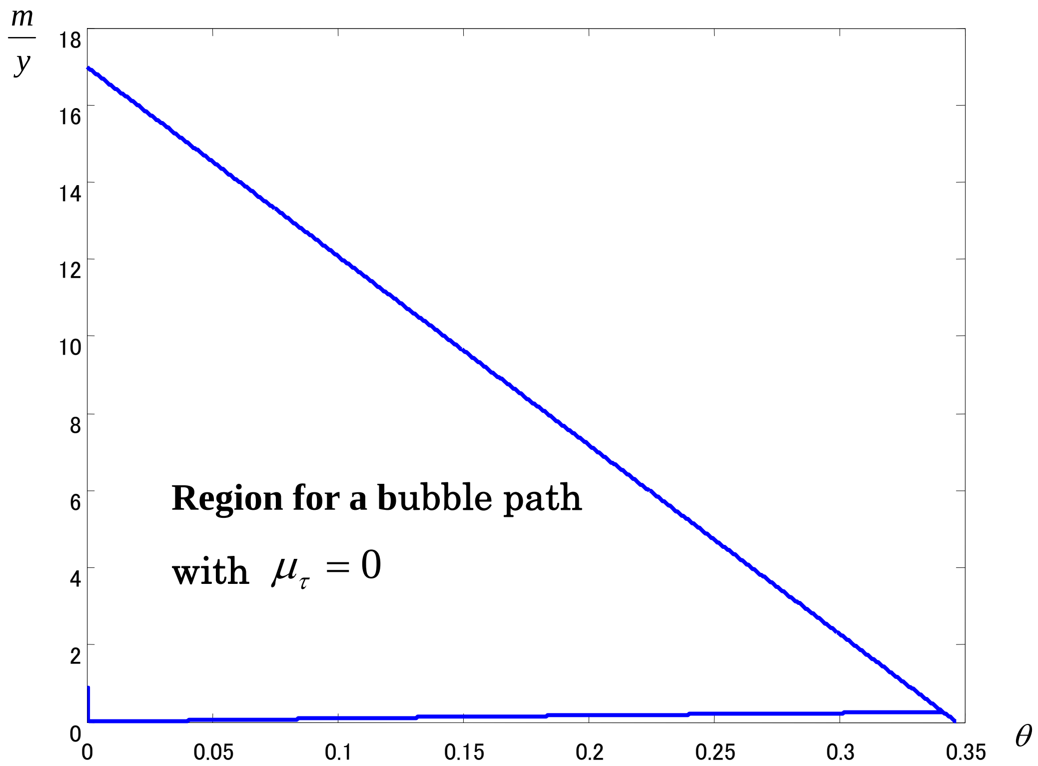
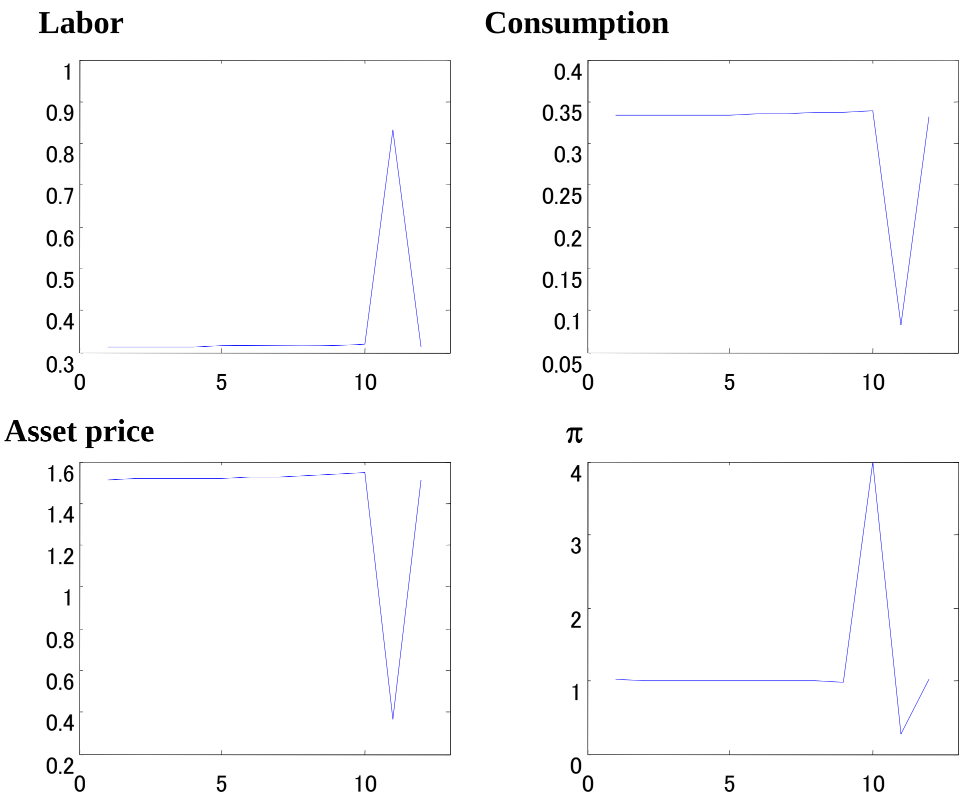


Figure 8. A bubble path with $\mu_\tau = 0$



time	Labor	Consumption	Asset price	π
1	0.31302	0.33302	1.5151	1.0101
2	0.31325	0.33325	1.5163	1.0089
3	0.31354	0.33354	1.5177	1.0073
4	0.31391	0.33391	1.5196	1.0054
5	0.31437	0.33437	1.5218	1.003
6	0.31494	0.33494	1.5247	1.0001
7	0.31565	0.33565	1.5283	0.9964
8	0.31654	0.33654	1.5327	0.99185
9	0.31765	0.33765	1.5382	0.98622
10	0.31903	0.33903	1.5451	3.9981
11(τ)	0.83308	0.083459	0.36646	0.28004
12(τ+1)	0.31207	0.33207	1.5103	1.0151