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Strategy-proof Sharing*

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Abstract

We consider the problem of sharing a good, where agents prefer more to less. In this environment, we prove that a sharing rule satisfies strategy-proofness if and only if it has the quasi-constancy property: no one changes her own share by changing her announcements. Next, by constructing a system of linear equations, we provide a way to find every strategy-proof sharing rule, and identify a necessary and sufficient condition for the existence of a non-constant, strategy-proof sharing rule. Finally, we show that it is only the equal-sharing rule that satisfies strategy-proofness and symmetry.

Keywords: Strategy-proofness, Bossiness, Non-constancy, Quasi-constancy, Symmetry.

JEL Classification Numbers: C72, D71.

1 Introduction

Consider a group of agents who are to share the operating cost of an organization. How do they share the operating cost? The agents usually pay membership dues that are common to all of them to cover the cost. Such a sharing rule is called the *equal-sharing rule*. However, the equal-sharing rule appears to be inappropriate in the sense that it does not at all reflect agents' types, such as intensities of preference. Why do the agents not use a sharing rule that mirrors differences in their types? When we consider the problem, it is important to keep in mind that each agent does not know about the other agents' types. Agents may have an incentive to gain by manipulating the sharing rule through misrepresentation of their types, because their *true* type is unknown to the other agents.

In this paper, we consider the problem of sharing a good, and search for *strategy-proof* sharing rules, where each agent prefers more to less.[1][2] Strategy-proofness is an incentive compatibility property that requires that agents should not benefit from misrepresenting their types irrespective of the types reported by other agents, which was introduced by the seminal papers of Gibbard (1973) and Satterthwaite (1975). The property seems attractive, but it is too strong a requirement in the sense that it rules out almost all rules in many environments.

A *constant sharing rule*, where the good is always split in a fixed ratio, is a familiar rule that satisfies strategy-proofness. In addition to the constant sharing rules, as is well known, there exists a *non-constant* sharing rule that satisfies strategy-proofness if there are three agents (see Example 1). The non-constant, strategy-proof sharing rule demonstrated in Example 1 is a *bossy* sharing rule, i.e., one where a change in an agent's types does not affect her *own* share, but affects *the other* agents' shares.[3] Besides the above sharing rules, is there a strategy-proof sharing rule? The answer is no, as shown by Theorem 1. The theorem tells us that a sharing rule satisfies strategy-proofness if and only if no agent can affect her share at all through misrepresentation of her types. This implies that it is only the bossy sharing rule that satisfies strategy-proofness and non-constancy.

Bossiness might appear unreasonable, as Satterthwaite and Sonnenschein (1981) state that "While we have not exhaustively considered this question, we have identified one substantial consideration that bears on nonbossiness's reasonableness and desirability. It relates to simplicity of design." However, the property of bossiness is inherent in some "nice" mechanisms, including in the Vickrey auction (Vickrey (1961)) and the Clarke–Groves mech-

[1] When considering the problem of sharing a cost, we alternatively assume that each agent prefers less to more.

[2] A sharing rule is a function that assigns a list of shares to each announcement of agents' types.

[3] The notion of bossiness was introduced by Satterthwaite and Sonnenschein (1981).

anisms (Clarke (1971) and Groves (1973)). Furthermore, *non*-bossiness is demanding, because non-bossiness together with strategy-proofness implies *coalitional strategy-proofness* in some environments such as pure exchange economies (see Barberà and Jackson (1995)) or the Shapley–Scarf housing markets (see Pàpai (2000)).[4]

In addition, consider the following example: There are three agents who are to share the cost of a project, where each agent prefers less to more. Suppose that each agent reports a different type and then their shares of the cost are $(0.3, 0.2, 0.5)$. What should be selected as the "fair" shares when agent 1's announcement is changed to the same as that of agent 2? Some might insist that the new shares should be $(0.2, 0.2, 0.6)$, since those who report the same types should be treated as the same. This rule violates strategy-proofness, because agent 1 benefits from the change in her announcement. Others may urge that the new shares should be $(0.3, 0.3, 0.4)$. This rule is bossy, but *not* manipulable.

As mentioned above, the bossy sharing rule is not so unreasonable, and thus, it is of interest to study how to find all of the strategy-proof sharing rules. In Section 4, we construct a system of linear equations by using Theorem 1 in the following way:

- Suppose that there are n agents having m types.
- Choose a ratio arbitrarily, which is the list of shares that agents receive when they announce type 1.
- Choose a ratio such that agent i's share remains unchanged, whenever only agent i changes her announcement.
- Iterate the above operation for all agents.

Thus, we obtain m^n linear equations in nm^{n-1} unknowns. By construction, we can find any strategy-proof sharing rule by solving the linear system. In Example 2, we demonstrate the linear system when there are three agents, each of whom has two types.

Next we identify a necessary and sufficient condition for the existence of a non-constant sharing rule by using the system of the linear equations. We investigate some properties of the linear system, and find a relationship between the constancy of the strategy-proof sharing rule and the dimension of the solution set of the linear system. In conjunction with the fact that the dimension of the solution space can be written as a function of the numbers of agents and of admissible types, the relationship implies a

[4]Coalitional strategy-proofness is a group incentive compatibility property that requires that no coalition of agents should be able to gain from *joint* misrepresentation. Note that coalitional strategy-proofness is a stronger requirement than strategy-proofness.

necessary and sufficient condition that guarantees the existence of the non-constant and strategy-proof sharing rule: there exists a sharing rule satisfying non-constancy and strategy-proofness if and only if there are at least three agents, each of whom has at least two types (Theorem 4). This makes a difference between the cases of two agents and of three or more agents (although the difference could be imagined from the example of Satterthwaite and Sonnenschein (1981) in the pure exchange economy).

Finally, we examine whether there is a non-constant sharing rule that is strategy-proof and *fair*. Our notion of fairness is such that if agents have the same types, then they receive equal shares, which is usually called *symmetry*. In Theorem 5, we prove that there is no sharing rule that satisfies symmetry, strategy-proofness, and non-constancy. The theorem leads to Corollary 1 which asserts that it is only the equal-sharing rule that is strategy-proof and symmetric. This may be a reason why the equal-sharing rule is used in practice for resolving the sharing problem.

The Related Literature

Sprumont (1991) considered the *division problem with single-peaked preferences*, and showed that a division rule satisfies strategy-proofness, Pareto efficiency, and anonymity if and only if it is the uniform allocation rule (e.g., see Sprumont (1991) or Barberà (2001) for details of the uniform allocation rule). Later, Ching (1994) weakened anonymity to symmetry. The sharing problem considered in our paper appears similar to the division problem with single-peaked preferences. Indeed, our problem could be regarded as a special case where each agent has the peak of her preference when she receives the entire good. Therefore, it is possible to consider Corollary 1 as a special case of Ching's result, since the uniform allocation rule is equivalent to the equal-sharing rule when each agent prefers more to less.

Our model is also analogous to the pure exchange economy model considered in Zhou (1991). Zhou showed that there is no mechanism that is strategy-proof, Pareto efficient, and non-inversely-dictatorial in two-agent pure exchange economies, and conjectured that a similar impossibility result could be proved in pure exchange economies with three or more agents.[5] His conjecture implies that there is a difference in the possibility of the existence of a non-constant, strategy-proof, and Pareto efficient mechanism in two-agent versus n-agent settings, where $n \geq 3$. Therefore, the difference implied by Theorem 4 is similar to the difference implied by Zhou's conjecture.

[5]However, Kato and Ohseto (2002) have recently proved that there exist some mechanisms that are strategy-proof, Pareto efficient, and non-inversely-dictatorial in pure exchange economies with four or more agents. Nevertheless, as noted in Kato and Ohseto (2002), Zhou's conjecture is still open in three-agent pure exchange economies.

This paper is organized as follows. Section 2 provides notation and definitions. In Section 3, we characterize strategy-proof sharing rules. A system of linear equations is constructed in Section 4. Section 5 identifies a necessary and sufficient condition for the existence of a non-constant and strategy-proof sharing rule. We search for a sharing rule that is fair and strategy-proof in Section 6, and Section 7 contains concluding remarks. Some proofs are given in the Appendix.

2 Notation and Definitions

Let $N := \{1, 2, \dots, n\}$ be the set of *agents*, where $2 \leq n < +\infty$. Let $X := \left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+ \middle| \sum_{i \in N} x_i = 1\right\}$ be the set of *ratios*, where agent $i \in N$ receives x_i, which we call agent i's *share*.

Let Θ_i be the set of possible *types* of agent $i \in N$. Each agent $i \in N$ has a *utility function* $u_i \colon X \times \Theta_i \to \mathbb{R}$. We assume that each agent $i \in N$ is selfish, i.e., $u_i(x; \theta_i)$ depends only on x_i for any $x \in X$ and any $\theta_i \in \Theta_i$. Let $\Theta_i^m := \left\{\theta_i^1, \theta_i^2, \dots, \theta_i^m\right\} \subset \Theta_i$ be a set of agent i's types: $\theta_i^k \in \Theta_i^m$ only if $u_i(\cdot; \theta_i^k)$ is a strictly increasing function of x_i. The *domain* is the set $\Theta^m := \Theta_1^m \times \Theta_2^m \times \cdots \times \Theta_n^m$. A *type profile* is a list $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \Theta^m$.

A *sharing rule* is a single-valued function $f \colon \Theta^m \to X$, which assigns a list of shares $x \in X$ to each type profile $\theta \in \Theta^m$. It will be convenient to write $f(\theta) = (f_1(\theta), f_2(\theta), \dots, f_n(\theta))$.

Now we introduce a property that the sharing rule is to satisfy. *Strategy-proofness* is an incentive compatibility property, which requires that no agent should be able to benefit from misrepresenting her types irrespective of the other agents' types.

Definition 1 (Strategy-proofness). A sharing rule f satisfies *strategy-proofness* if, for all $\theta \in \Theta^m$ and all $i \in N$, there is no $\theta_i' \in \Theta_i^m$ such that

$$u_i(f(\theta_i', \theta_{-i}); \theta_i) > u_i(f(\theta); \theta_i).$$

A *constant sharing rule* is a sharing rule satisfying strategy-proofness.

Definition 2 (Constant Sharing Rules). A sharing rule f is a *constant sharing rule* if, for some $x \in X$, $f(\theta) = x$ for any $\theta \in \Theta^m$.

A *dictatorial sharing rule*, which always assigns the entire good to a given agent, is a special case of the constant sharing rule. In order to distinguish non-constant sharing rules from the constant sharing rules, we often impose the following condition.

Definition 3 (Non-constancy). A sharing rule f satisfies *non-constancy* if, for some $\theta, \theta' \in \Theta^m$,

$$f(\theta) \neq f(\theta').$$

3 Strategy-proof Sharing Rules

In this section, we investigate what kinds of sharing rules satisfy strategy-proofness. Aside from constant sharing rules, there is a *non-constant* sharing rule that satisfies strategy-proofness, as demonstrated in Example 1.

Example 1. Suppose that there are three agents, 1, 2, and 3, who are to share a good worth \$100 to each of them. Furthermore, suppose that $m = 2$, i.e., for each agent, the set of types consists of only two types. Then, a non-constant and strategy-proof sharing rule is the following:

$$\begin{cases} \bar{f}(\theta_1^1, \theta_2^1, \theta_3^1) = x^1 = (0.7, 0.2, 0.1) \\ \bar{f}(\theta_1^2, \theta_2^1, \theta_3^1) = x^2 = (0.7, 0.1, 0.2) \\ \bar{f}(\theta_1^1, \theta_2^2, \theta_3^1) = x^3 = (0.4, 0.2, 0.4) \\ \bar{f}(\theta_1^2, \theta_2^2, \theta_3^1) = x^4 = (0.4, 0.1, 0.5) \\ \bar{f}(\theta_1^1, \theta_2^1, \theta_3^2) = x^5 = (0.5, 0.4, 0.1) \\ \bar{f}(\theta_1^2, \theta_2^1, \theta_3^2) = x^6 = (0.5, 0.3, 0.2) \\ \bar{f}(\theta_1^1, \theta_2^2, \theta_3^2) = x^7 = (0.2, 0.4, 0.4) \\ \bar{f}(\theta_1^2, \theta_2^2, \theta_3^2) = x^8 = (0.2, 0.3, 0.5). \end{cases}$$

The sharing rule is illustrated in Figure 1. ■

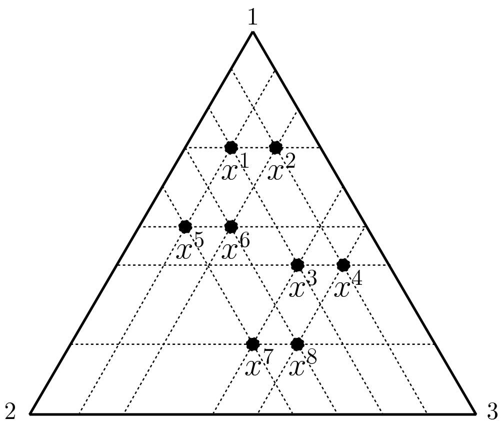

Figure 1: A Non-constant and Strategy-proof Sharing Rule

Example 1 shows that it is possible to design a non-constant sharing rule that satisfies strategy-proofness if there are three agents.[6] The following theorem provides a full characterization of strategy-proof sharing rules.

[6]We shall show in Example 2 how to construct the non-constant, strategy-proof sharing rule.

Theorem 1. *A sharing rule f satisfies strategy-proofness if and only if*

$$f_i(\theta) = f_i(\theta'_i, \theta_{-i})$$

for all $\theta \in \Theta^m$, all $i \in N$, and all $\theta'_i \in \Theta^m_i$.

Proof. The *only if* part: Suppose to the contrary that $f_i(\theta) \neq f_i(\theta'_i, \theta_{-i})$ for some $\theta \in \Theta^m$, some $i \in N$, and some $\theta'_i \in \Theta^m_i$. Without loss of generality, we assume $f_i(\theta) > f_i(\theta'_i, \theta_{-i})$. Since $u_i(\cdot; \theta'_i)$ is a strictly increasing function of x_i, we obtain

$$u_i(f(\theta); \theta'_i) > u_i(f(\theta'_i, \theta_{-i}); \theta'_i),$$

contradicting strategy-proofness.

The *if* part: It is easy to check that if $f_i(\theta) = f_i(\theta'_i, \theta_{-i})$ for all $\theta \in \Theta^m$, all $i \in N$, and all $\theta'_i \in \Theta^m_i$, then f satisfies strategy-proofness. $\square$

Theorem 1 tells us that a sharing rule satisfies strategy-proofness if and only if each agent never changes her own share by misrepresenting her types. This does not lead to the constancy of the sharing rule, because it might be a *bossy* sharing rule, i.e., one where each agent could change *someone else*'s share through misrepresentation of her types, even though she cannot affect her *own* share. Nevertheless, the bossy sharing rule is *quasi-constant* in the sense that each agent never affects her own share by changing her announcements. In this sense, Theorem 1 could be deemed to be an impossibility result.

It follows from Theorem 1 that non-bossiness is inconsistent with strategy-proofness and non-constancy. This reveals a stark contrast between our model and other models such as the pure exchange economy model or the model considered in Sprumont (1991), because non-bossiness is consistent with strategy-proofness and non-constancy in the models (see Barberà and Jackson (1995) for the pure exchange economy model, and Barberà, Jackson, and Neme (1997) for Sprumont's model).

4 The Linear System

We construct a system of linear equations to find all strategy-proof sharing rules. Let $f \colon \Theta^m \to X$ be a strategy-proof sharing rule. Then, by Theorem 1, we have the following:

$$\begin{aligned} f(\theta^1_1, \theta^1_2, \theta^1_3, \ldots, \theta^1_{n-1}, \theta^1_n) &= (x^1_1, x^1_2, x^1_3, \ldots, x^1_{n-1}, x^1_n) \\ f(\theta^2_1, \theta^1_2, \theta^1_3, \ldots, \theta^1_{n-1}, \theta^1_n) &= (x^1_1, x^2_2, x^2_3, \ldots, x^2_{n-1}, x^2_n) \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
f(\theta_1^m,\theta_2^1,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^1,x_2^m,x_3^m,\ldots,x_{n-1}^m,x_n^m)\\
f(\theta_1^1,\theta_2^2,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^2,x_2^1,x_3^{m+1},\ldots,x_{n-1}^{m+1},x_n^{m+1})\\
f(\theta_1^2,\theta_2^2,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^2,x_2^2,x_3^{m+2},\ldots,x_{n-1}^{m+2},x_n^{m+2})\\
&\vdots\\
f(\theta_1^m,\theta_2^2,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^2,x_2^m,x_3^{m+m},\ldots,x_{n-1}^{m+m},x_n^{m+m})\\
&\vdots\\
f(\theta_1^1,\theta_2^m,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^m,x_2^1,x_3^{m(m-1)+1},\ldots,x_{n-1}^{m(m-1)+1},x_n^{m(m-1)+1})\\
f(\theta_1^2,\theta_2^m,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^m,x_2^2,x_3^{m(m-1)+2},\ldots,x_{n-1}^{m(m-1)+2},x_n^{m(m-1)+2})\\
&\vdots\\
f(\theta_1^m,\theta_2^m,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^1) &= (x_1^m,x_2^m,x_3^{m(m-1)+m},\ldots,x_{n-1}^{m(m-1)+m},x_n^{m(m-1)+m})\\
&\vdots\\
f(\theta_1^m,\theta_2^m,\theta_3^m,\ldots,\theta_{n-1}^m,\theta_n^1) &= (x_1^{m^{n-2}},x_2^{m^{n-2}},x_3^{m^{n-2}},\ldots,x_{n-1}^{m^{n-2}},x_n^{m^{n-1}})\\
f(\theta_1^1,\theta_2^1,\theta_3^1,\ldots,\theta_{n-1}^1,\theta_n^2) &= (x_1^{m^{n-2}+1},x_2^{m^{n-2}+1},x_3^{m^{n-2}+1},\ldots,x_{n-1}^{m^{n-2}+1},x_n^1)\\
&\vdots\\
f(\theta_1^m,\theta_2^m,\theta_3^m,\ldots,\theta_{n-1}^m,\theta_n^m) &= (x_1^{m^{n-1}},x_2^{m^{n-1}},x_3^{m^{n-1}},\ldots,x_{n-1}^{m^{n-1}},x_n^{m^{n-1}}),
\end{aligned}
$$

where $x_i^k \geq 0$.

Since $\sum_{i\in N} f_i = 1$, we obtain m^n linear equations as follows:

$$
\begin{gathered}
x_1^1+x_2^1+x_3^1+\cdots+x_{n-1}^1+x_n^1=1\\
x_1^1+x_2^2+x_3^2+\cdots+x_{n-1}^2+x_n^2=1\\
\vdots\\
x_1^1+x_2^m+x_3^m+\cdots+x_{n-1}^m+x_n^m=1\\
x_1^2+x_2^1+x_3^{m+1}+\cdots+x_{n-1}^{m+1}+x_n^{m+1}=1\\
x_1^2+x_2^2+x_3^{m+2}+\cdots+x_{n-1}^{m+2}+x_n^{m+2}=1\\
\vdots\\
x_1^2+x_2^m+x_3^{m+m}+\cdots+x_{n-1}^{m+m}+x_n^{m+m}=1\\
\vdots\\
x_1^m+x_2^1+x_3^{m(m-1)+1}+\cdots+x_{n-1}^{m(m-1)+1}+x_n^{m(m-1)+1}=1\\
x_1^m+x_2^2+x_3^{m(m-1)+2}+\cdots+x_{n-1}^{m(m-1)+2}+x_n^{m(m-1)+2}=1\\
\vdots\\
x_1^m+x_2^m+x_3^{m(m-1)+m}+\cdots+x_{n-1}^{m(m-1)+m}+x_n^{m(m-1)+m}=1
\end{gathered}
$$

$$\vdots$$

$$x_1^{m^{n-2}} + x_2^{m^{n-2}} + x_3^{m^{n-2}} + \cdots + x_{n-1}^{m^{n-2}} + x_n^{m^{n-1}} = 1$$

$$x_1^{m^{n-2}+1} + x_2^{m^{n-2}+1} + x_3^{m^{n-2}+1} + \cdots + x_{n-1}^{m^{n-2}+1} + x_n^{1} = 1$$

$$\vdots$$

$$x_1^{m^{n-1}} + x_2^{m^{n-1}} + x_3^{m^{n-1}} + \cdots + x_{n-1}^{m^{n-1}} + x_n^{m^{n-1}} = 1,$$

where $x_i^k \geq 0$.

By construction, solving the system of the m^n linear equations in nm^{n-1} unknowns, we can find every strategy-proof sharing rule. To handle the linear system easily, we put the equations into matrix form:

$$\left(\begin{array}{cccccccccccccccccccccccccccccccccc}
1&0&\cdots&0&\cdots&0&0&\cdots&0&1&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&1&0&\cdots&0&0&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
1&0&\cdots&0&\cdots&0&0&\cdots&0&0&1&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0&1&\cdots&0&0&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots\\
1&0&\cdots&0&\cdots&0&0&\cdots&0&0&0&\cdots&1&\cdots&0&0&\cdots&0&\cdots&0&0&\cdots&1&0&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
0&1&\cdots&0&\cdots&0&0&\cdots&0&1&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0&0&\cdots&0&1&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
0&1&\cdots&0&\cdots&0&0&\cdots&0&0&1&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0&0&\cdots&0&0&1&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
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0&0&\cdots&0&\cdots&0&1&\cdots&0&0&0&\cdots&0&\cdots&0&1&\cdots&0&\cdots&1&0&\cdots&0&0&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&0\\
\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\ddots&\vdots\\
0&0&\cdots&0&\cdots&0&0&\cdots&1&0&0&\cdots&0&\cdots&0&0&\cdots&1&\cdots&0&0&\cdots&0&0&0&\cdots&0&\cdots&0&0&\cdots&0&\cdots&1
\end{array}\right)
\begin{pmatrix}
x_1^1\\ x_1^2\\ \vdots\\ x_1^m\\ \vdots\\ x_1^{m^{n-2}}\\ x_1^{m^{n-2}+1}\\ \vdots\\ x_1^{m^{n-1}}\\ x_2^1\\ x_2^2\\ \vdots\\ x_2^m\\ \vdots\\ x_2^{m^{n-2}}\\ x_2^{m^{n-2}+1}\\ \vdots\\ x_2^{m^{n-1}}\\ \vdots\\ x_n^1\\ x_n^2\\ \vdots\\ x_n^m\\ x_n^{m+1}\\ x_n^{m+2}\\ \vdots\\ x_n^{2m}\\ \vdots\\ x_n^{m(m-1)+1}\\ x_n^{m(m-1)+2}\\ \vdots\\ x_n^{m(m-1)+m}\\ \vdots\\ x_n^{m^{n-1}}
\end{pmatrix}
= \mathbf{1}.$$

To simplify notation, let A denote the $m^n \times nm^{n-1}$ coefficient matrix, and $\boldsymbol{x}$ denote the $nm^{n-1} \times 1$ matrix.

The following example is helpful in understanding the linear system.

Example 2. Consider again the situation described in Example 1. Let f be a sharing rule that satisfies strategy-proofness. Then, by Theorem 1, we have the following:

$$
\begin{aligned}
f(\theta_1^1, \theta_2^1, \theta_3^1) &= (x_1^1, x_2^1, x_3^1)\\
f(\theta_1^2, \theta_2^1, \theta_3^1) &= (x_1^1, x_2^2, x_3^2)\\
f(\theta_1^1, \theta_2^2, \theta_3^1) &= (x_1^2, x_2^1, x_3^3)\\
f(\theta_1^2, \theta_2^2, \theta_3^1) &= (x_1^2, x_2^2, x_3^4)\\
f(\theta_1^1, \theta_2^1, \theta_3^2) &= (x_1^3, x_2^3, x_3^1)\\
f(\theta_1^2, \theta_2^1, \theta_3^2) &= (x_1^3, x_2^4, x_3^2)\\
f(\theta_1^1, \theta_2^2, \theta_3^2) &= (x_1^4, x_2^3, x_3^3)\\
f(\theta_1^2, \theta_2^2, \theta_3^2) &= (x_1^4, x_2^4, x_3^4),
\end{aligned}
$$

where $x_i^k \geq 0$. Since $\sum_{i \in N} f_i = 1$, we have the following equations:

$$
\begin{aligned}
x_1^1 + x_2^1 + x_3^1 &= 1\\
x_1^1 + x_2^2 + x_3^2 &= 1\\
x_1^2 + x_2^1 + x_3^3 &= 1\\
x_1^2 + x_2^2 + x_3^4 &= 1\\
x_1^3 + x_2^3 + x_3^1 &= 1\\
x_1^3 + x_2^4 + x_3^2 &= 1\\
x_1^4 + x_2^3 + x_3^3 &= 1\\
x_1^4 + x_2^4 + x_3^4 &= 1,
\end{aligned}
$$

where $x_i^k \geq 0$.

We solve the system of the eight linear equations in 12 unknowns to find strategy-proof sharing rules. The system of the linear equations is expressed in matrix form:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1^1\\ x_1^2\\ x_1^3\\ x_1^4\\ x_2^1\\ x_2^2\\ x_2^3\\ x_2^4\\ x_3^1\\ x_3^2\\ x_3^3\\ x_3^4
\end{pmatrix}
= \mathbf{1}.
$$

Solving the linear system, we have

$$\begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \\ x_2^1 \\ x_2^2 \\ x_2^3 \\ x_2^4 \\ x_3^1 \\ x_3^2 \\ x_3^3 \\ x_3^4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, any strategy-proof sharing rule f is written as

$$\begin{cases} f(\theta_1^1, \theta_2^1, \theta_3^1) = (x_1^1, x_2^1, x_3^1) = (1 - \alpha_1 - \alpha_3, \alpha_1 - \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 - \alpha_5) \\ f(\theta_1^2, \theta_2^1, \theta_3^1) = (x_1^1, x_2^2, x_3^2) = (1 - \alpha_1 - \alpha_3, \alpha_1, \alpha_3) \\ f(\theta_1^1, \theta_2^2, \theta_3^1) = (x_1^2, x_2^1, x_3^3) = (1 - \alpha_1 - \alpha_5, \alpha_1 - \alpha_4 + \alpha_5, \alpha_4) \\ f(\theta_1^2, \theta_2^2, \theta_3^1) = (x_1^2, x_2^2, x_3^4) = (1 - \alpha_1 - \alpha_5, \alpha_1, \alpha_5) \\ f(\theta_1^1, \theta_2^1, \theta_3^2) = (x_1^3, x_2^3, x_3^1) = (1 - \alpha_2 - \alpha_3, \alpha_2 - \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 - \alpha_5) \\ f(\theta_1^2, \theta_2^1, \theta_3^2) = (x_1^3, x_2^4, x_3^2) = (1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3) \\ f(\theta_1^1, \theta_2^2, \theta_3^2) = (x_1^4, x_2^3, x_3^3) = (1 - \alpha_2 - \alpha_5, \alpha_2 - \alpha_4 + \alpha_5, \alpha_4) \\ f(\theta_1^2, \theta_2^2, \theta_3^2) = (x_1^4, x_2^4, x_3^4) = (1 - \alpha_2 - \alpha_5, \alpha_2, \alpha_5), \end{cases}$$

for some $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \big\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in [0,1]^5 \big| \ \alpha_1 + \alpha_3 \leq 1, \alpha_2 + \alpha_3 \leq 1, \alpha_4 \leq \alpha_1 + \alpha_5 \leq 1, \alpha_4 \leq \alpha_2 + \alpha_5 \leq 1, \text{ and } \alpha_5 \leq \alpha_3 + \alpha_4 \leq 1\big\}$.

The strategy-proof sharing rule introduced in Example 1 is given by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.1, 0.3, 0.2, 0.4, 0.5)$. ■

5 A Necessary and Sufficient Condition

In this section, we identify a necessary and sufficient condition for the existence of a non-constant, strategy-proof sharing rule. We first provide the following lemma, which is concerned with the properties of the coefficient matrix A.

Lemma 1. *Consider the linear system $A\boldsymbol{x} = \mathbf{1}$. Then the following statements hold whenever $n \geq 2$:*

(i) $\operatorname{rank} A = m^n - (m-1)^n$.

(ii) The dimension of the solution space of the linear system is

$$nm^{n-1} - \{m^n - (m-1)^n\}.$$

The proof of Lemma 1 is given in the Appendix. We next look for the relationship between the constancy of the strategy-proof sharing rule and the dimension of the solution set of the linear system. The following theorem is a fundamental result, which follows from the fact that $n-1$ linear independent vectors are necessary to express all of the constant sharing rules, each of which is a typical strategy-proof sharing rule.

Theorem 2. *Consider the linear system* $A\boldsymbol{x} = \mathbf{1}$. *Then the dimension of the solution set of the linear system is greater than or equal to* $n-1$.

The proof of Theorem 2 appears in the Appendix. Theorem 2 tells us that in order for every strategy-proof sharing rule to be obtained as a solution of the linear system, it is necessary that the dimension of its solution space is at least $n-1$. This leads to the following theorem, which states that $(n-1)$-dimensional solution space is not enough for a non-constant sharing rule to be represented as a solution of the linear system.

Theorem 3. *Consider the linear system* $A\boldsymbol{x} = \mathbf{1}$. *Then, only the constant sharing rule satisfies strategy-proofness if and only if the dimension of the solution set of the linear system is equal to* $n-1$.

The proof of Theorem 3 is in the Appendix. Theorem 3 implies that the existence of the non-constant and strategy-proof sharing rule depends on the dimension of the solution set of the linear system. Combined with Lemma 1, Theorem 3 implies that it also depends on the numbers of agents and of admissible types, whether or not there exists a non-constant, strategy-proof sharing rule, which is formally stated in Theorem 4 below.

Theorem 4. *Consider the linear system* $A\boldsymbol{x} = \mathbf{1}$. *Then, there exists a non-constant and strategy-proof sharing rule if and only if* $n \geq 3$ *and* $m \geq 2$.

Theorem 4 indicates that non-constant, strategy-proof sharing rules as well as *all* of the constant sharing rules appear as solutions to the linear equations whenever there are three or more agents, each of whom has at least two types. Furthermore, the theorem implies that more complicated sharing rules emerge as either the numbers of agents or of types increase, since the dimension of the solution space of the linear system becomes large as either of these numbers grows.

Before proceeding to the proof of Theorem 4, we present two lemmas.

Lemma 2. *Let* $g(s)$ *be a polynomial of degree* 3. *If* g *satisfies*

(i) $g(1) \geq 0$ *and*

(ii) $g'(s) > 0$ *for any* $s \geq 1$,

then $g(s) > 0$ *for any* $s \geq 2$.

Lemma 3. *Let $g(s)$ be a polynomial of degree $l \geq 4$. If g satisfies*

(i) $g(1) \geq 0$,

(ii) $g^i(1) > 0$ *for any* i *with* $1 \leq i \leq l-3$, *and*

(iii) $g^{l-2}(s) > 0$ *for any* $s \geq 1$,

then $g(s) > 0$ *for any* $s \geq 2$, *where* $g^i(s) := \frac{\mathrm{d}^i g(s)}{\mathrm{d}s^i}$.

The proof of Lemma 2 is analogous to that of Lemma 3, which is in the Appendix. Now we prove Theorem 4.

Proof of Theorem 4. By the contrapositive of Theorem 3, Theorem 4 is equivalent to the following statement: the dimension of the solution set of the linear system is *not* $n-1$ if and only if $n \geq 3$ and $m \geq 2$. Then, together with Lemma 1-(ii) and Theorem 2, Theorem 4 is also equivalent to the following statement:

$$nm^{n-1} - \{m^n - (m-1)^n\} > n-1 \text{ if and only if } n \geq 3 \text{ and } m \geq 2. \quad (*)$$

Thus, we prove $(*)$ instead of the original statement.

The *if* part: Given $n \geq 2$ and $m \geq 1$, define a continuous function g as follows:

$$\begin{aligned} g(m) &:= \left\{nm^{n-1} - \{m^n - (m-1)^n\}\right\} - (n-1) \\ &= (m-1)^n - m^n + nm^{n-1} - n + 1. \end{aligned}$$

In order to prove the *if* part of $(*)$, for any integer $n \geq 3$, it is sufficient to show that $g(m) > 0$ whenever $m \geq 2$.

Case 1: $n \geq 4$.

By Lemma 3, in order to prove that $g(m) > 0$ for any $m \geq 2$, it suffices to verify that (i) $g(1) \geq 0$, (ii) $g^i(1) > 0$ for all i with $1 \leq i \leq n-3$, and (iii) $g^{n-2}(m) > 0$ for all $m \geq 1$. Differentiating $g(m)$ i times, we get

$$g^i(m) = \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}\left\{(m-1)^{n-i} - m^{n-i} + (n-i)m^{n-(i+1)}\right\}.$$

First, we check $g(1) \geq 0$.

$$\begin{aligned} g(1) &= (1-1)^n - 1^n + n \cdot 1^{n-1} - n + 1 \\ &= 0 \geq 0. \end{aligned}$$

Second, we verify that $g^i(1) > 0$ for all i with $1 \leq i \leq n-3$.

$$g^i(1) = \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}\left\{(1-1)^{n-i} - 1^{n-i} + (n-i) \cdot 1^{n-(i+1)}\right\}$$

$$= \underbrace{n(n-1)(n-2)\cdots(n-(i-1))}_{i}(n-(i+1)).$$

Since $1 \leq i \leq n-3$, it must hold that $4 \leq (n-(i-1)) \leq n$ and $2 \leq (n-(i+1)) \leq n-2$. Hence, we conclude that $g^i(1) > 0$ for all i with $1 \leq i \leq n-3$.

Finally, we confirm that $g^{n-2}(m) > 0$ for all $m \geq 1$.

$$\begin{aligned} g^{n-2}(m) &= \underbrace{n(n-1)(n-2)\cdots(n-((n-2)-1))}_{n-2} \\ &\qquad \times \left\{(m-1)^{n-(n-2)} - m^{n-(n-2)} + (n-(n-2))m^{n-((n-2)+1)}\right\} \\ &= \underbrace{n(n-1)(n-2)\cdots 3}_{n-2} \times \left\{(m-1)^2 - m^2 + 2m\right\} \\ &= \underbrace{n(n-1)(n-2)\cdots 3}_{n-2} \times 1 > 0. \end{aligned}$$

Therefore, $g^{n-2}(m) > 0$ for all $m \geq 1$.

Case 2: $n = 3$.

By Lemma 2, in order to show that $g(m) > 0$ for any $m \geq 2$, it is sufficient to check that (i) $g(1) \geq 0$ and (ii) $g'(m) > 0$ for any $m \geq 1$. In a way similar to Case 1, we can verify that g fulfills (i) and (ii).

The *only if* part: Suppose not, then $n = 2$ or $m = 1$. It is easy to check that the inequality $nm^{n-1} - \{m^n - (m-1)^n\} > n-1$ does not hold when $n = 2$ or $m = 1$. □

Theorem 4 gives a condition that is necessary and sufficient for the existence of a sharing rule satisfying non-constancy and strategy-proofness. It turns out that, under the realistic assumption that each agent has more than one type, we can design non-constant and strategy-proof sharing rules whenever there are at least three agents, while we can never do so when there are only two agents. This makes a critical difference between the two-agent and n-agent cases, where $n \geq 3$. The result parallels the conjecture of Zhou (1991), who states that there exists a rule that satisfies non-constancy, strategy-proofness, and Pareto efficiency in n-agent pure exchange economies, where $n \geq 3$, whereas there does not exist such a rule in two-agent pure exchange economies.[7]

[7] To be precise, Zhou (1991) conjectured that a rule satisfies strategy-proofness and Pareto efficiency if and only if it is inversely-dictatorial in pure exchange economies. Note that there exists an inversely-dictatorial and non-constant rule if there are three or more agents, while every inversely-dictatorial rule is constant (because it is dictatorial) when there are only two agents (see Zhou (1991) or Kato and Ohseto (2002) for details).

6 Fairness

In this section, we search for rules satisfying strategy-proofness and *fairness*. As the notion of fairness, we adopt *symmetry*, which is one of the weakest properties that pertain to fairness. Symmetry requires that if agents announce identical types, they should receive the same shares.

Definition 4 (Symmetry). A sharing rule f satisfies *symmetry* if, for all $\theta \in \Theta^m$ and all $i, j \in N$, if $\theta_i = \theta_j$, then $f_i(\theta) = f_j(\theta)$.

The following theorem asserts that symmetry, strategy-proofness, and non-constancy are jointly inconsistent.

Theorem 5. *There is no sharing rule f satisfying symmetry, strategy-proofness, and non-constancy.*

Proof. Suppose to the contrary that there does exist a sharing rule f that satisfies symmetry, strategy-proofness, and non-constancy. Consider $\theta^1 = (\theta_1^1, \theta_2^1, \theta_3^1, \ldots, \theta_n^1) \in \Theta^m$. Then, by symmetry, we have $f(\theta^1) = (1/n, \ldots, 1/n)$.

Step 1: $f(\theta_i', \theta_{-i}^1) = (1/n, \ldots, 1/n)$ for all $i \in N$ and all $\theta_i' \in \Theta_i^m$.
Suppose not, then, for some $i \in N$ and some $\theta_i' \in \Theta_i^m$, we have $f(\theta_i', \theta_{-i}^1) \neq (1/n, \ldots, 1/n)$.

Case 1-1: $f_i(\theta_i', \theta_{-i}^1) \neq 1/n$.
Theorem 1 implies $f_i(\theta^1) = f_i(\theta_i', \theta_{-i}^1) = 1/n$: a contradiction.

Case 1-2: $f_h(\theta_i', \theta_{-i}^1) \neq 1/n$ for some $h \in N \setminus \{i\}$.
By the argument of Case 1-1, we have $f_i(\theta_i', \theta_{-i}^1) = 1/n$. Symmetry implies that $f_g(\theta_i', \theta_{-i}^1) = f_h(\theta_i', \theta_{-i}^1)$ for any $g, h \in N \setminus \{i\}$. Since $\sum_{i \in N} f_i(\theta_i', \theta_{-i}^1) = 1$, these imply that $f_g(\theta_i', \theta_{-i}^1) = f_h(\theta_i', \theta_{-i}^1) = 1/n$ for any $g, h \in N \setminus \{i\}$: a contradiction.

Step 2: $f(\theta_i', \theta_j'', \theta_{-i,j}^1) = (1/n, \ldots, 1/n)$ for all $i \in N$, all $j \in N$, all $\theta_i' \in \Theta_i^m$, and all $\theta_j'' \in \Theta_j^m$.
Suppose not, then there exist $i \in N$, $j \in N$, $\theta_i' \in \Theta_i^m$, and $\theta_j'' \in \Theta_j^m$ such that $f(\theta_i', \theta_j'', \theta_{-i,j}^1) \neq (1/n, \ldots, 1/n)$.

Case 2-1: $f_i(\theta_i', \theta_j'', \theta_{-i,j}^1) \neq 1/n$.
By Step 1, it holds that $f(\theta_j'', \theta_{-j}^1) = (1/n, \ldots, 1/n)$. It follows from Theorem 1 that $f_i(\theta_j'', \theta_{-j}^1) = f_i(\theta_i', \theta_j'', \theta_{-i,j}^1) = 1/n$: a contradiction.

Case 2-2: $f_j(\theta_i', \theta_j'', \theta_{-i,j}^1) \neq 1/n$.
This case follows from an argument similar to Case 2-1.

Case 2-3: $f_h(\theta_i', \theta_j'', \theta_{-i,j}^1) \neq 1/n$ for some $h \in N \setminus \{i, j\}$.

The arguments of Cases 2-1 and 2-2 imply $f_i(\theta'_i, \theta''_j, \theta^1_{-i,j}) = f_j(\theta'_i, \theta''_j, \theta^1_{-i,j}) = 1/n$. By symmetry, it must hold that $f_g(\theta'_i, \theta''_j, \theta^1_{-i,j}) = f_h(\theta'_i, \theta''_j, \theta^1_{-i,j})$ for any $g, h \in N \setminus \{i, j\}$. Since $\sum_{i \in N} f_i(\theta'_i, \theta''_j, \theta^1_{-i,j}) = 1$, it follows that $f_g(\theta'_i, \theta''_j, \theta^1_{-i,j}) = f_h(\theta'_i, \theta''_j, \theta^1_{-i,j}) = 1/n$ for any $g, h \in N \setminus \{i, j\}$: a contradiction.

Step 3: $f(\theta'_i, \theta''_j, \theta'''_k, \theta^1_{-i,j,k}) = (1/n, \ldots, 1/n)$ for all $i \in N$, all $j \in N$, all $k \in N$, all $\theta'_i \in \Theta^m_i$, all $\theta''_j \in \Theta^m_j$, and all $\theta'''_k \in \Theta^m_k$.
The argument for Step 3 is analogous to the arguments for Steps 1 and 2.

Iteration of similar arguments for further agents establishes that, for any $\tilde{\theta} \in \Theta^m$,

$$f(\tilde{\theta}) = f(\theta^1) = (1/n, \ldots, 1/n),$$

which contradicts non-constancy. $\square$

The proof of Theorem 5 implies that if a sharing rule satisfies symmetry and strategy-proofness, then it is a *equal-sharing rule*. Hence, as we formalize below, we can conclude that it is only the equal-sharing rule that satisfies symmetry and strategy-proofness.

Definition 5 (The Equal-Sharing Rule). A sharing rule f is the *equal-sharing rule* if, for all $\theta \in \Theta^m$,

$$f(\theta) = (1/n, \ldots, 1/n).$$

Corollary 1. *A sharing rule f satisfies symmetry and strategy-proofness if and only if it is the equal-sharing rule.*

Corollary 1 parallels the result of Ching (1994) in the division problem with single-peaked preferences, which states that a rule satisfies symmetry, strategy-proofness, and Pareto efficiency if and only if it is the uniform rule.[8] Ching's result is a generalization of Sprumont's characterization that asserts that a rule satisfies anonymity (or equivalently *envy-freeness*), strategy-proofness, and Pareto efficiency if and only if it is the uniform rule (Sprumont (1991)).[9]

It is easy to show that only the equal-sharing rule satisfies envy-freeness. Combined with Corollary 1, this implies that envy-freeness is equivalent to the conjunction of symmetry and strategy-proofness in our environment. The equivalence between envy-freeness and symmetry plus strategy-proofness fails to hold in Sprumont's environment. Therefore, it is a difference between our environment and Sprumont's whether the equivalence holds or not.

[8] As mentioned in the Introduction, the uniform rule is equivalent to the equal-sharing rule in our environment.

[9] Envy-freeness is a requirement that each agent should never prefer someone else's share to her own, which was first introduced by Foley (1967).

7 Conclusion

In this paper, we have characterized the class of strategy-proof sharing rules, and provided a way to find all of the strategy-proof sharing rules. In Theorem 1, we have shown that only strategy-proof sharing rules have the quasi-constancy property: each agent never changes her own share through misrepresentation of her types. The theorem implies that only strategy-proof and non-bossy sharing rules are constant. Hence, combined with the fact that strategy-proofness plus non-bossiness implies coalitional strategy-proofness, the theorem leads to an impossibility result: there is no sharing rule that satisfies coalitional strategy-proofness and non-constancy. Thus, Theorem 1 seems to be a possibility result, but it has a somewhat negative implication.

In Section 5, we have established a fundamental result concerning the dimension of the solution space of the linear system constructed in Section 4: every strategy-proof sharing rule can be represented as a solution of the linear system, only when the dimension of the solution set is greater than or equal to the number of agents minus one. Furthermore, we have established that more *non-constant*, strategy-proof sharing rules emerge, as the dimension of the solution space, which is determined by the numbers of agents and types, becomes greater than the number of agents minus one. Thus, we have shown that there are non-constant, strategy-proof sharing rules, when there are at least three agents who each have at least two types. However, we have not yet found an algorithm for easily finding such sharing rules (although we know that it is possible to find them by solving the linear system constructed in Section 4). It would be an interesting area of further research to provide such an algorithm.

The model considered in this paper is related to Sprumont's model (Sprumont (1991)) and to the pure exchange economy model considered in Zhou (1991), Barberà and Jackson (1995), Kato and Ohseto (2002), and others. Here, we have obtained some results similar to the ones given by them. However, we have provided two results, each of which holds only in our model: one is the inconsistency between non-constancy and strategy-proofness plus non-bossiness (or equivalently the inconsistency between non-constancy and coalitional strategy-proofness); and the other is the equivalence between envy-freeness and strategy-proofness plus symmetry. The difference is due to the fact that, in our model, no one is indifferent between any pair of outcomes such that she receives distinct shares.

Strategy-proofness is closely related to Nash implementability, since strategy-proofness is implied by *monotonicity* that is both necessary and sufficient for Nash implementation in our environment.[10] It is easy to show that mono-

[10]Necessity follows immediately from Maskin (1999) who showed that monotonicity is necessary for Nash implementation. In the case of three or more agents, sufficiency also follows from Maskin (1999), who proved that monotonicity and no veto power are sufficient for Nash implementation, together with the fact that no veto power is automatically satis-

tonicity is equivalent to constancy in our environment. Therefore, every strategy-proof sharing rule is Nash implementable when there are only two agents, since it is a constant sharing rule. On the other hand, when there are three or more agents, not all strategy-proof sharing rules are Nash implementable. Indeed, *none* of the bossy and strategy-proof sharing rules are implementable in Nash equilibria, because they violate monotonicity in our environment. In short, only constant sharing rules are Nash implementable no matter how many agents there are.

It is also easy to check that, in our environment, monotonicity is equivalent to the *rectangular property* which is a necessary and sufficient condition for *secure implementation*, i.e., double implementation in Nash and dominant strategy equilibria (see Saijo, Sjöström, and Yamato (2003) for details of secure implementation). Hence, we reach a conclusion similar to one about Nash implementation: only constant sharing rules are secure implementable, whereas none of the non-constant and strategy-proof sharing rules are secure implementable. As already remarked, constant sharing rules are distinguished from the other sharing rules by many properties, such as fairness, coalitional strategy-proofness, implementability, etc. This appears to be a reason why non-constant sharing rules are not used in practice, even if they satisfy strategy-proofness.

In this paper, we have searched for *bossy* and strategy-proof rules by constructing a system of linear equations. The way of finding such rules developed here could help in the search for bossy, strategy-proof rules in other environments, including in pure exchange economies where *non-bossy*, strategy-proof rules have been sought.

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fied in our environment. In the two-agent case, sufficiency follows from the fact that only the constant sharing rule satisfies strategy-proofness which is implied by monotonicity, a necessary condition for Nash implementation.

Appendix

In the Appendix, we first provide a remark, which concerns constant sharing rules, and then provide some proofs.

Remark 1. Every constant sharing rule can be written as a vector

$$(\underbrace{\bar{x}_1, \bar{x}_1, \ldots, \bar{x}_1}_{m^{n-1}}, \underbrace{\bar{x}_2, \bar{x}_2, \ldots, \bar{x}_2}_{m^{n-1}}, \underbrace{\bar{x}_3, \bar{x}_3, \ldots, \bar{x}_3}_{m^{n-1}}, \ldots, \underbrace{\bar{x}_n, \bar{x}_n, \ldots, \bar{x}_n}_{m^{n-1}})^t$$

for some $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n) \in X$, where the constant sharing rule always assigns $\bar{x}_1$ to agent 1, $\bar{x}_2$ to agent 2, $\bar{x}_3$ to agent 3, …, $\bar{x}_n$ to agent n.

Proof of Lemma 1-(i). Given $n \geq 2$ and $m \geq 1$, define matrix A^r by

$$a^r_{pq} = \begin{cases} a_{pq} & \text{if } q \in \left\{(r-1)m^{n-1}+1, (r-1)m^{n-1}+2, \ldots, rm^{n-1}\right\}, \\ 0 & \text{otherwise,} \end{cases}$$

where a^r_{pq} and a_{pq} denote the pq-th elements of A^r and A, respectively. Then, we obtain the following matrices $A^1, A^2, \ldots, A^n$.

$$A^1 = \begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix},$$

$$A^2 = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix},$$

$$\vdots$$

$$
A^n = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}.
$$

Note that $A^1 + A^2 + \cdots + A^n = A$. Let $\binom{v}{w} := \frac{v!}{w!(v-w!)}$.

Consider matrix A^1. By means of Gaussian elimination, we obtain

$$\operatorname{rank} A^1 = m^{n-1}.$$

Consider $(A^1 + A^2)$. By Gaussian elimination, we have

$$
\begin{aligned}
\operatorname{rank}(A^1 + A^2) &= 2m^{n-1} - (m^{n-2}) \\
&= 2m^{n-1} - \binom{2}{2} m^{n-2}(-1)^2.
\end{aligned}
$$

Consider $(A^1 + A^2 + A^3)$. By applying Gaussian elimination, we get

$$
\begin{aligned}
\operatorname{rank}(A^1 + A^2 + A^3) &= 3m^{n-1} - (3m^{n-2} - m^{n-3}) \\
&= 3m^{n-1} - \left\{ \binom{3}{2} m^{n-2}(-1)^2 + \binom{3}{3} m^{n-3}(-1)^3 \right\} \\
&= 3m^{n-1} - \sum_{h=2}^{3} \binom{3}{h} m^{n-h}(-1)^h.
\end{aligned}
$$

Thus, by the construction of A, we can find that

$$
\operatorname{rank}(A^1 + A^2 + \cdots + A^r) = \begin{cases} m^{n-1} & \text{if } r = 1, \\ rm^{n-1} - \displaystyle\sum_{h=2}^{r} \binom{r}{h} m^{n-h}(-1)^h & \text{if } r \geq 2. \end{cases}
$$

Therefore, we establish that

$$
\begin{aligned}
\operatorname{rank} A &= \operatorname{rank}(A^1 + A^2 + \cdots + A^n) \\
&= nm^{n-1} - \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h \\
&= nm^{n-1} - \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + (m^n - m^n)
\end{aligned}
$$

$$
\begin{aligned}
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h - nm^{n-1} + m^n \right\} \\
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + \frac{n!}{1!(n-1)!} m^{n-1}(-1)^1 + \frac{n!}{0!n!} m^n(-1)^0 \right\} \\
&= m^n - \left\{ \sum_{h=2}^{n} \binom{n}{h} m^{n-h}(-1)^h + \binom{n}{1} m^{n-1}(-1)^1 + \binom{n}{0} m^n(-1)^0 \right\} \\
&= m^n - \sum_{h=0}^{n} \binom{n}{h} m^{n-h}(-1)^h \\
&= m^n - (m-1)^n,
\end{aligned}
$$

whenever $n \geq 2$. □

Proof of Lemma 1-(ii). Let $\boldsymbol{c}_1 := (\underbrace{1, 1, \dots, 1}_{m^{n-1}}, 0, 0, \dots, 0)^t$ be a constant sharing rule where agent 1 gets the entire share of the good. Since the constant sharing rule satisfies strategy-proofness, $\boldsymbol{c}_1$ is a particular solution of the linear system $A\boldsymbol{x} = \mathbf{1}$. So, the solution set of the linear system is the affine space

$$
\left\{ \boldsymbol{x} \in \mathbb{R}^{nm^{n-1}} \,\middle|\, \boldsymbol{x} = \boldsymbol{c}_1 + \boldsymbol{w} \text{ for some } \boldsymbol{w} \in \mathrm{Null}(A) \right\}.
$$

Since the dimension of the affine space is equal to that of $\mathrm{Null}(A)$, and since $\dim \mathrm{Null}(A)$ is equal to the number of variables nm^{n-1} minus $\mathrm{rank}(A)$, the dimension of the solution space is equal to $nm^{n-1} - \{m^n - (m-1)^n\}$. □

Proof of Theorem 2. Suppose to the contrary that the dimension of the solution space of the linear system is less than $n-1$, i.e., $\dim \mathrm{Null}(A) < n-1$. Except for the particular solution $\boldsymbol{c}_1$ defined in the proof of Lemma 1-(ii), the linear system $A\boldsymbol{x} = \mathbf{1}$ must have $n-1$ kinds of solutions such that

$$
\begin{aligned}
\boldsymbol{c}_2 &= (\underbrace{0, 0, \dots, 0}_{m^{n-1}}, \underbrace{1, 1, \dots, 1}_{m^{n-1}}, \underbrace{0, 0, \dots, 0}_{m^{n-1}}, \dots, \underbrace{0, 0, \dots, 0}_{m^{n-1}})^t, \\
\boldsymbol{c}_3 &= (\underbrace{0, 0, \dots, 0}_{m^{n-1}}, \underbrace{0, 0, \dots, 0}_{m^{n-1}}, \underbrace{1, 1, \dots, 1}_{m^{n-1}}, \dots, \underbrace{0, 0, \dots, 0}_{m^{n-1}})^t, \\
&\vdots \\
\boldsymbol{c}_n &= (\underbrace{0, 0, \dots, 0}_{m^{n-1}}, \underbrace{0, 0, \dots, 0}_{m^{n-1}}, \underbrace{0, 0, \dots, 0}_{m^{n-1}}, \dots, \underbrace{1, 1, \dots, 1}_{m^{n-1}})^t,
\end{aligned}
$$

because each of the solutions $\boldsymbol{c}_2, \boldsymbol{c}_3, \dots, \boldsymbol{c}_n$ is a constant sharing rule, which satisfies strategy-proofness. It follows from the definition of the solution set provided in the proof of Lemma 1-(ii) that the vectors

$$\begin{aligned}
\boldsymbol{c}_2 - \boldsymbol{c}_1 &= (\underbrace{-1,-1,\dots,-1}_{m^{n-1}}, \underbrace{1,1,\dots,1}_{m^{n-1}}, \underbrace{0,0,\dots,0}_{m^{n-1}}, \dots, \underbrace{0,0,\dots,0}_{m^{n-1}})^t,\\
\boldsymbol{c}_3 - \boldsymbol{c}_1 &= (\underbrace{-1,-1,\dots,-1}_{m^{n-1}}, \underbrace{0,0,\dots,0}_{m^{n-1}}, \underbrace{1,1,\dots,1}_{m^{n-1}}, \dots, \underbrace{0,0,\dots,0}_{m^{n-1}})^t,\\
&\vdots\\
\boldsymbol{c}_n - \boldsymbol{c}_1 &= (\underbrace{-1,-1,\dots,-1}_{m^{n-1}}, \underbrace{0,0,\dots,0}_{m^{n-1}}, \underbrace{0,0,\dots,0}_{m^{n-1}}, \dots, \underbrace{1,1,\dots,1}_{m^{n-1}})^t
\end{aligned}$$

are all contained in $\mathrm{Null}(A)$. Since the $n-1$ vectors $\boldsymbol{c}_2-\boldsymbol{c}_1, \boldsymbol{c}_3-\boldsymbol{c}_1, \dots, \boldsymbol{c}_n-\boldsymbol{c}_1$ are linearly independent, $\dim \mathrm{Null}(A) = n-1$: a contradiction because we have assumed that $\dim \mathrm{Null}(A) < n-1$. □

Proof of Theorem 3. The *if* part: Suppose not, then there exists a non-constant sharing rule satisfying strategy-proofness. Let $\boldsymbol{c}'$ denote the non-constant sharing rule. Then $\boldsymbol{c}_2 - \boldsymbol{c}_1, \boldsymbol{c}_3 - \boldsymbol{c}_1, \dots, \boldsymbol{c}_n - \boldsymbol{c}_1$, and $\boldsymbol{c}' - \boldsymbol{c}_1$ are linearly independent; otherwise, for some $(r_2, r_3, \dots, r_n)$, it must hold that

$$\begin{aligned}
\boldsymbol{c}' - \boldsymbol{c}_1 &= r_2(\boldsymbol{c}_2 - \boldsymbol{c}_1) + r_3(\boldsymbol{c}_3 - \boldsymbol{c}_1) + \cdots + r_n(\boldsymbol{c}_n - \boldsymbol{c}_1)\\
\boldsymbol{c}' &= \{1 - (r_2 + r_3 + \cdots + r_n)\}\boldsymbol{c}_1 + r_2\boldsymbol{c}_2 + r_3\boldsymbol{c}_3 + \cdots + r_n\boldsymbol{c}_n,
\end{aligned}$$

which contradicts the fact that c' is a non-constant sharing rule. Consequently, $\mathrm{Null}(A)$ has n linear independent vectors $\boldsymbol{c}_2-\boldsymbol{c}_1, \boldsymbol{c}_3-\boldsymbol{c}_1, \dots, \boldsymbol{c}_n-\boldsymbol{c}_1$, and $\boldsymbol{c}' - \boldsymbol{c}_1$, so $\dim \mathrm{Null}(A) = n$: a contradiction because $\dim \mathrm{Null}(A) = n-1$.

The *only if* part: Suppose that only the constant sharing rule satisfies strategy-proofness. Then the linear system $A\boldsymbol{x} = \boldsymbol{1}$ must have the solutions $\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n$ defined in the proofs of Lemma 1-(ii) and Theorem 2, because each of the solutions is a constant sharing rule satisfying strategy-proofness. It follows from the same argument as in the proof of Theorem 2 that $\dim \mathrm{Null}(A) = n-1$. Since any other constant sharing rule can be written as $\boldsymbol{c}_1$ plus a linear combination of $\boldsymbol{c}_2 - \boldsymbol{c}_1, \boldsymbol{c}_3 - \boldsymbol{c}_1, \dots, \boldsymbol{c}_n - \boldsymbol{c}_1$, the dimension of $\mathrm{Null}(A)$ still remains $n-1$. □

Proof of Lemma 3. Since $g^{l-3}(1) > 0$ and $g^{l-2}(s) > 0$ for all $s \geq 1$ by Conditions (ii) and (iii) respectively, it must hold that $g^{l-3}(s) > 0$ for all $s \geq 1$. In conjunction with $g^{l-4}(1) > 0$, this implies that $g^{l-4}(s) > 0$ for all

$s \geq 1$. Similarly together with $g^{l-5}(1) > 0$, this implies that $g^{l-5}(s) > 0$ for all $s \geq 1$. Iterations of this argument implies that $g^1(s) > 0$ for all $s \geq 1$. Combining Condition (i), i.e., $g(1) \geq 0$, we conclude that $g(s) > 0$ for any $s \geq 2$. $\square$

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