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Firm growth and Laplace distribution: The importance of large jumps[1]

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Abstract

Recent empirical studies have shown that firm growth rate distribution is not Gaussian but closely follows a Laplace distribution. This robust feature of the growth rate distribution challenges existing models based on Gibrat's model because it predicts a Gaussian distribution. First, we analyze more than 100,000 Japanese firms and empirically show that the Laplace shape can be observed for the Japanese firms. Then, by using the theory of stochastic processes, we theoretically show that the absence of jumps causes the discrepancy between Gibrat's model and the Laplace shape. In particular, based on the Laplace shape and the law of proportionate effect, we show that the firm growth process is a jump process. In other words, firm growth cannot be explained by the consequence of many small shocks but is determined by a few large jumps. The widely observed Laplace distribution reflects this jump property of firm growth dynamics.

Keywords: Firm growth; Gibrat's model; Law of proportionate effect; Laplace distribution; Variance Gamma process.
JEL classification: D21; D22; L10.

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1 Introduction

How do firms grow? Is there any empirical law governing firm growth patterns? Empirical growth paths are very complicated and, at first glance, there appears to be no law governing the dynamics. For example, Ashton (1926), who studies British textile firms, writes, “In their growth they obey no one law. A few apparently undergo a steady expansion ... . With others, increase in size takes place by a sudden leap” (pp.572–573). However, although it is difficult to predict individual firms' growth, a series of empirical studies have shown that there are several statistical regularities in firm size and growth dynamics (see, e.g. Sutton (1997); Coad (2009); Dosi et al. (2016)). One of the most important regularities is firm size distribution, which is rightly skewed and fat-tailed. A cornerstone in the literature to explain this regularity is Gibrat's model (Gibrat (1931)), in which firm size and its growth rate are assumed to be statistically independent of each other (known as *the law of proportionate effect*). In this model, firm size is the result of independent multiplicative shocks and, by taking the logarithm, the distribution of the log of firm size converges to a Gaussian distribution by the central limit theorem (CLT). Although Gibrat's main concern is to explain the right skewness of the firm size distribution, his model also makes a prediction about the growth rate distribution because firm size is the cumulative result of the underlying growth dynamics. To be precise, since firm growth is the product of successive shocks within a given interval, his model predicts that growth rates defined by the log difference of firm size follow a Gaussian distribution.

However, although it has been empirically shown that the law of proportionate effect itself, that is, the independence between firm size and growth rate, is a good working hypothesis,[1] recent studies have shown that the prediction of a Gaussian growth rate distribution is not empirically supported. Since Stanley et al. (1996), subsequent studies have revealed that the growth rate distribution is not Gaussian but closely follows a Laplace distribution, which is sharply peaked around the center and has a fatter tail than a Gaussian distribution. Amaral et al. (1997) analyze all publicly traded US manufacturing firms and find significant departure from Gaussian distribution. Bottazzi et al. (2002) and Bottazzi and Secchi (2003) analyze Italian firms and show that the growth rate distribution is very close to a Laplace distribution (see also Dosi (2007); Coad (2009); Dosi et al. (2016)).[2] More interestingly, the same distribution shape can be observed at a more disaggregated level. Bottazzi et al. (2007) show that this distribution shape is invariant across Italian manufacturing subsectors. Lunardi et al. (2014) develop a statistical test to check whether the distribution shape is due to the intrinsic feature of firm growth dynamics or the aggregation effect of heterogeneous firms, showing that firms belonging to the same subsector are sufficiently homogeneous and their growth rate distributions at the individual level can be well approximated by a common Laplace distribution. These results imply that the underlying growth process generating the Laplace distribution is totally different from Gibrat's model. Therefore, the observed Laplace shape challenges existing models based on Gibrat's model as well as our understanding of firm growth dynamics.

The aim of our study is to explore the observed Laplace shape by using a stochastic approach. In particular, we take a different approach from previous studies in that we do not provide a micro-foundation for firm growth or its optimization problem but rather spell out a set of properties that such a model must satisfy to be consistent with the empirical evidence. Beginning with the generalization of Gibrat's model, we show that a stochastic process consistent with the two empirical facts, that is, the law of proportionate effect and Laplace distribution, is unique and a pure jump process (the *variance Gamma process*). This process is

[1] For a survey of empirical results about the law of proportionate effect, see Santarelli et al. (2006), who conclude that “[o]ne cannot conclude that the Law is generally valid nor that it is systematically rejected” (p.43). For example, several empirical studies reject the law of proportionate effect by showing that smaller firms grow faster than larger firms (e.g., Hall (1987); Dunne and Hughes (1994); Audretsch et al. (1999); Calvo (2006)) and that the growth dynamics of large firms are less volatile than those of smaller firms (e.g., Bottazzi and Secchi (2006b); Secchi et al. (2018)). On the other hand, some researchers (e.g., Lotti et al. (2003, 2009); Fotopoulos and Giotopoulos (2010)) argue that the law of proportionate effect holds for the population of mature firms over the minimum efficient scale. Related to this literature, Daunfeldt and Elert (2013) examine in which industries this law is likely to hold.

The aim of our study is *not* to decide this long-term controversy, that is, whether the law of proportionate effect is empirically valid or should be rejected. The point is that, as the controversy suggests, the law of proportionate effect can be used for the description of firm growth dynamics, at least, as a first approximation, by excluding start-up firms or by focusing on particular industries.

[2] Related to these works, Alfarano et al. (2012) and Mundt et al. (2015) show that profit rates also follow a Laplace distribution. In their studies, a statistical equilibrium is considered to explain the distribution shape. Although the relationship between these two distributions is worth exploring in future research, our study focuses on growth rate distribution and examines the implications for the distribution shape.

completely characterized by discrete changes (i.e., jumps) without any continuous changes, and in particular, mostly determined by a few large jumps. Put differently, firm growth is not the consequence of many small shocks but determined by a few big successes (or big failures). The observed Laplace shape reflects this jump property. This is sharply contrast with Gibrat's model because Gibrat's model implicitly assumes a continuous process as the underlying firm growth process and excludes the possibility of jumps. We show that the discrepancy between Gibrat's model and the observed Laplace distribution is due to the absence of jumps, and that the jump property is an indispensable feature of firm growth dynamics. The robustness of this jump property is further examined by considering cases in which the growth rate distribution has a fatter tail than a Laplace distribution, as suggested by Bottazzi et al. (2011) and Buldyrev et al. (2007a,b). Our analysis shows that the jump property of the firm growth process holds even in these cases, that is, that a small number of large jumps have disproportional impacts on firm growth.

The study most closely related to ours is Bottazzi and Secchi (2006a), which introduces a *success brings success* type of dynamics to explain the Laplace shape: a successful firm has higher probability of achieving future success. This positive feedback mechanism generates big leaps and the resulting distribution converges to a Laplace distribution. However, their model relies on a questionable assumption: "the process of opportunity assignment is repeated anew each year, i.e., that no memory of the previous year's assignment is retained when the new year's opportunities are assigned" (p.251). In other words, their model assumes that on the one hand, growth opportunities are correlated within a year, but on the other hand, this feedback is abruptly cut off at the end of the year. This assumption is crucial in their model because otherwise, a single lucky firm would end up gaining all the opportunities and eventually diverge in size as time passes. By contrast, we show that by considering the jump property, our generalized model based on the law of proportionate effect becomes consistent with the observed Laplace distribution without relying on the feedback assumption. Put differently, it is not the absence of a feedback mechanism but that of the jump property that is responsible for the failure of Gibrat's model. In particular, our model is the minimum and straightforward generalization of Gibrat's model, being consistent with the observed Laplace distribution and leaving the essential parts unchanged.

The absence of the jump property is not unique to Bottazzi and Secchi (2006a) but can be found in various fields of the literature. For example, in Klette and Kortum (2004), a firm consists of many independent divisions and firm growth is the sum of independent innovations of each division.[3] As in Gibrat (1931), their model implicitly assumes a continuous movement of firm growth dynamics because the effect of each innovation is very small relative to the size of large firms. By contrast, our analysis suggests that if firm growth is the result of innovations, radical innovation corresponding to large jumps is of crucial importance for firm growth, as emphasized in the management literature (see Ettlie et al. (1984); Chandy and Tellis (2000); Leifer et al. (2000)).[4] Such *granularity* of shocks has received increasing attention in recent years (e.g., Gabaix (2011)).[5] The granular hypothesis, which is originally developed for the explanation of aggregate fluctuations, means that if firm size is very heterogeneous (e.g., Zipf's law), idiosyncratic shocks to firms are not averaged out but shocks to large firms generate non-trivial aggregate fluctuations. This idea can be applied to firm growth dynamics in a similar manner; if a firm does not entirely consist of many small components but contains some granular components, the impact of a shock to each component is not homogeneous, and firm growth is largely explained by shocks to a handful of such granular components. Although our derivation does not assume granular components but is based only on two empirical facts (i.e., the law of proportionate effect and Laplace distribution), the jump property is consistent with the granular hypothesis in that firm growth cannot be divided into many small shocks. Another possible interpretation of the jump property is the lumpiness of firm investment. For example, in labor economics, Elsby and Michaels (2013) and Kaas and Kircher (2014) show that firms do not adjust their workforces immediately owing to sunk costs and would make adjustments only if the gain from doing so were sufficiently large. In the literature on capital investment, the lumpiness of investment activity at the plant level is a stylized empirical fact (e.g., Doms and

[3]Klette and Kortum (2004) write, "...the evolution of the entire firm is obtained by summing the evolution of these independent divisions, each behaving as a firm starting with a single product would" (p.995).

[4]In the management literature, it is emphasized that radical innovation, which is defined as the development or application of something fundamentally new that creates a wholly new industry or causes a complete transformation of the market structure, is crucial to the long-term growth of firms. Leifer et al. (2000) note that once a radical innovation has been introduced into the existing market, "products based on one technology were undermined by radically new ones — and incremental improvement to the old technology has done little more than delay the eventual rout" (p.3).

[5]The connection of our analysis to this literature was kindly suggested by an anonymous referee.

Dunne (1998); Cooper et al. (1999); Thomas (2002)). Although we cannot identify which mechanism causes the jump property of firm growth dynamics and the complete description of the growth process with all these features is beyond the scope of this study, our analysis suggests that considering such discrete movement in models is indispensable for further understanding of firm growth dynamics. This is a key contribution of our study to the literature.

The remainder of this paper is organized as follows. Section 2 analyzes Japanese firms, showing that the growth rate distribution is different from a Gaussian distribution but is close to a Laplace distribution. Section 3 reviews Gibrat's model and develops a general framework for firm growth processes. Section 4 discusses the robustness of the jump property by analyzing cases in which the growth rate distribution is fatter tailed than a Laplace distribution is. The conclusions are summarized in Section 5. Mathematical definitions and concepts are summarized in the Appendix.

2 Empirical Evidence

This section provides empirical evidence of the Laplace shape of the growth rate distribution by analyzing Japanese firms. Our dataset consists of annual observations of firms over 2003–2009 and 2012–2017, compiled by the Teikoku Data Bank (TDB).[6] The TDB database is one of the most comprehensive firm-level surveys in Japan, covering more than 100,000 firms with at least 40 employees across all sectors. We separate the database into manufacturing and non-manufacturing sectors. In what follows, the definition of firm size is the total sales of a firm. Let S_t be the sales of a firm at time t and growth rate g_t be the log differences of S_t, that is, $g_t := \log S_{t+1} - \log S_t$.[7] Tables 1 and 2 report the number of observations and descriptive statistics.[8]

Figure 1 shows the kernel density estimates of the growth rate distribution for manufacturing firms in 2003 (i.e., g_{2003}). For comparison, fitted Gaussian and Laplace densities are plotted; a Laplace distribution (denoted by μ_{Lap}) is given by

$$\mu_{Lap}(dx) := \frac{1}{2a} \exp(-\frac{|x-\gamma|}{a})dx. \tag{1}$$

As previous studies have found, the growth rate distribution clearly shows marked departure from a Gaussian distribution and displays a tent-shaped form which is sharply peaked around the center and has a fatter tail.[9] This distribution can be well described by a Laplace distribution. Figures 2 and 3 show the kernel density estimates for manufacturing and non-manufacturing firms over 2003–2008 and 2012–2016. As shown in these figures, this tent-shaped form is stable over the entire period. Furthermore, this tent-shaped form can be observed at a more disaggregated level. Figure 4 shows the kernel density estimates for six manufacturing subsectors (chemicals, iron & steel, fabricated metal, general-purpose machinery, electrical machinery, and transportation equipment), which are totally different from a Gaussian distribution but close to a Laplace distribution.[10]

To explore this distribution shape further, we consider a wider class of distributions called the *Subbotin*

[6] The sample periods of the global financial crisis and subsequent great recession are excluded from our sample. Although the effect of the financial crisis on the distribution is another important issue, this study focuses on the stable feature of the growth rate distribution.

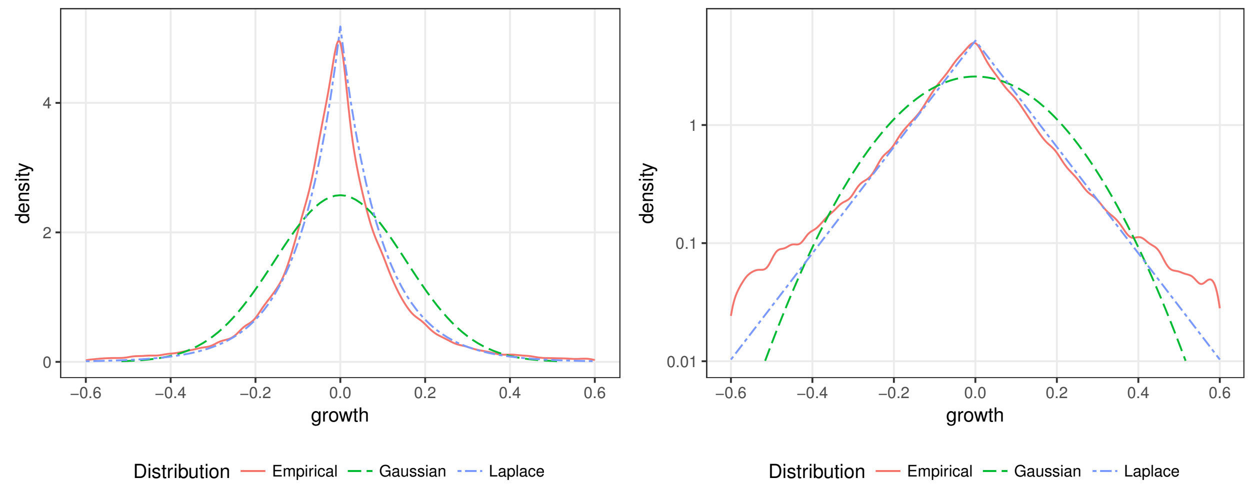
[7] Interestingly, the Laplace distribution emerges across different measures of firm size such as the number of employees or value added. See Bottazzi et al. (2007).

[8] Since the TDB database is based on survey, a non-negligible proportion (around 10%) of firms do not report their correct sales values but approximate values (e.g., "it is almost at the level of the previous year"). For such firms, the value of the previous year is used as the current value in the dataset. To confirm that the Laplace shape is unrelated to this data problem, we exclude from our sample those firms whose sales are the same in three successive years.

[9] Indeed, this significant departure can be checked by the Kolmogorov–Smirnov test. The null hypothesis of normality is rejected at the 1% significance level.

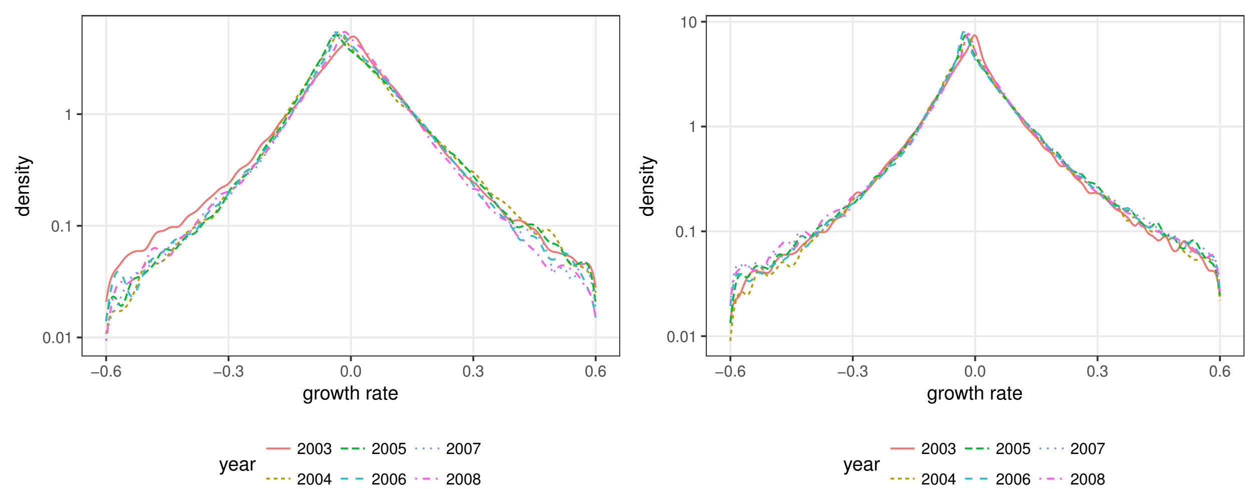
[10] One might consider that this distribution shape and a fatter tail are not due to internal growth, such as innovations, but due to external factors, such as mergers and acquisitions. However, previous studies have shown that the distribution shape is due to an internal growth process. For example, Bottazzi and Secchi (2006a) build *superfirms*, which are a union of entities undertaking such external changes, and show the Laplace shape of the growth rate distribution of these superfirms. In Appendix A.1.1, we reconfirm this point and show that internal growth is related to the observed Laplace distribution.

Figure 1: Kernel density estimation and fitted density functions.



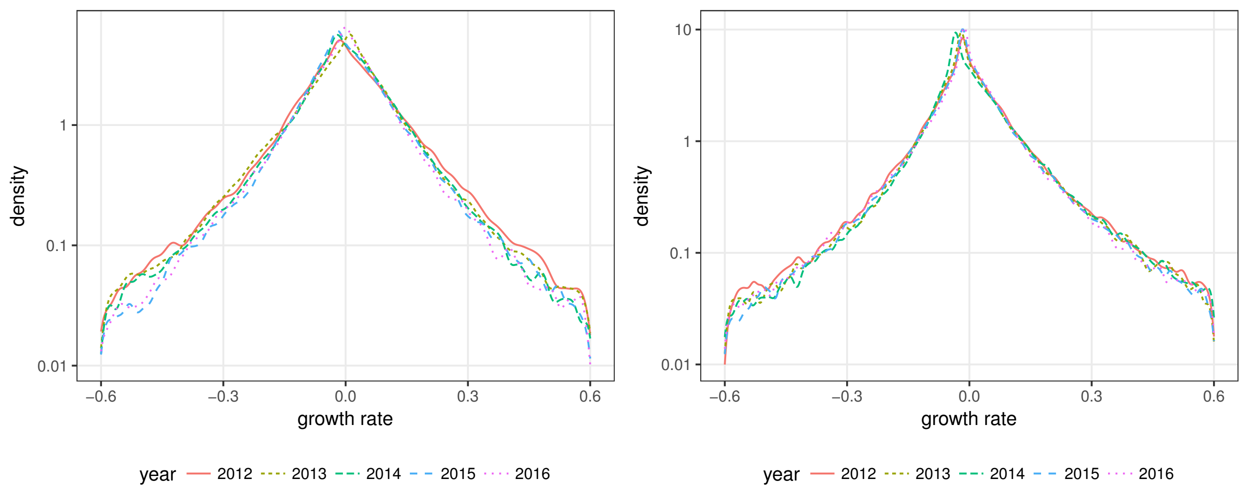
Note: In the left panel, the solid line (*Empirical*) represents the kernel density estimation for manufacturing firms in 2003. Bandwidth is chosen following the method in Scott (1992), using a factor of 1.06. *Gaussian* (*Laplace*) refers to the Gaussian (Laplace) fit with standard deviation = .155 (a = .0965 in Equation (1)). Subtracting the sample mean from growth rates, we apply the maximum likelihood method to obtain the parameter estimates. In the right panel, the same graph is plotted in logarithmic scale.

Figure 2: Kernel density estimation of the growth rate distributions in 2003–2008.



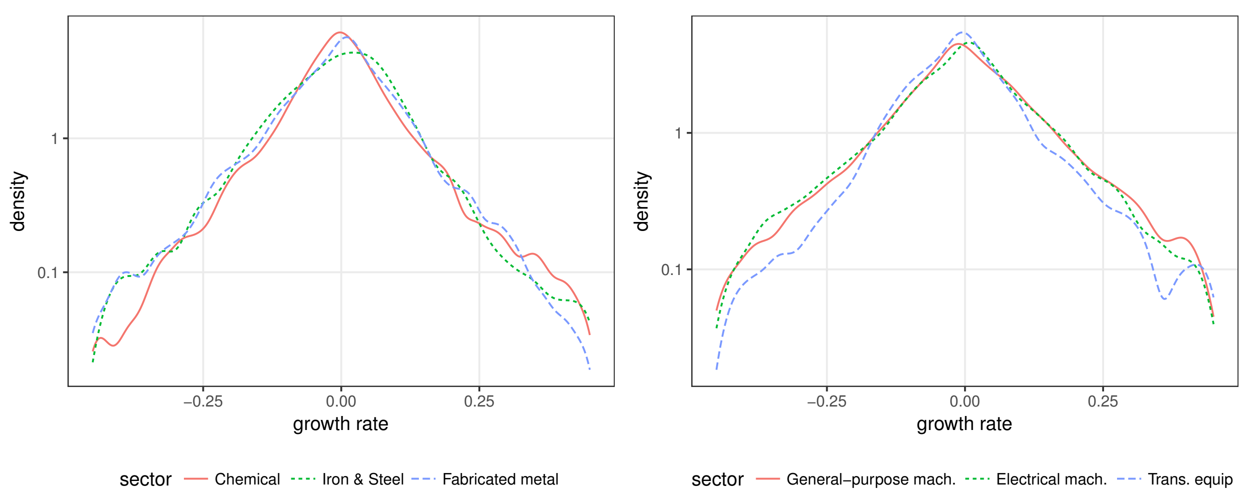
Note: The left (right) panel is for manufacturing (non-manufacturing) firms. The estimation method is the same as in Figure 1.

Figure 3: Kernel density estimation of the growth rate distributions in 2012–2016.



Note: The left (right) panel is for manufacturing (non-manufacturing) firms. The estimation method is the same as in Figure 1.

Figure 4: Kernel density estimation for six manufacturing subsectors in 2016.



Note: In the left panel, the kernel density estimates for chemicals, iron & steel, and fabricated metal are shown. In the right panel, the kernel density estimates for general-purpose machinery, electrical machinery, and transportation equipment are shown. The estimation method is the same as in Figure 1.

family, which is introduced by Bottazzi et al. (2002).[11] The Subbotin family is defined as follows:

$$\mu_{Sub}(dx) := \frac{1}{2ab^{1/b}\Gamma(1/b+1)} \exp(-\frac{1}{b}|\frac{x-\gamma}{a}|^b)dx, \tag{2}$$

where Γ is the Gamma function. The Subbotin family includes Gaussian and Laplace distributions as special cases; when $b = 2$ ($b = 1$), μ_{Sub} corresponds to a Gaussian (Laplace) distribution. Intuitively, μ_{Sub} is more sharply peaked around the center and has a fatter tail as the shape parameter b becomes smaller.

By using our dataset, we estimate the parameters a and b in Equation (2) by the maximum likelihood method. Tables 1 and 2 report the estimation results, showing that the shape parameter b in all cases is significantly lower than 2 (the Gaussian case) and is close to 1 (the Laplace case), especially for manufacturing firms. In particular, Table 2 shows that this property can be observed at the subsectoral level. Consistent with previous studies, a Laplace distribution can well approximate the growth rate distribution for Japanese firms. For non-manufacturing firms, the shape parameter b is significantly lower than 1, suggesting that the tail of the growth rate distribution is fatter than the tail of a Laplace distribution. This point is further discussed in Section 4.

Table 1: Descriptive statistics and maximum likelihood estimates in 2003–2008 and 2012–2016.

	# obs.	mean	s.d.	$\hat{a}$	s.e. of $\hat{a}$	$\hat{b}$	s.e. of $\hat{b}$
Manufacturing							
2003	27,276	−.0090	.155	.0965	.00078	.839	.0090
2004	27,136	.0294	.149	.100	.00077	.952	.0099
2005	29,573	.0404	.147	.102	.00073	.989	.0098
2006	30,010	.0301	.144	.0942	.00069	.913	.0089
2007	30,331	.0390	.141	.0946	.00068	.946	.0092
2008	30,553	.0179	.139	.0888	.00066	.880	.0086
2012	30,121	.0152	.153	.0985	.00074	.885	.0089
2013	30,008	−.0100	.148	.0910	.00070	.829	.0083
2014	29,862	.0232	.141	.0889	.00067	.871	.0086
2015	29,913	.0218	.134	.0843	.00063	.872	.0085
2016	29,863	.0009	.132	.0750	.00062	.735	.0079
Non-manufacturing							
2003	65,996	.0001	.145	.0707	.00043	.600	.0046
2004	65,946	.0164	.145	.0826	.00044	.770	.0050
2005	75,317	.0232	.151	.0863	.00042	.770	.0046
2006	76,790	.0273	.148	.0851	.00041	.783	.0046
2007	77,856	.0275	.151	.0871	.00042	.782	.0046
2008	79,487	.0175	.151	.0824	.00040	.726	.0043
2012	88,993	.0165	.146	.0790	.00037	.723	.0040
2013	91,053	.0198	.140	.0758	.00034	.736	.0039
2014	92,349	.0344	.139	.0807	.00035	.808	.0042
2015	93,971	.0168	.137	.0722	.00033	.706	.0037
2016	95,306	.0129	.138	.0692	.00032	.665	.0035

Note: The fourth and sixth columns represent the maximum likelihood estimates of the parameters in Equation (2) and the fifth and seventh columns represent the standard errors of the estimated parameters. Observations of $|g_t| \geq 0.7$ are removed as outliers. We confirm that the conclusion of our analysis does not depend on outlier criteria.

Although the Laplace shape of the growth rate distribution is remarkable, one of the concerns about the usage of stochastic models to explain this distribution shape is that a stochastic model implicitly assumes that

[11] This distribution family can be generalized to asymmetric cases. See Bottazzi and Secchi (2011), who examine the properties of this distribution family's maximum likelihood estimates and apply them to electricity market, foreign exchange market, and stock market data.

Table 2: Descriptive statistics and maximum likelihood estimates for six manufacturing subsectors in 2016.

	# obs.	mean	s.d.	$\hat{a}$	s.e. of $\hat{a}$	$\hat{b}$	s.e. of $\hat{b}$
Chemicals	1,632	.0051	.123	.0696	.0024	.762	.032
Iron & steel	1,397	−.0437	.125	.0900	.0029	1.05	.048
Fabricated metal	3,468	−.0090	.127	.0810	.0018	.878	.026
General-purpose machinery	4,334	.0145	.161	.1046	.0021	.891	.024
Electrical machinery	3,221	−.0057	.154	.1003	.0024	.878	.029
Transportation equipment	1,895	.0066	.136	.0815	.0025	.800	.032

Note: See the explanation in Table 1.

Table 3: Descriptive statistics and maximum likelihood estimates for six subsamples.

	# obs.	mean	s.d.	$\hat{a}$	s.e. of $\hat{a}$	$\hat{b}$	s.e. of $\hat{b}$
Manufacturing							
$\log_{10}(S_t) < 2.5$	1,730	−0.0417	0.166	0.103	0.0032	0.854	0.033
$2.5 \leq \log_{10}(S_t) < 3.0$	8,280	−0.00652	0.135	0.0754	0.0012	0.732	0.014
$3.0 \leq \log_{10}(S_t) < 3.5$	10,668	0.00672	0.130	0.0815	0.0010	0.855	0.014
$3.5 \leq \log_{10}(S_t) < 4.0$	5,552	0.0109	0.121	0.0757	0.0013	0.859	0.020
$4.0 \leq \log_{10}(S_t) < 4.5$	2,348	0.00757	0.131	0.0765	0.0021	0.801	0.027
$4.5 \leq \log_{10}(S_t)$	1,285	0.00170	0.121	0.0736	0.0027	0.840	0.039
Non-manufacturing							
$\log_{10}(S_t) < 2.5$	9,473	−0.0129	0.162	0.0684	0.0011	0.555	0.0089
$2.5 \leq \log_{10}(S_t) < 3.0$	29,940	0.0112	0.135	0.0619	0.00053	0.606	0.0056
$3.0 \leq \log_{10}(S_t) < 3.5$	29,817	0.0177	0.139	0.0746	0.00060	0.712	0.0068
$3.5 \leq \log_{10}(S_t) < 4.0$	16,354	0.0204	0.131	0.0743	0.00079	0.767	0.010
$4.0 \leq \log_{10}(S_t) < 4.5$	6,564	0.0171	0.125	0.0736	0.0012	0.804	0.016
$4.5 \leq \log_{10}(S_t)$	3,158	0.0155	0.123	0.0686	0.0017	0.756	0.022

Note: S_t is firm size measured by sales (in million yen). See the explanation in Table 1.

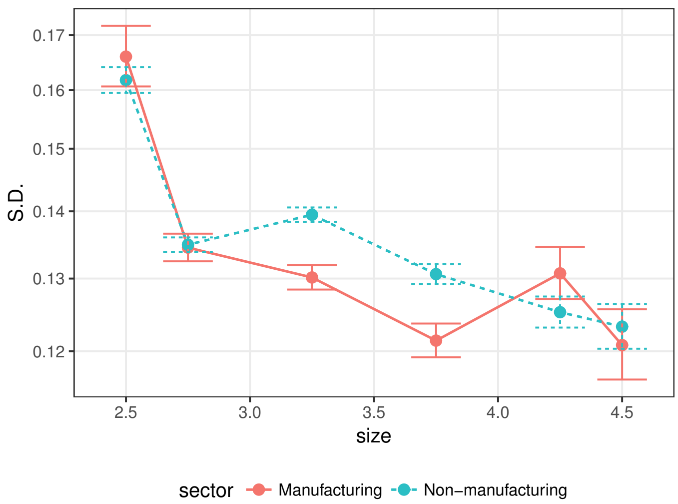
growth rates for all firms are drawn from a common probability distribution across firms. Put differently, one can argue that the observed Laplace shape may be due to the aggregation of heterogeneous firms. Indeed, previous studies (e.g., Bottazzi and Secchi (2006b); Secchi et al. (2018)) show that the growth rate distribution is not independent from firm size but the variance of growth rates is negatively correlated with firm size. In other words, the firm growth dynamics of larger firms are less volatile than those of smaller firms. Figure 5 plots the standard deviation and firm size by decomposing our samples into six parts by firm size, showing the negative relationship between firm size and variance.[12] Therefore, it is unlikely that the growth rates of all firms follow a common distribution.

However, Figure 6 shows that the growth rate distribution for each bin has a tent-shaped form. This point can be confirmed by the maximum likelihood estimates of a and b reported in Table 3. In other words, although the growth rate distributions for different size classes are not described by a common distribution, they can be well described by a common functional form with different parameters. In particular, this finding suggests the possibility that if the growth rate is properly standardized for each firm, the standardized growth rate across firms may follow a common distribution.

To check this possibility formally, we perform a statistical test proposed by Lunardi et al. (2014). This test

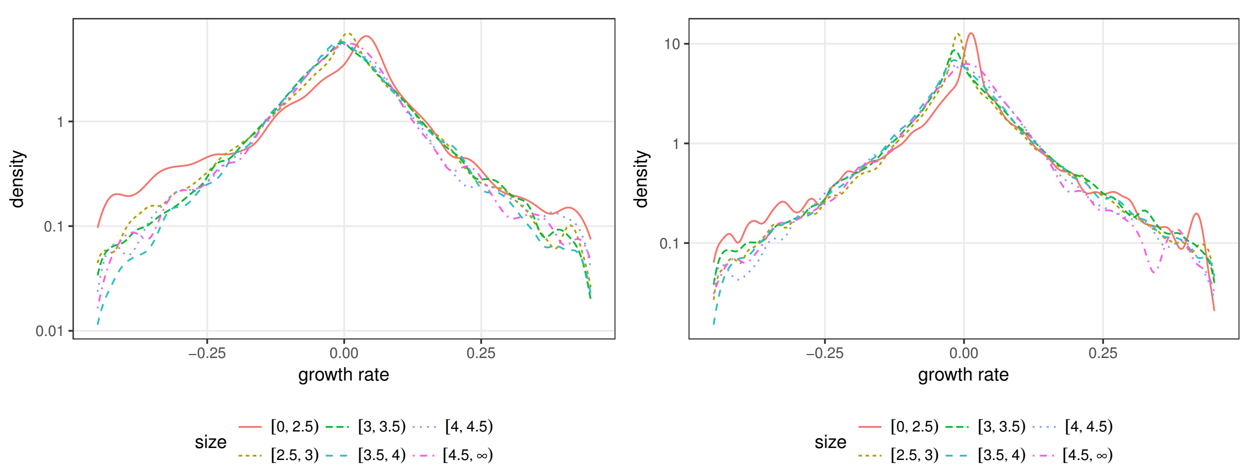
[12]Secchi et al. (2018) show that the relationship is described by $\log(\sigma_{g_t}) = -\beta \log(S_t)$, where $\beta \approx 0.18$ for most of the countries analyzed. However, according to Secchi et al. (2018), Japan is an exceptional case in that the slope of this relationship β is relatively flat and not significantly different from zero.

Figure 5: Standard deviation and firm size.



Note: The horizontal axis is $\log_{10}(S_t)$. Samples are decomposed into six subsamples by firm size: $\log_{10}(S_t) < 2.5$, $2.5 \leq \log_{10}(S_t) < 3.0$, , $3.0 \leq \log_{10}(S_t) < 3.5$, $3.5 \leq \log_{10}(S_t) < 4.0$, $4.0 \leq \log_{10}(S_t) < 4.5$, and $4.5 \leq \log_{10}(S_t)$. The standard deviations and their 95% confidence intervals for each subsamples are shown.

Figure 6: Kernel density estimations for the six subsamples.



Note: The left (right) panel is for manufacturing (non-manufacturing) firms. See the explanations in Figures 2 and 5.

can be used to discriminate whether the Laplace shape of the growth rate distribution is due to the intrinsic feature of the firm growth process or the aggregation of heterogeneous firms. The null hypothesis is that the standardized growth rate $g_{i,t}^{std} := \frac{g_{i,t}-m_i}{\sigma_i}$ is drawn from a common distribution across firms, where m_i and σ_i^2 are the mean and variance of firm i's growth rates. Namely, firm idiosyncrasy of growth rate is captured by the first two moments under the null hypothesis. Then, we check whether the empirical distribution of $g_{i,t}^{std}$ pooled together for all i is statistically different from the null distribution. For comparison between the two distributions, we use four test statistics: D (Kolmogorov), A^2 (Anderson-Darling), W^2 (Cramer-Von Mises), and U^2 (Watson).[13]

We apply this test to the six major manufacturing subsectors for 2003-2008 and 2012-2016 ($T = 11$).[14] As the null distribution, we consider μ_{Sub} with $b = 2$ (Gaussian), $b = 1$ (Laplace), and $b = b^*$ at which test statistics takes the lowest value. In other words, at $b = b^*$, the empirical distribution is relatively more similar to the null distribution than at other values of b. Table 4 reports the results of this test. First, it shows that for all subsectors, the Gaussian hypothesis is rejected at the 1% significance level. As expected, the growth rate distribution is statistically different from a Gaussian distribution even when the mean and variance of growth rate for each firm are controlled. For the Laplace case, the null hypothesis is rejected for all subsectors except chemicals. However, this result is largely due to the fact that the number of our observations is large enough to detect small deviation of the actual distribution from the null distribution rather than due to firms' heterogeneity. Indeed, at $b = b^*$, the null hypothesis is not rejected for four out of six subsectors. In other words, at $b = b^*$, we cannot detect the heterogeneity of the growth rate distributions for the four subsectors. This result implies that, although there are differences across subsectors, the standardized growth rate distributions for firms belonging to the same subsector, as a rule, can be viewed as homogeneous. Note that b^* for all subsectors are within the range of $[1.0, 1.3]$ and relatively close to the Laplace case. At least as long as the population of firms is considered at the subsector level, the distribution shape cannot be reduced to the aggregation effect of heterogeneous firms, and therefore, it is meaningful to analyze the (standardized) growth rate distribution as a Laplace distribution based on the homogeneous assumption.[15] In the following section, we develop our stochastic model based on this assumption.

3 Firm Growth Model

3.1 Gibrat's model

Let us begin with reviewing Gibrat's model. Gibrat (1931) assumes that firm size S_t evolves over time according to the following equation:

$$S_{t+1} = (1 + \epsilon_t)S_t = (1 + \epsilon_t)(1 + \epsilon_{t-1})...(1 + \epsilon_1)S_1, \tag{3}$$

where $\{\epsilon_s\}_{s=1,...,t}$ are independent shocks to the firm. In other words, firm size S_t is the product of multiplicative shocks ϵ_s and the initial size. By taking the logarithm, Equation (3) can be written as $\log S_{t+1} = \sum_{s=1}^{t} \log(1 + \epsilon_s) + \log S_1 \approx \sum_{s=1}^{t} \epsilon_s + \log S_1$. If t is sufficiently large and $\log S_1$ becomes negligible, the distribution of $\log S_t$ becomes Gaussian given that the CLT conditions are satisfied. Therefore, the firm size distribution is a lognormal distribution. Similarly, the growth rate over some time period g_t

[13] If the four empirical test statistics are significantly larger than a critical value calculated under the null hypothesis, it suggests that the empirical distribution is statistically different from the null distribution, and therefore, we reject the null hypothesis.

[14] Here, we combine the two sample periods (2003-2008 and 2012-2016) and focus on firms surviving the entire sample period because if the time span of panel data (i.e., T) is too short, the power of this test becomes very weak. The number of firms analyzed is given in Table 4.

[15] In Appendix A.1.2, we apply Lunardi et al. (2014) test to 4-digit industries. Results show that, one the one hand, the Gaussian hypothesis ($b = 2$) is rejected for about 75% of the 4-digit industries, but on the other hand, the Laplace hypothesis ($b = 1$) is *not* rejected for about 60% of the 4-digit industries. These results suggest that at this disaggregated level, firms in each 4-digit industry are more homogeneous, and therefore, the hypothesis that standardized growth rates follow a common Laplace distribution is more likely to hold.

Table 4: Results of Lunardi et al. (2014) test.

	statistics	$b = 2$ (Gaussian) empirical	1%	$b = 1$ (Laplace) empirical	1%	$b = b^*$ empirical	1%
Chemicals	D	0.0399	0.0100	**0.00767**	**0.0118**	**0.00753**	**0.0118**
Number of firms = 1008	A^2	38.126	0.673	**0.336**	**1.137**	**0.260**	**1.120**
$b^* = 1.04$	W^2	5.969	0.179	**0.0886**	**0.292**	**0.0617**	**0.278**
	U^2	5.969	0.168	**0.0880**	**0.241**	**0.0610**	**0.228**
Iron & steel	D	0.0262	0.0107	0.0190	0.0127	**0.00709**	**0.0116**
Number of firms = 865	A^2	12.798	0.583	5.460	1.051	**0.111**	**0.766**
$b^* = 1.29$	W^2	2.031	0.177	1.024	0.285	**0.0638**	**0.222**
	U^2	2.030	0.165	1.023	0.234	**0.0622**	**0.193**
Fabricated metal	D	0.0259	0.00720	0.0165	0.00863	**0.00569**	**0.00786**
Number of firms = 1940	A^2	36.444	0.835	9.197	1.315	1.175	1.076
$b^* = 1.24$	W^2	5.662	0.178	1.598	0.286	**0.155**	**0.233**
	U^2	5.657	0.167	1.586	0.233	**0.145**	**0.204**
General machinery	D	0.0291	0.00639	0.0210	0.00757	0.00716	0.00705
Number of firms = 2469	A^2	41.643	0.908	16.119	1.384	2.168	1.098
$b^* = 1.26$	W^2	6.559	0.180	2.681	0.293	0.279	0.221
	U^2	6.525	0.167	2.607	0.233	0.229	0.194
Electric machinery	D	0.0297	0.00692	0.0119	0.00816	**0.00499**	**0.00768**
Number of firms = 2126	A^2	51.312	0.878	5.982	1.341	**1.086**	**1.199**
$b^* = 1.17$	W^2	7.884	0.182	1.040	0.285	**0.102**	**0.247**
	U^2	7.873	0.171	1.019	0.231	**0.0858**	**0.209**
Transportation equipment	D	0.0310	0.00956	0.0170	0.0116	0.0170	0.0110
Number of firms = 1077	A^2	27.342	0.688	5.034	1.179	2.533	1.016
$b^* = 1.18$	W^2	4.549	0.179	1.026	0.291	0.584	0.253
	U^2	4.493	0.168	0.905	0.239	0.484	0.214

Note: Each entry in the "empirical" column (1%) reports the observed value (critical value at the 1% significance level) of four test statistics: D (Kolmogorov), A^2 (Anderson-Darling), W^2 (Cramer-Von Mises), and U^2 (Watson). Entries printed in boldface mean *no rejection* of the null hypothesis, that is, the observed value of test statistics is smaller than the corresponding critical value at the 1% significance level. For fabricated metal at $b = b^*$, Anderson-Darling statistic A^2 rejects the null but other statistics do not.

defined by the log difference of firm size can be written as the sum of independent shocks:

$$g_t := \log S_{t+1} - \log S_t = \sum_{i=1}^{n} \log(1 + \epsilon_t^i) \approx \sum_{i=1}^{n} \epsilon_t^i, \tag{4}$$

where the time interval between $t+1$ and t is divided into n parts. By using the same argument, the growth rate distribution converges to a Gaussian distribution when n is large.

In short, the basic assumptions of Gibrat's model are summarized as follows.

1. The growth rate of a firm is independent of its initial size.
2. The successive growth rates of a firm are independent of each other.[16]
3. The growth rate of a firm consists of many small shocks that satisfy the CLT conditions.

The law of proportionate effect refers to Assumptions 1 and 2. Although it is empirically shown that the law of proportionate effect does not strictly hold in every case, it can be viewed as a good working hypothesis and well describe the growth dynamics, especially for larger and mature firms. By contrast, as discussed in the previous section, the Laplace shape is widely observed both for smaller and larger firms, and at different disaggregated levels, even when the growth rate is standardized by individual firms' mean and variance. This empirical fact clearly contradicts Gibrat's model, and suggests that the observed Laplace distribution is due to the violation of Assumption 3 rather than the violation of the law of proportionate effect. In the following subsection, we generalize Gibrat's model by relaxing Assumption 3, keeping the law of proportionate effect unchanged.

3.2 Generalization

Let X_t be the log of firm size ($X_t := \log S_t$) and growth rate g_t be the difference of X_t ($g_t := X_{t+1} - X_t$). Suppose that firm size satisfies the law of proportionate effect and evolves according to Equation (3). For generalization, we separately consider two related aspects of Gibrat's model: the firm growth process (stochastic process) and growth rate distribution. In Gibrat's model, the growth rate distribution is Gaussian. As a mirror image of this distribution, the process X_t in Gibrat's model is a Brownian motion in the continuous time scale. Therefore, it is necessary to generalize both Brownian motion and the CLT.

Table 5 summarizes our strategy, which consists of three steps. First, by generalizing the CLT, we show that under the law of proportionate effect (i.e., Assumptions 1 and 2), the growth rate distribution does not necessarily converge to a Gaussian distribution but to a member of a distribution family called *infinitely divisible distributions*. This family includes both Gaussian and Laplace distributions as special cases. Second, we introduce *Lévy processes*, which are a class of stochastic processes with independent and stationary increments. Brownian motion is one of the Lévy processes. In our analysis, Brownian motion and Gaussian distribution in Gibrat's model are generalized to Lévy processes and infinitely divisible distributions, respectively. Finally, by using the relationship between infinitely divisible distributions and Lévy processes, we identify the Lévy process corresponding to a Laplace distribution, which is the variance Gamma process. Therefore, the firm growth dynamics can be explored by examining the property of the variance Gamma process.

Let us begin with the generalization of the CLT by relaxing Assumption 3. Suppose that firm growth consists of independent random shocks (Assumptions 1 and 2) but does not satisfy the CLT condition (Assumption 3). Intuitively, this means that in our model, a firm grows by various reasons such as the introduction of new products, quality improvements, and effective advertising, and their impacts on firm growth may differ. Some shocks may have a disproportional impact, and therefore, the CLT cannot be applied because random shocks are not identically distributed. A more general limit theorem on the sums of independent but not necessarily identically distributed random variables, as in our case, is proven by Khintchine (1937) (Theorem 2 in A.2). This theorem shows that the distribution of sums does not necessarily

[16] We assume that the independence of multiplicative shocks holds for every time scale; ϵ_t^i and ϵ_t^j for $i \neq j$ in Equation (4) are independent of each other for any n. This assumption is stronger than the simple independence between firm size and growth rate. However, this assumption is a natural extension in the continuous time scale and can be found in the literature; for example, Bottazzi et al. (2002) says, "[t]his is indeed a straightforward conjecture" (p.710).

Table 5: One-to-one correspondence.

<Firm growth process>		<Growth rate distribution>
Brownian motion	$\Longleftrightarrow$	Gaussian distribution
Lévy processes	$\Longleftrightarrow$	Infinitely divisible distributions
Variance Gamma process	$\Longleftrightarrow$	Laplace distribution

Note: The first row represents Gibrat's model, in which the underlying process is Brownian motion in a continuous scale and the resulting growth rate distribution is Gaussian. The second row is our generalized model; under the law of proportionate effect, Brownian motion and a Gaussian distribution are generalized to Lévy processes and infinitely divisible distributions, respectively. The last row shows that by using the one-to-one correspondence between infinitely divisible distributions and Lévy processes, we can identify the Lévy process corresponding to a Laplace distribution. This process is the variance Gamma process, representing the firm growth dynamics.

converge to a Gaussian distribution but rather to an *infinitely divisible distribution* (see Definition 1 in A.2). Put differently, if firm growth is composed of a large number of independent but not necessarily identically distributed shocks, the resulting growth rate distribution is an infinitely divisible distribution. The CLT can be seen as a special case of this theorem in which an additional assumption (Assumption 3) is imposed such that the resulting distribution of sums converges to a particular subclass of infinitely divisible distributions, i.e., Gaussian distribution.

Each infinitely divisible distribution can be fully characterized by using the *Lévy–Khintchine formula*. Let $\hat{\mu}$ be the characteristic function of a distribution μ, that is, $\hat{\mu}(z) := E[e^{izX}] = \int_{\mathbb{R}} e^{izx}\mu(dx),\ z \in \mathbb{R}$. The Lévy–Khintchine formula means that the characteristic function of infinitely divisible distributions can be represented as follows:

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}Az^2 + i\gamma z + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_D(x))\nu(dx)\right],\ z \in \mathbb{R}, \tag{5}$$

where A is a non-negative constant, ν is a measure on $\mathbb{R}$ (called a *Lévy measure*), and $\gamma \in \mathbb{R}$.[17] The point is that an infinitely divisible distribution is fully characterized by three components $\{A, \nu, \gamma\}$, called a *generating triplet*. In other words, given a generating triplet $\{A, \nu, \gamma\}$, the corresponding infinitely divisible distribution with $\{A, \nu, \gamma\}$ is uniquely determined. For example, if $\nu = 0$, $\hat{\mu}$ corresponds to the characteristic function of a Gaussian distribution with mean γ and variance A. This means that the Gaussian growth rate distribution is not the direct consequence of the law of proportionate effect (Assumptions 1 and 2) because a Gaussian distribution is a special case of infinitely divisible distributions with $\nu = 0$. Without Assumption 3, the growth rate distribution may converge to an infinitely divisible distribution with $\nu \neq 0$. Therefore, the departure of the growth rate distribution from Gaussian does not contradict the law of proportionate effect and, as we see in the remainder of this subsection, a Laplace distribution is indeed an infinitely divisible distribution.

We show some properties of a Laplace distribution relevant to our analysis. One such property is that a random variable X drawn from a Laplace distribution can be decomposed into positive and negative parts, $Y^1, Y^2 \geq 0$, that is, $X = Y^1 - Y^2$, where Y^1, Y^2 follow a common exponential distribution μ_{exp}. This property can be shown as follows. Let $\hat{\mu}_{Lap}$ be the characteristic function of a Laplace distribution with $\gamma = 0$. The characteristic function $\hat{\mu}_{Lap}$ can be explicitly written as[18]

$$\hat{\mu}_{Lap}(z) = \frac{1}{1 + a^2z^2} = \frac{1}{(1 + iaz)(1 - iaz)}. \tag{6}$$

Note that $\hat{\mu}_{Lap}(z)$ is the product of $(1 - iaz)^{-1}$ and $(1 + iaz)^{-1}$, which means that a random variable drawn from μ_{Lap} can be represented by the sum of two independent random variables whose characteristic functions are given by $(1 - iaz)^{-1}$ and $(1 + iaz)^{-1}$, respectively, because of the property of the characteristic

[17] ν satisfies $\nu(0) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty$. In Equation (5), 1_D is an indicator function and D is the closed unit ball.
[18] See, for example, Sato (1999), p.98.

function $E[e^{iz(X^1+X^2)}] = E[e^{izX^1}]E[e^{izX^2}]$ when X^1 and X^2 are independent of each other. It is known that $(1-iaz)^{-1}$ is the characteristic function of an exponential distribution μ_{exp}:

$$\mu_{exp}(dx) := 1_{[0,\infty)}\frac{1}{a}\exp(-\frac{x}{a})dx. \tag{7}$$

Therefore, by using the fact that if the characteristic function of a random variable Y is given by $(1-iaz)^{-1}$, the characteristic function of $-Y$ is given by $(1+iaz)^{-1}$, we can show that $X = Y^1 - Y^2$, where $Y^1, Y^2 \geq 0$ follow μ_{exp}. If a random variable X represents the growth rate, this means that Y^1, Y^2 represent positive and negative growth, respectively.

Next, we consider the Lévy–Khintchine representation of a Laplace distribution μ_{Lap}. Since a Laplace distribution can be represented as the convolution of two exponential distributions, the Lévy–Khintchine representation of μ_{Lap} is uniquely determined by that of an exponential distribution μ_{exp}. The characteristic function $\hat{\mu}_{exp}$ has the following representation:[19]

$$\hat{\mu}_{exp}(z) = \frac{1}{(1-iaz)} = \exp\left[\int_0^\infty (e^{izx}-1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \tag{8}$$

where $A = 0, \nu(dx) = 1_{(0,\infty)}(x)x^{-1}e^{-\frac{x}{a}}dx$ in Equation (5).[20] Therefore, the generating triplet of a Laplace distribution is given by $A = 0, \nu(dx) = 1_{(0,\infty)}(x)x^{-1}e^{-\frac{x}{a}}dx + 1_{(-\infty,0)}(x)|x|^{-1}e^{\frac{x}{a}}$.[21] This is in contrast to the Gaussian case, in which $A > 0$ and Lévy measure ν is identically equal to zero. The meaning of this difference becomes clear in the next subsection.

From here, we consider the generalization of Brownian motion under the law of proportionate effect (the left-hand side of Table 5). Recall that $X_t := \log(S_t)$ in our analysis and the growth rate $X_{t+1} - X_t = \log(S_{t+1}) - \log(S_t)$ is the sum of independent shocks. In the literature on stochastic processes, such an independent additive process is called a Lévy process (see Definition 3 in A.3). To be precise, Lévy processes are a class of stochastic processes characterized by the stationary independent increment property: if $\{X_t\}_{t\geq 0}$ is a stochastic process, for an arbitrary choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.[22] This property is equivalent to the law of proportionate effect. Indeed, the Brownian motion in Gibrat's model is a Lévy process because an increment $X_{t+1} - X_t$ is an independent random variable. An important feature of Lévy processes is that the sample path of a Lévy process can be discontinuous, that is, jump processes are also considered. To provide a rough idea of the jump property, let us consider a compound Poisson process defined by

$$X_t := \sum_{i=1}^{N_t} J_i,$$

where N_t is a Poisson process with intensity $\lambda > 0$ and J_i is an independent random variable with distribution function σ. In other words, jumps arrive randomly at the Poisson rate $\lambda > 0$ and X_t jumps by J_i. See Figure 7, in which a typical sample path of the compound Poisson process is shown. Note that X_t can be written as follows:

$$X_t = X_s + \sum_{i=N_s+1}^{N_t} J_i.$$

This equation means that, given the lack of memory of Poisson processes, X_t can be considered as the sum

[19]See, for example, Sato (1999), p.45.

[20]The term $\int_{\mathbb{R}}(-izx1_D(x))\nu(dx)$ becomes irrelevant because $\int_{|x|\leq 1}|x|\ \nu(dx) < \infty$ in our case.

[21]Here, we use the fact that the Lévy measure of a Laplace distribution is the sum of the Lévy measures of two exponential distributions because a Laplace distribution is the convolution of the two exponential distributions.

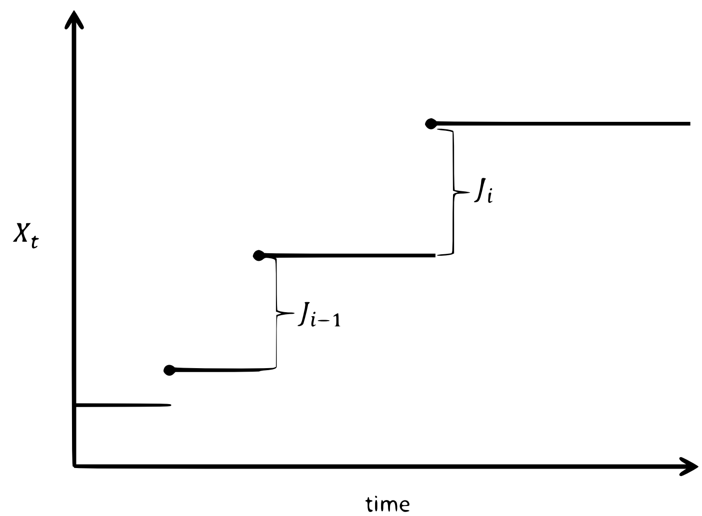
[22]Because of the stationary independent increment property, growth rate $X_{t+1} - X_t$ has the same distribution as $X_1 - X_0$. Note that in Definition 3 of Lévy processes in the A.3, X_0 is set to 0 without loss of generality and therefore $X_1 - X_0 = X_1$ a.s. Put differently, $X_t = X_t - X_0$ can be considered as the growth rate over t periods as well as the logarithm of firm size at t. In the following analysis, we explore the distribution properties of X_t and X_1, which are interpreted as the growth rates over t and one period as well as the logarithm of firm size at t and 1, respectively.

of X_s and an independent copy of X_{t-s}. In other words, $X_t - X_s$ is a stationary independent increment, and therefore, the compound Poisson process is a Lévy process. The characteristic function of the compound Poisson process is given by[23]

$$\hat{\mu}_{X_t}(z) := E[e^{izX_t}] = \exp\left[t\lambda \int_{\mathbb{R}} (e^{izx} - 1)\sigma(dx)\right]. \tag{9}$$

Note that since the compound Poisson process satisfies the stationary independent increment property, this jump process is consistent with the law of proportionate effect. In other words, the law of proportionate effect itself does not exclude the possibility of discrete movements like jumps. In light of this fact, Assumption 3 in Gibrat's model is a strong a priori restriction on the underlying firm growth process because it excludes jump processes and considers only continuous processes (i.e., Brownian motion). By considering the possibility of jumps in the process, we can broaden the scope of possible candidates for firm growth dynamics.

Figure 7: Sample path of the compound Poisson process.



Note: The stochastic process X_t jumps by J_i at the time of the shock, and stays constant until the occurrence of the next shock.

Finally, we discuss the one-to-one relationship between infinitely divisible distributions and Lévy processes. It can be shown that if an infinitely divisible distribution μ is given, there uniquely exists a corresponding Lévy process $\{X_t\}_{t\geq 0}$ such that the probability distribution of X_1 is μ.[24] For example, as in Gibrat's model, when a Gaussian distribution is given, Brownian motion is the corresponding Lévy process. Similarly, when a Laplace distribution is given, there uniquely exists a Lévy process generating the Laplace

[23]

$$\begin{aligned} E[e^{izX_t}] &= \sum_{n=0}^{\infty} P[N_t = n] E[e^{iz\sum_{j=1}^{n} J_j}] \\ &= \sum_{n=0}^{\infty} e^{-\lambda t}(n!)^{-1}(\lambda t)^n \hat{\sigma}(z)^n \\ &= \exp(\lambda t(\hat{\sigma}(z) - 1)) \\ &= \exp\left[t\lambda \int_{\mathbb{R}} (e^{izx} - 1)\sigma(dx)\right] \end{aligned}$$

Here, we used the independence property of J_j.

[24] See Corollary 11.6 in Sato (1999).

distribution. This process is called the variance Gamma process in the financial literature.[25] Because this relationship is one-to-one, there is no Lévy process generating the Laplace distribution other than the variance Gamma process. Put differently, under the law of proportionate effect (i.e., Lévy process), the variance Gamma process is a unique process consistent with the observed Laplace distribution. This means that the properties of firm growth dynamics can be revealed by examining the properties of the variance Gamma process. In particular, since an infinitely divisible distribution is determined by the generating triplet $\{A, \nu, \gamma\}$, the variance Gamma process is also characterized by $\{A, \nu, \gamma\}$.

3.3 Variance Gamma process

This subsection shows sample path properties of the variance Gamma process. For this purpose, we use the *Lévy–Itō decomposition* (for details, see Theorem 4 in A.3), which states that a Lévy process X_t can be decomposed into a jump part X_t^1 and a continuous part X_t^2, and X_t can be represented as the sum of these two parts $X_t = X_t^1 + X_t^2$. The characteristic function of the jump part X_t^1 is given by

$$\hat{\mu}_{X_t^1}(z) := E[e^{izX_t^1}] = \exp[t\int_{\mathbb{R}}(e^{izx} - 1)\nu(dx)].$$

In particular, if $\nu(\mathbb{R}) < \infty$, the jump part X_t^1 is a compound Poisson process with $\nu = \lambda\sigma$ (see Equation (9)). In other words, the sample path of X_t^1 is characterized by discrete movements as in Figure 7, and the jump structure of X_t^1 is completely determined by the Lévy measure ν. For example, the frequency of jumps larger than x^* is given by $\nu(x^*, \infty) = \lambda\sigma(x^*, \infty)$.

By contrast, the characteristic function of the continuous part X_t^2 is given as follows:

$$\hat{\mu}_{X_t^2}(z) := E[e^{izX_t^2}] = \exp[-\frac{1}{2}tAz^2 + it\gamma z].$$

This is the characteristic function of a Brownian motion with drift γ, which is a continuous stochastic process, that is, X_t^2 has no jumps and its value changes continuously. A Lévy process X_t is the sum of these two processes.

Let us return to the variance Gamma process. Suppose that X_t is the variance Gamma process corresponding to the observed Laplace distribution. Since a Laplace distribution can be expressed by the convolution of two exponential distributions, X_t can be represented as the difference between two independent and identically distributed processes Y_t^1 and Y_t^2:

$$X_t = Y_t^1 - Y_t^2,$$

where the characteristic function of $Y_t^i \geq 0, i = 1, 2$ is given by (see Equation (8))

$$\hat{\mu}_{Y_t^i} := E[e^{izY_t^i}] = \exp\left[t\int_0^\infty (e^{izx} - 1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \quad i = 1, 2. \tag{10}$$

In other words, firm growth can be divided into two components, a positive growth process Y_t^1 and a negative one Y_t^2. Since Y_t^1 and Y_t^2 are independent and identically distributed, we focus on Y_t^1 and denote it by Y_t in the following.

Note that Equation (10) shows that for Y_t, $A = 0$ and $\nu(dx) = 1_{(0,\infty)}(x)x^{-1}e^{-\frac{x}{a}}dx$ in the Lévy-Khintchine representation, suggesting that Y_t has no continuous part by the Lévy–Itō decomposition. Namely, the growth process Y_t is completely determined by jumps as in Figure 7. This is in sharp contrast to Brownian motion in Gibrat's model ($A \neq 0$, $\nu = 0$), which has only a continuous part (see Table 6). This shows a qualitative difference between Gibrat's model and our models described by Equation (10), and in this sense, the variance Gamma process can be seen as the opposite end of Brownian motion. By way of illustration, imagine that firm growth is the result of sales growth of the firm's n ($\gg 0$) products. In Gibrat's model, there are many products among n products whose sales grow but the growth of each product is very small relative to the

[25] The variance Gamma process was first introduced in option pricing theory by Madan and Seneta (1990). The mathematical properties of the process and its generalization, pure jump processes, are explored by Ferguson and Klass (1972).

growth of total sales. In other words, there are no particular products that greatly contribute to the growth of total sales. Because of this nature, the CLT can be applied, and the firm growth process is characterized by gradual changes without any jumps. By contrast, in our model described by Equation (10), the impact of the growth of each product is heterogeneous, and some *hit* products account for most of the growth of total sales, leading to discrete changes in firm size, i.e., jumps. The observed Laplace distribution suggests that such jumps rather than continuous changes are crucial in firm growth dynamics.

Finally, we characterize the jump property of the variance Gamma process by a series of compound Poisson processes. Consider a compound Poisson process $Y_{t,n}$ represented by the following characteristic function:

$$\hat{\mu}_{Y_{t,n}}(z) := E[e^{izY_{t,n}}] = \exp\left[t\int_{\epsilon_n}^{\infty}(e^{izx}-1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \quad \epsilon_n > 0, \tag{11}$$

This process consists of jumps of Y_t larger than ϵ_n.[26] To be precise, $Y_{t,n}$ is expressed as follows:

$$Y_{t,n} = \sum_{i=0}^{N_t} J_i, \tag{12}$$

where N_t is a Poisson process with intensity $\lambda = \int_{\epsilon_n}^{\infty} x^{-1}e^{-\frac{x}{a}}dx$ and J_i is an independent random variable with probability density given by $\lambda^{-1}\frac{e^{-\frac{x}{a}}}{x}dx$ on $[\epsilon_n, \infty)$ (see Equation (9)). We can calculate the intensity λ of $Y_{t,n}$, that is, the frequency of jumps larger than ϵ_n, by using the estimate $\hat{a} = .0965$ in Table 1. For example, a positive growth shock larger than 2.5% has intensity $\nu(0.025, \infty) = 1.06$, which means that such a shock occurs, on average, once a year. For shocks $\geq 8\%$ ($\geq 10\%$), the intensity $\nu(0.08, \infty) = 0.295$ ($\nu(0.1, \infty) = 0.207$). Firms frequently experience such large shocks comparable to annual growth rates, and a handful of such large jumps mostly determine firm growth. This is the property of firm growth dynamics represented by the observed Laplace distribution.

Table 6: Lévy–Itō decomposition.

	Jump part X_t^1	Continuous part X_t^2
Brownian motion	$X_t^1 = 0$ $(\nu = 0)$	$X_t^2 = X_t$
Variance Gamma process	$X_t^1 = X_t$	$X_t^2 = 0$ $(A = 0)$

Note: A Lévy process X_t is represented as the sum of jump and continuous processes, that is, $X_t = X_t^1 + X_t^2$. Brownian motion has only a continuous part X_t^2 and the variance Gamma process has only a jump part X_t^1.

4 Robustness of the jump property

In the previous section, we show that firm growth dynamics are characterized by jumps, given that the growth rate distribution is a Laplace distribution. One might be concerned about the robustness of this property, that is, whether this property holds in cases in which the growth rate distribution slightly deviates from a Laplace distribution. Indeed, our empirical analysis in Section 2 shows that in some cases, the tail of the growth rate distribution is fatter than that of a Laplace distribution. In particular, the shape parameter b for non-manufacturing firms is substantially lower than 1 for all the years. In the remainder of this section, by using the *tail equivalence* of Lévy processes (for details, see A.4), we show that these fatter tails are consistent with and further support the jump property.

First, define the right tail of a measure μ by $\overline{\mu}(x) := \mu(x, \infty)$.[27] For the Laplace distribution, this can

[26] Indeed, comparing Equations (10) and (11), we observe that $Y_{t,n}$ converges to Y_t as ϵ_n goes to zero.
[27] The analysis developed in this section can also be applied to the left tail of distributions in the same manner.

be written as

$$\overline{\mu}_{Lap}(x) = e^{-cx}.$$

Here, c is some constant and we omit the normalization constants irrelevant to our analysis. We consider two fatter-tailed distributions than this tail. One is the distribution exhibiting power law behavior (e.g., Buldyrev et al. (2007a,b)):

$$\overline{\mu}_{pow}(x) = x^{-\alpha}, \quad \text{for } x > x^*,$$

where x^* is some large value. The other is the so-called Weibull distribution:

$$\overline{\mu}_{\mathrm{Wei}}(x) = e^{-cx^b}, \quad \text{for } x > x^*,$$

where $0 < b < 1$. Note that the Weibull distribution with $0 < b < 1$ is very similar to the Subbotin family. Since the density function is given by e^{-cx^b} for the Subbotin family and $x^{b-1}e^{-cx^b}$ for the Weibull distribution, it is practically difficult to distinguish these distributions by using empirical data. The decay of the distribution is largely determined by e^{-cx^b}, especially when b is close to 1. Note that both distributions converge to the Laplace distribution as $b \to 1$. Since the Weibull distribution is more tractable, we consider the Weibull distribution with $0 < b < 1$ as the class having a fatter tail than the Laplace distribution, instead of the Subbotin family.

The question to address here is the following: do the Lévy processes corresponding to these distributions have a similar jump property to the variance Gamma process? Since only the tail of the distribution is specified, the sample path properties of the corresponding Lévy processes cannot be fully identified. However, the frequency of large jumps can be identified solely by the tail of the distribution. In other words, the tail of the Lévy measure ν can be identified by the tail of the distribution, as shown by Theorem 6 and Proposition 7 in the Appendix A.4.

Specifically, A.4 shows that $\overline{\mu}_{pow}$ ($\overline{\mu}_{\mathrm{Wei}}$) and $\overline{\nu}_{pow}$ ($\overline{\nu}_{\mathrm{Wei}}$) decay at almost the same rate, that is, $\lim_{x\to\infty} \overline{\mu}_{pow}(x)/\overline{\nu}_{pow}(x) = 1$ ($\lim_{x\to\infty} \overline{\mu}_{\mathrm{Wei}}(x)/\overline{\nu}_{\mathrm{Wei}}(x) = 1$), where we denote by ν_{pow} (ν_{Wei}) the Lévy measure of μ_{pow} (μ_{Wei}). Recall that the frequency of the jumps of a Lévy process is fully determined by its Lévy measure. Thus, we can estimate the rate at which large jumps occur by the tail of the observed growth rate distribution; a frequency of jumps larger than some large value x^*, $\nu(x^*, \infty)$, can be approximated by $\mu(x^*, \infty)$. Therefore, a fatter tail of the growth rate distribution suggests the existence of large jumps in the underlying growth process. In particular, in light of the fact that $b = 1$ corresponds to a Laplace distribution, the Laplace case can be considered as a boundary case. As long as the growth rate distribution is described by a Laplace distribution or distributions with fatter tails, we can conclude that large jumps are crucial for firm growth dynamics. Therefore, the recent finding that a growth rate distribution has a fatter tail than a Laplace distribution is consistent with and further supports the importance of large jumps.

5 Conclusions

Firm growth is an engine for economic growth and has been a central topic in economics. The importance of statistical regularities cannot be overemphasized because they reflect some robust properties and provide clues for better understanding of firm growth dynamics. In recent years, a series of empirical studies about growth rate distribution has opened a new field of inquiry, showing that growth rate distribution is quite different from Gaussian distribution but close to Laplace distribution. Indeed, our analysis showed that the growth rate distribution for Japanese firms is also well described by a Laplace distribution. This empirical fact is totally inconsistent with Gibrat's model and challenges our understanding of firm growth dynamics. The present study tackled this problem.

Our analysis starts with the law of proportionate effect but does not impose the CLT condition that random shocks constituting firm growth are identically distributed (Assumption 3). In other words, we allow for the possibility that there are some shocks having disproportional impact on the firm growth. In this general setting, the growth rate distribution does not necessarily converge to a Gaussian distribution but to a member of a more general distribution family called infinitely divisible distributions. Indeed, a

Laplace distribution belongs to this distribution family. This means that the law of proportionate effect itself does not contradict the observed Laplace distribution, and it is the CLT condition that leads to a discrepancy between Gibrat's model and the Laplace shape. The same is true for the underlying firm growth process because the growth rate distribution and firm growth dynamics are two sides of the same coin. If we assume the CLT condition and a Gaussian distribution, it is equivalent to assuming that the underlying growth process is continuous without jumps. However, without this condition, an independent additive stochastic process with jumps is also an eligible candidate for the firm growth process. Indeed, the unique process corresponding to the Laplace distribution is a pure jump process without any continuous changes. In particular, this process is mostly determined by a few large jumps. Our analysis showed that the CLT condition is not innocuous but strong a priori assumption, and that the jump property is an indispensable feature of firm growth dynamics.

The jump property of firm growth dynamics has several economic interpretations. If firm growth is explained by innovations, our finding suggests that the impact of each innovation is not homogeneous. There are major innovations that contribute greatly to firm growth while the contribution of many others innovations is negligible. This interpretation is consistent with the role of radical innovation, as discussed in management literature (e.g., Ettlie et al. (1984); Chandy and Tellis (2000); Leifer et al. (2000)). In contrast to increment innovations, a few radical innovations determine long-term firm growth. The jump property can also be interpreted in a different context; especially in the macroeconomic literature, the granular hypothesis initiated by Gabaix (2011) has received increasing attention in recent years. Consistent with the granular hypothesis, our analysis suggests that the assumption that a firm consists of many small components cannot be justified but granularity must be considered. Another possible explanation consistent with the jump property is lumpy behavior; the workforce adjustment (e.g., Elsby and Michaels (2013); Kaas and Kircher (2014)) and the lumpiness of capital investment activity at the plant level (e.g., Doms and Dunne (1998); Cooper et al. (1999); Thomas (2002)) have become stylized facts in the literature.

The complete description of the growth process with all these features is beyond the scope of this study, and we cannot identify the main mechanism generating jumps. However, what we showed in this paper is that such a mechanism generating jumps is crucial for further understanding of firm growth dynamics. The observed Laplace distribution can be viewed as evidence of this fact. By using Ashton's terminology presented in the Introduction, we conclude that *sudden leaps* are fundamental for firm growth dynamics.

A Appendix

A.1 Additional empirical evidence

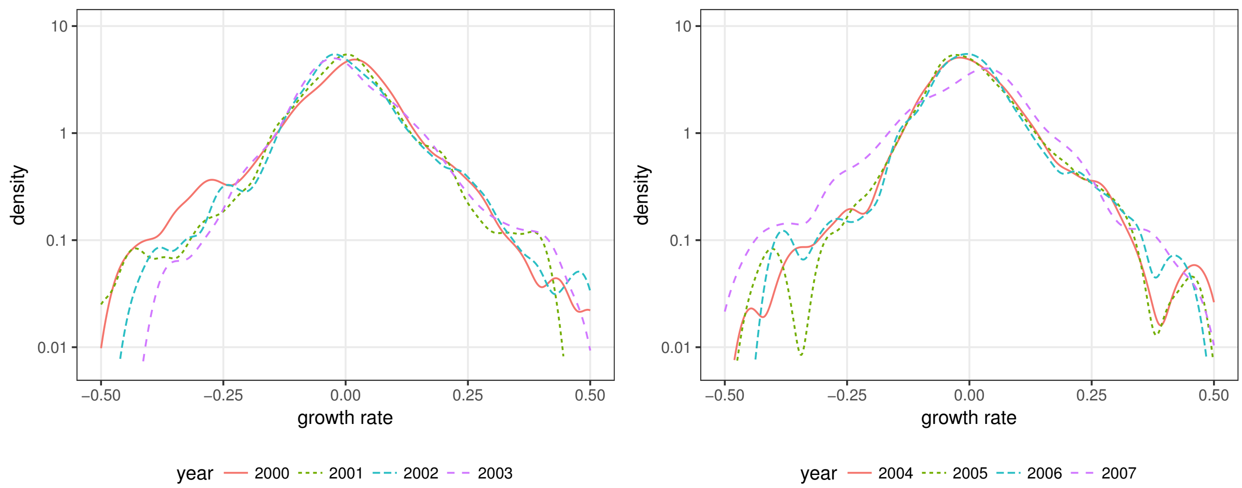
A.1.1 External and internal growth

In this section, we provide additional evidence on the Laplace shape of the growth rate distribution. We show that this distribution shape can be observed even when we exclude firms that expand their size by external activities, such as acquisitions. Consistent with the findings of previous studies (e.g., Bottazzi and Secchi (2006a)), this result suggests that the distribution shape is closely related to the internal growth process, as opposed to the external growth process.

The database used in this section covers publicly traded Japanese firms over 2000–2008 and has information on whether a firm changes the scope of consolidation.[28] Here, the definition of firm size is the total sales of a consolidated firm. We exclude from our analysis firms that undertake changes in the scope of consolidation. Table 7 summarizes the descriptive statistics of the growth rates of the remaining firms. Figure 8 shows the kernel density estimates of the growth rate distribution. Consistent with the findings in Section 2, the empirical distribution in Figure 8 is tent-shaped. As in Section 2, we use the Subbotin family and estimate its parameters by adopting the maximum likelihood method. The results in Table 7 show that the shape parameter b is close to 1, that is, a Laplace distribution. In addition, by using the log-likelihood ratio test, we find that the null hypothesis of $b = 2$, that is, a Gaussian distribution, is rejected for all cases at the 1% significance level. Therefore, we conclude that the distribution shape approximated by a Laplace distribution reflects an important aspect of internal growth.

[28]This database is compiled by Nikkei NEEDS.

Figure 8: Kernel density estimation of the growth rate distribution in 2000–2007.



Note: The estimation method is the same as in Figure 1.

Table 7: Descriptive statistics and maximum likelihood estimates in 2000–2007.

	# obs.	mean	s.d.	$\hat{a}$	s.e. of $\hat{a}$	$\hat{b}$	s.e. of $\hat{b}$
2000	791	−.0310	.125	.0865	.0039	.977	.062
2001	780	−.0040	.112	.0746	.0034	.927	.059
2002	807	.0261	.114	.0784	.0034	.986	.059
2003	795	.0516	.115	.0841	.0036	1.070	.066
2004	814	.0509	.111	.0773	.0033	1.006	.059
2005	858	.0583	.109	.0758	.0032	1.002	.058
2006	932	.0424	.111	.0712	.0030	.896	.049
2007	922	−.0450	.138	.1067	.0042	1.171	.069

Note: See the explanation in Table 1.

A.1.2 Lunardi et al. (2014) test for 4-digit industry level

In Section 2, we apply Lunardi et al. (2014) test to six major manufacturing subsectors (2-digit industry level). In general, heterogeneity across firms is mitigated at a more disaggregated level, and we can obtain, by further disaggregation, additional evidence suggesting that the observed Laplace shape is not a consequence of aggregation but an intrinsic feature of firm growth dynamics. To be concrete, we consider the 4-digit industry level and apply Lunardi et al. (2014) test to 64 4-digit industries, each of which has at least 50 firms. Results are given in Tables 8 and 9, showing that the Gaussian hypothesis (i.e., $b = 2$) is rejected for 48/64 (= 75.0%) 4-digit industries, that is, all of the four test statistics (i.e., D (Kolmogorov), A^2 (Anderson-Darling), W^2 (Cramer-Von Mises), and U^2 (Watson)) reject the null hypothesis. Only for 6/64 (= 9.4%) 4-digit industries, the four test statistics do not reject the Gaussian hypothesis. Note that the power of this statistical test is strong enough to reject the null hypothesis with our small sample size. On the other hand, the Laplace hypothesis ($b = 1$) is rejected for 11/64 (= 17.2%) 4-digit industries and not rejected for 39/64 (= 60.9%) 4-digit industries by the four test statistics. In other words, for these 39/64 4-digit industries, we cannot find any statistical departure from the hypothesis that growth rate for each firm follow a Laplace distribution.

As easily found, there are differences across different industries, and there may be subsectors or 4-digit industries that are not well described by a Laplace distribution. However, these results suggest that the Laplace shape is widely observed even at this disaggregated level and cannot be reduced to a simple aggregation effect. Therefore, as a benchmark model for firm growth dynamics, the Laplace hypothesis based on the homogeneity assumption can be a good alternative to the Gaussian hypothesis at this disaggregated level.

A.2 Infinitely divisible distributions

Definition 1 *A probability distribution μ on $\mathbb{R}$ is infinitely divisible if for any positive integer n, there is a probability distribution μ_n on $\mathbb{R}$ such that $\mu = \mu_n^n$, where μ^n is the n-fold convolution of the probability distribution μ with itself.*

The infinite divisibility of μ means that a random variable drawn from μ can be expressed by the sum of an arbitrary number of independent and identically distributed random variables drawn from the distribution μ_n. This family of distributions includes, for example, Gaussian and stable distributions.

Next, we introduce a generalization of the CLT. A fundamental limit theorem on the sums of independent random variables is proven by Khintchine (1937).

Theorem 2 *Let $\{Z_{nk}\}$ be a null array*[29] *on $\mathbb{R}$ with row sums $S_n = \sum_{k=1}^{r_n} Z_{nk}$. If, for some $b_n \in \mathbb{R}, n = 1, 2, ...,$ the distribution of $S_n - b_n$ converges to a distribution μ, then μ is infinitely divisible* (see Sato (1999), p.47).

It is not assumed that the random variables composing the sum S_n are identically drawn from some common distribution. This theorem means that if a random variable is composed of a large number of independent random variables and has a limiting distribution, this distribution must be infinitely divisible.

A.3 Lévy processes and Lévy–Itō decomposition

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 3 *A stochastic process $\{X_t\}_{t \geq 0}$ on $\mathbb{R}$ is a Lévy process if the following conditions are satisfied.*

1. *$X_0 = 0$ a.s.*
2. *Independent increment property: For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.*
3. *The distribution of $X_{t+s} - X_t$ does not depend on t.*

[29] A double sequence of random variables $\{Z_{nk} : n = 1, 2, ...; k = 1, 2, ..., r_n; r_n \to \infty\}$ on $\mathbb{R}$ is called a null array if for each fixed n, $Z_{n1}, Z_{n2}, ..., Z_{nr_n}$ are independent, and for any $\epsilon > 0, \lim_{n\to\infty} \max\limits_{1\leq k\leq r_n} P[|Z_{nk}| > \epsilon] = 0$.

Table 8: Results of Lunardi et al. (2014) test for 4-digit industry level.

	4-digit industry	# firms	$b = 2$	$b = 1$
Chemicals	Miscellaneous industrial inorganic chemicals (2819)	79	All	**None**
	Plastics (2826)	54	D, W^2, U^2	**None**
	Miscellaneous industrial organic chemicals (2829)	51	All	**None**
	Paints and printing ink (2851)	85	All	D
	Soaps and miscellaneous oil and fat products (2852)	72	**None**	U^2
	Drugs and medicines (2870)	241	All	All
	Cosmetics, perfumes and fragrances (2891)	102	All	**None**
	Miscellaneous chemical and allied products (2899)	112	All	**None**
Iron & steel	Steel materials, except made by smelting furnaces and steel works with rolling facilities (3313)	131	All	A^2, W^2, U^2
	Iron castings (3320)	112	All	**None**
	Steel castings, secondary forgings and steel forgings (3331)	83	All	All
	Miscellaneous iron and steel (3339)	90	All	D, W^2, U^2
	Non-ferrous die castings (3372)	99	All	**None**
	Electric wire and cable (3380)	89	All	**None**
Fabricated metal	Tableware, cutlery, hand tools and hardware (3420)	130	All	U^2
	Heating apparatus and plumbing supplies (3430)	123	W^2, U^2	A^2, U^2
	Fabricated construction-use metal products (3441)	138	All	**None**
	Fabricated architectural metal products, except structural hardware (3442)	181	All	**None**
	Fabricated plate work, sheet metal work and pallets (3443)	162	All	All
	Heat treated metal (3451)	53	**None**	**None**
	Coating metal products, galvanized and other hot-dip coated metal products, engraving on metal and miscellaneous treatment of metal surface (3452)	288	All	**None**
	Stamped and pressed aluminum products and aluminum alloys (3453)	92	All	**None**
	Stamped and pressed metal products, except aluminum and aluminum alloys (3454)	416	All	**None**
	Fabricated wire products (3470)	60	All	**None**
	Bolts, nuts, rivets, machine screws and wood screws (3480)	131	**None**	All
	Metallic springs (3491)	69	W^2, U^2	**None**
	Miscellaneous fabricated metal products (3499)	50	D, W^2, U^2	**None**

Note: The name of 4-digit industries are shown in the first column (4-digit TDB industry classification code is in the parenthesis). The third ($b = 2$) and fourth ($b = 1$) columns show the Gaussian and Laplace hypothesis, respectively. D (Kolmogorov), A^2 (Anderson-Darling), W^2 (Cramer-Von Mises), and U^2 (Watson) in the elements show the test statistics that reject the hypothesis at the 1% level. **None** (All) means that none (all) of the four test statistics reject the null hypothesis.

Table 9: Results of Lunardi et al. (2014) test for 4-digit industry level.

	4-digit industry	# firms	$b = 2$	$b = 1$
General-purpose machinery	Agricultural machinery and equipment (3520)	70	All	A^2, W^2
	Machinery and equipment for construction and mining (3530)	105	All	All
	Metal machine tools and metal working machinery (3541)	196	All	All
	Parts and accessories for metal working machines and machine tools, except machinists' precision tools, molds and dies (3543)	60	All	**None**
	Machinists' precision tools, except powder metallurgy products (3544)	129	All	**None**
	Food processing machinery and equipment (3561)	60	**None**	**None**
	Printing, bookbinding and paper converting machinery (3564)	56	W^2, U^2	**None**
	Miscellaneous special industry machinery (3569)	185	All	D, A^2
	Pumps and pumping equipment, air compressors, gas compressors and blowers (3571)	100	A^2, W^2, U^2	All
	Conveyors and conveying equipment (3573)	135	W^2, U^2	A^2, U^2
	Oil hydraulic and pneumatic equipment (3574)	79	All	**None**
	Mechanical power transmission equipment, except ball and roller bearings (3576)	121	All	D, U^2
	Chemical machinery and its equipment (3578)	127	All	All
	Miscellaneous general industry machinery and equipment (3579)	58	All	**None**
	Refrigerating machines and air conditioning apparatus (3582)	56	All	**None**
	Office machines, miscellaneous office, service industry and household machines (3589)	112	All	**None**
	Valves and fittings (3591)	99	**None**	All
	Ball and roller bearings (3592)	79	All	**None**
	Molds and dies, parts and accessories (3595)	235	All	**None**
	Packing machines (3596)	53	**None**	**None**
	Machine shops (jobbing and repair) (3599)	151	All	**None**
Electrical machinery	Wiring devices and supplies (3611)	99	All	**None**
	Generators, motors, power and distribution transformers, and electrical control equipment (3613)	346	All	A^2, W^2, U^2
	Auxiliary equipment for internal combustion engines (3614)	60	W^2, U^2	**None**
	Miscellaneous industrial electrical apparatus (3619)	63	All	**None**
	Household electric appliances (3620)	69	W^2, U^2	**None**
	Electric bulbs and lighting fixtures (3650)	87	D, W^2, U^2	A^2, W^2, U^2
	Communication equipment and related products (3661)	236	All	**None**
	Electronic parts and devices (3662)	640	All	All
	Electron tubes, semiconductor devices and integrated circuits (3663)	104	All	**None**
	Electric measuring instruments (3670)	138	All	**None**
	Computer and electronic equipment (3680)	251	All	A^2, W^2, U^2
Transportation equipment	Motor vehicles bodies and trailers (3712)	62	All	**None**
	Internal combustion engines for motor vehicles (3713)	150	All	U^2
	Motor vehicles drive and control equipment (3714)	221	All	**None**
	Motor vehicles parts and accessories (3719)	401	All	All
	Shipbuilding and repairing (3731)	66	All	**None**

Note: See the explanation in Table 8.

4. *It is stochastically continuous.*[30]

5. *There is $\Omega_0 \in \mathcal{F}$ with $P[\Omega_0] = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.*

Theorem 4 *Lévy–Itō decomposition: let $\{X_t\}_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with a generating triplet $\{A, \nu, \gamma\}$. Let $\{J(B) : B \in \mathcal{B}(H)\}$ be a Poisson random measure defined by*

$$J(B, \omega) = \int_B J(ds \times dx, \omega) := \#\{s : (s, X_s(\omega) - X_{s-}(\omega)) \in B\}, \text{ for } \omega \in \Omega,$$

where $H := (0, \infty) \times (\mathbb{R} \setminus \{0\})$ and $\mathcal{B}(H)$ is the Borel σ-algebra of H. Suppose that the Lévy process satisfies $\int_0^1 |x| \; \nu(dx) < \infty$. Then, there exists $\Omega_1 \in \mathcal{F}$ with $P[\Omega_1] = 1$ such that for any $\omega \in \Omega_1$,

$$X_t^1(\omega) := \int_{(0,t]\times(\mathbb{R}\setminus\{0\})} xJ(ds \times dx, \omega) \tag{13}$$

is defined for all $t \geq 0$. The process is a Lévy process on $\mathbb{R}$ such that

$$\hat{\mu}_{X_1^1}(z) := E[e^{izX_1^1}] = \exp[\int_{\mathbb{R}} (e^{izx} - 1)\nu(dx)].$$

Define

$$X_t^2(\omega) := X_t(\omega) - X_t^1(\omega), \text{ for } \omega \in \Omega_1.$$

There exists $\Omega_2 \in \mathcal{F}$ with $P[\Omega_2] = 1$ such that for any $\omega \in \Omega_2$, $X_t^2(\omega)$ is continuous in t and $\{X_t^2\}$ is a Lévy process on $\mathbb{R}$ such that

$$\hat{\mu}_{X_1^2}(z) := E[e^{izX_1^2}] = \exp[-\frac{1}{2}Az^2 + i\gamma z]. \tag{14}$$

The two processes $\{X_t^1\}$ and $\{X_t^2\}$ are independent (see Sato (1999), p.121).

Note that $J(B)$ counts the number of jumps, that is, discontinuous points s such that $X_s(\omega) \neq X_{s-}(\omega)$ whose size is within B. Thus, the integral in (13) is the weighted sum of these jumps and therefore, X_t^1 is a jump process as shown in Figure 7. By contrast, the characteristic function in (14) tells us that X_t^2 is a Brownian motion with drift γ, which is a continuous stochastic process. This theorem means that a Lévy process can be decomposed into these two processes.

A.4 Tail equivalence

Let us begin with some notations used in the following. We denote the η-exponential moment of μ by $\tilde{\mu}(\eta) := \int_{-\infty}^{\infty} e^{\eta x}\mu(dx)$, where $\eta \geq 0$. $f(r) \sim g(r)$ means that $\lim_{r\to\infty} f(r)/g(r) = 1$. The convolution of distributions μ and ρ is denoted by $\mu * \rho$. We introduce two classes of distributions $\mathbf{L}(\eta)$ and $\mathbf{S}(\eta)$ as follows.

Definition 5 *Let μ be a distribution on $\mathbb{R}$. Suppose that $\overline{\mu}(r) > 0$ for all $r \in \mathbb{R}$.*

1. *$\mu \in \mathbf{L}(\eta)$ if $\overline{\mu}(r + a) \sim e^{a\eta}\overline{\mu}(r)$ for all $a \in \mathbb{R}$.*

2. *$\mu \in \mathbf{S}(\eta)$ if $\mu \in \mathbf{L}(\eta)$, $\tilde{\mu}(\eta) < \infty$, and $\overline{\mu * \mu}(r) \sim 2\tilde{\mu}(\eta)\overline{\mu}(r)$.*

If $\mu \in \mathbf{L}(0)$, the tail of μ shows slower decay than the tail of a Laplace distribution; $\lim_{x\to\infty} e^{\epsilon x}\overline{\mu}(x) = \infty$ for $\mu \in \mathbf{L}(0)$ and each $\epsilon > 0$ (see Embrechts et al. (1997), p.41; Pakes (2004)).

[30] A stochastic process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}$ is called stochastically continuous if for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s\to t} P[|X_s - X_t| > \epsilon] = 0.$$

There is a close relationship between $\mathbf{S}(\eta)$ and Lévy processes. We denote the Lévy measure of an infinitely divisible distribution by ν. Let $\nu_c(dx)$ denote the jump (larger than c) distribution for $\overline{\nu}(c) > 0$:

$$\nu_c(dx) := \frac{1}{\overline{\nu}(c)} 1_{(c,\infty)}(x)\nu(dx), \quad c > 0.$$

Watanabe (2008) proves the following theorem (see also Pakes (2004, 2007)).

Theorem 6 *Let $\eta \geq 0$. Let μ be an infinitely divisible distribution on $\mathbb{R}$. Then, the following are equivalent:*

1. $\mu \in \mathbf{S}(\eta)$;
2. $\nu_1 \in \mathbf{S}(\eta)$;
3. $\nu_1 \in \mathbf{L}(\eta)$, $\tilde{\mu}(\eta) < \infty$, *and* $\overline{\mu}(x) \sim \tilde{\mu}(\eta)\overline{\nu}(x)$.

This theorem characterizes the class $\mathbf{S}(\eta)$ of infinitely divisible distributions and shows that the tail of a distribution is determined by the tail of the Lévy measure.

Let us return to the relationship between this theorem and the firm growth rate distribution. In Section 4, we propose two classes of distributions having a fatter tail: the power law tail and Weibull distributions. Let μ_{Wei} be a Weibull distribution with $0 < b < 1$. Define $\mu^+_{\text{Wei}}(dx) := 1_{[0,\infty)}(x)\mu_{\text{Wei}}(dx) + \mu_{\text{Wei}}(-\infty, 0)\delta_0(dx)$, where δ_0 is a Dirac measure. In other words, μ^+_{Wei} is a distribution with support $[0, \infty)$ and a right tail identical to that of μ_{Wei}. Corollary 2.1 in Pakes (2004) implies that for $\eta \geq 0$, $\mu_{\text{Wei}} \in \mathbf{S}(\eta)$ if and only if $\mu^+_{\text{Wei}} \in \mathbf{S}(\eta)$. Example 1.4.3 in Embrechts et al. (1997) shows that distributions with a right tail decaying at the same rate as a Weibull distribution are in $\mathbf{S}(0)$. Therefore, $\mu^+_{\text{Wei}} \in \mathbf{S}(0)$ and $\mu_{\text{Wei}} \in \mathbf{S}(0)$.

Next, we consider the distributions with a power-law tail and apply the same argument. Corollary 1.3.2 in Embrechts et al. (1997) states that distributions with a power-law tail on $[0, \infty)$ are in the class $\mathbf{S}(0)$. Given Corollary 2.1 in Pakes (2004), this result implies that $\mu_{pow} \in \mathbf{S}(0)$. Therefore, we reach the following proposition.

Proposition 7 *Both classes of distributions ($\overline{\mu}_{\text{Wei}}(x) = e^{-cx^b}$, $0 < b < 1$ and $\overline{\mu}_{pow}(x) = x^{-\alpha}$) are included in $\mathbf{S}(0)$.*

Therefore, in either distribution (i.e., μ_{pow} and μ_{Wei}), we can apply Theorem 6 with $\eta = 0$. Since $\eta = 0$ implies that $\tilde{\mu} = 1$, we find that $\overline{\mu}(x) \sim \overline{\nu}(x)$, that is, $\lim_{x\to\infty} \overline{\mu}(x)/\overline{\nu}(x) = 1$. Thus, the empirical distribution μ and Lévy measure ν decay at almost the same rate. In other words, we can estimate the frequency of large jumps by estimating the tail of the empirical growth rate distribution.

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