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Existence and Uniqueness of Equilibrium for a Spatial Model of Social Interactions

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Abstract

We extend Beckmann's spatial model of social interactions to the case of a two-dimensional spatial economy involving a large class of utility functions, accessing costs, and space-dependent amenities. We show that spatial equilibria derive from a potential functional. By proving the existence of a minimiser of the functional, we obtain that of spatial equilibrium. Under mild conditions on the primitives of the economy, the functional is shown to satisfy displacement convexity, a concept used in the theory of optimal transportation. This provides a variational characterisation of spatial equilibria. Moreover, the strict displacement convexity of the functional ensures the uniqueness of spatial equilibrium. Also, the spatial symmetry of equilibrium is derived from that of the spatial primitives of the economy. Several examples illustrate the scope of our results. In particular, the emergence of multiplicity of equilibria in the circular economy is interpreted as a lack of convexity of the problem.

Keywords: Social interaction, Spatial equilibria, Multiple cities, Optimal transportation, Displacement convexity

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1. INTRODUCTION

Since Marshall [1920], it is known that both market and non-market forces play an important role in shaping the distribution of economic activities across space. The new economic geography literature has reemphasised the role of localised pecuniary externalities mediated by the market in a general equilibrium framework, see Krugman [1991]. Social interactions through face-to-face contacts also contribute to the gathering of individuals in villages, agglomerations, or cities, see Glaeser and Scheinkman [2003]. In Beckmann [1976], the urban structure results from the interplay between a spatial communication externality and the land market.

When studying the role of agglomeration forces on the urban structure, the existing literature traditionally relies on specific functional forms regarding utility functions or transportation costs. New economic geography models make a wide use of Dixit-Stiglitz or quadratic preferences over manufacturing varieties and of 'icerberg' transport costs, see Fujita et al. [1999] and Ottaviano et al. [2002]. In Beckmann's spatial model of social interactions, the preference for land is logarithmic and the cost of accessing agents is linear, see Fujita and Thisse [2002].

More recently, some efforts have been made to build models allowing for more general preferences over goods, with internal or external increasing returns to scale. For instance, some works have extended the CES preferences used in monopolistic competition to the case of variable elasticity of substitution, see Behrens and Murata [2007], and to the case of additively separable preferences across varieties, see Zhelobodko et al. [2012]. Also, in a multi-district model with external increasing returns in the spirit of Fujita and Ogawa [1982], Lucas and Rossi-Hansberg [2002] have proved the existence of a symmetric spatial equilibrium for a large class of economies. Despite these various efforts in extending models addressing agglomeration forces mediated by the market mechanism, little progress has been made to extend further spatial models where agglomeration externalities are driven by non-market forces. The aim of this paper is to fill up this gap by addressing the existence and uniqueness of equilibrium for general spatial economies involving social interactions.

Our main results are the following. We generalise Beckmann's spatial model of social interactions to the case of a two-dimensional spatial economy involving a large class of preferences for land, accessing costs, and space-dependent amenities. We prove the existence and the uniqueness of spatial equilibrium. So as to get our results, we start our analysis by providing conditions under which spatial equilibria derive from a potential. Stated differently, we build a functional of which the critical points correspond to the spatial equilibria of the economy. In this context, the conditions ensuring the existence of a minimiser of the functional also ensure the existence of a spatial equilibrium of the economy. As the functional is not convex in the usual sense, we introduce another notion of convexity, referred to as *displacement convexity*, a concept widely used in the theory of optimal transportation. Under mild conditions on the primitives of the economy, the functional is shown to be displacement convex, and we obtain an equivalence between the minimisers of the functional and the spatial equilibria of the economy. This provides a variational characterisation of spatial equilibria. Moreover, if the functional displays strict displacement convexity, we get the uniqueness of minimiser, and hence that of spatial equilibrium. Also, the spatial symmetry of equilibrium is derived from that of the spatial primitives of the economy. We present several examples with the purpose of illustrating the scope of our existence and uniqueness results. In particular, the case of one- or two-dimensional geographical spaces, linear or quadratic accessing costs, and linear or power residence costs are examined. Finally, the case of a circular spatial economy is revisited so as to illustrate the role of non-convexities in explaining the emergence of multiple equilibria. A direct method allows us to derive all the spatial equilibria arising along the circle. The analysis completes the work initiated by Mossay and Picard [2011].

The remainder of the paper is organised as follows. Section 2 presents the economic environment and generalises Beckmann's spatial model of social interactions. In Section 3, we prove the existence of a spatial equilibrium. Section 4 is devoted to the variational characterisation and the uniqueness of equilibrium, as well as its spatial symmetry properties. In Section 5, we present several examples of spatial economies so as to illustrate the scope of our results. Section 6 is devoted to the analysis of the circular economy. Section 7 summarises the main results of the paper and concludes.

2. SPATIAL MODEL

In this Section we present the economic environment. We consider a closed spatial economy $\mathcal{E}$ extending along a one- or two-dimensional geographical space $\mathcal{K} \subset \mathbb{R}^d$, $d = 1, 2$. A unit-mass of agents is distributed according to the spatial density $\lambda : \mathcal{K} \to \mathbb{R}_+$ with $\int_{\mathcal{K}} \lambda(x)\,\mathrm{d}x = 1$. Agents meet each other so as to benefit from social contacts. The social utility $S(x)$ that an agent in location $x \in \mathcal{K}$ derives from interacting with other agents is given by

$$S(x) = B - \int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y \tag{2.1}$$

where the constant B denotes the total benefit from interacting with other agents and $W : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ the cost of accessing them. To ensure that social interactions are global, B is assumed to be large enough, $B > \max_x \int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y$.

As agents in location $x \in \mathcal{K}$ also consume a composite good z and some land space s, their utility $\mathcal{U}$ is given by

$$\mathcal{U}(s,z,x) \;=\; z + u(s) + S(x) + A(x)$$

where S is the social utility defined in Expression (2.1), $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ the utility of land consumption, and $A : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\}$ the spatial distribution of amenities. The budget constraint faced by agents is

$$z + R(x)\ s \;=\; Y$$

where Y is the income of agents (e.g., the endowment of the composite good) and $R(x)$ the land rent in location x.

As is usual in the urban economics literature, we assume the presence of an absentee landlord who collects the rent paid by agents. Also, we assume that land has no alternate use other than residence. The agent's bid rent function in location x is defined as the maximum rent that an agent is willing to pay for residing in that location,

$$\psi(x,\bar{\mathcal{U}}) = \max_{s,z} \frac{Y-z}{s} \quad \text{such that} \quad \mathcal{U}(s,z,x) = \bar{\mathcal{U}}\ .$$

Assumption 1 (Utility of land consumption) *The utility of land consumption $u \in C^2(\mathbb{R}_+)$ is concave.*

Lemma 1 (Spatial indirect utility function) *Under Assumption 1, the spatial indirect utility function U is given by*

$$U(x) = Y + B - v(\lambda(x)) - \int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y + A(x) \tag{2.2}$$

where the residence cost v defined by

$$v(\lambda) = \frac{1}{\lambda}\ u'\left(\frac{1}{\lambda}\right) - u\left(\frac{1}{\lambda}\right)$$

is an increasing function of the spatial distribution λ.

Proof: The bid-rent $\psi(x,\bar{\mathcal{U}})$ can be rewritten as $\max_s (Y + u(s) + S(x) + A(x) - \bar{\mathcal{U}})/s$. Let $\hat{s}(x,\bar{\mathcal{U}})$ denote the bid-maximising consumption of land. The corresponding first-order condition is then given by $u'(\hat{s})\hat{s} - (Y + u(\hat{s}) + S(x) + A(x) - \bar{\mathcal{U}}) = 0$.

The land market equilibrium condition ($\lambda(x) = 1/\hat{s}$) allows to define the spatial indirect utility function $U(x) = Y - v(\lambda(x)) + S(x) + A(x)$, where the residence cost v is defined by $v(\lambda) = (1/\lambda)u'(1/\lambda) - u(1/\lambda)$. Finally, we have $v'(\lambda) = -(1/\lambda^3)u''(\lambda) > 0$ as u is concave. *Q.E.D.*

The spatial indirect utility $U(x)$ corresponds to the utility available to agents located in x once the land market is in equilibrium. Its Expression (2.2) involves three non-constant terms: the accessing cost $\int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y$, the residence cost $v(\lambda)$, and the space-dependent amenities A.

Let $\mathcal{M}(\mathcal{K})$ denote the set of absolutely continuous spatial densities over $\mathcal{K}$ with respect to the Lebesgue measure. In this context, we define a spatial equilibrium of the economy $\mathcal{E}$ as follows.

DEFINITION 1 (Spatial equilibrium) A spatial distribution of agents $\lambda \in \mathcal{M}(\mathcal{K})$ constitutes a *spatial equilibrium* of the economy $\mathcal{E}$ if there exists $\bar{U}$ such that

$$\begin{cases} U(x) \leq \bar{U} & \text{for almost every } x \in \mathcal{K}, \\ U(x) = \bar{U} & \text{for almost every } x \in \mathcal{K} \text{ such that } \lambda(x) > 0. \end{cases} \tag{2.3}$$

Interestingly, the spatial equilibrium condition (2.3) can be restated as follows.

PROPOSITION 1 *Suppose that the utility of land consumption satisfies Assumption 1 and the condition* $\lim_{\lambda\to 0} v(\lambda) = 0$*. Then the spatial distribution of agents* λ *is a spatial equilibrium of the economy* $\mathcal{E}$ *if and only if the residence cost* v *corresponds to*

$$v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+ \tag{2.4}$$

PROOF: First, suppose the residence cost v satisfies $v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+$. Under Assumption 1, the residence cost v is increasing. Hence, $v(\lambda > 0) > 0$, which implies

$$\begin{cases} v(\lambda(x)) \geq Y - \bar{U} + S(x) + A(x), & \text{for almost every } x \in \mathcal{K}, \\ v(\lambda(x)) = Y - \bar{U} + S(x) + A(x), & \text{for almost every } x \in \mathcal{K} \text{ such that } \lambda(x) > 0, \end{cases}$$

and the spatial density λ constitutes a spatial equilibrium of the economy $\mathcal{E}$.

Conversely, by using the spatial equilibrium condition (2.3) and the expression of the spatial indirect utility (2.2), when $\lambda(x) > 0$, $v(\lambda(x)) = Y - \bar{U} + S(x) + A(x) > 0$ while $\lim_{\lambda\to 0} v(\lambda) = 0$. This can be summarised by $v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+$. *Q.E.D.*

3. EXISTENCE OF EQUILIBRIUM

In this Section we address the following issue. Can the spatial equilibria of the economy $\mathcal{E}$ be derived from a potential? More specifically, is it possible to relate the search of spatial equilibrium to the optimisation of a functional which could be interpreted as the measure of some global cost. Games of which the equilibria can be derived from the optimisation of a potential are referred to as *potential games* as introduced by Monderer and Shapley [1996]. From an analytical point of view, the equilibria of such games correspond to critical points of the potential.

Here, our approach is similar to that developed in the literature on potential games. We build a potential functional $\mathcal{F}$ which depends on the spatial distribution λ and turns out to have the following property: the first-order condition to the optimisation problem $\min_\lambda \mathcal{F}[\lambda]$ corresponds to the spatial equilibrium condition (2.3). Let $V : \mathbb{R}_+ \to \mathbb{R}_+$ be a primitive of v. The functional $\mathcal{F} : \mathcal{M}(\mathcal{K}) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{F}[\lambda] = \mathcal{V}[\lambda] + \mathcal{W}[\lambda] + \mathcal{A}[\lambda] \tag{3.1}$$

where λ denotes a spatial density in $\mathcal{M}(\mathcal{K})$ and the terms $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{W}$ are defined respectively by[1]

$$\mathcal{V}[\lambda] = \int_{\mathcal{K}} V[\lambda(x)]\,\mathrm{d}x\,, \quad \mathcal{A}[\lambda] = -\int_{\mathcal{K}} A(x)\lambda(x)\,\mathrm{d}x$$
$$\text{and} \quad \mathcal{W}[\lambda] = \frac{1}{2}\iint_{\mathcal{K}\times\mathcal{K}} W(x-y)\lambda(x)\lambda(y)\,\mathrm{d}x\,\mathrm{d}y$$

Assumption 2 (Spatial symmetry)

- *The geographical space $\mathcal{K}$ is symmetric: for all $x \in \mathcal{K}, -x \in \mathcal{K}$,*
- *The accessing cost W is even: $W(x) = W(-x)$, for all $x \in \mathcal{K}$.*

We now consider the minimisation of $\mathcal{F}$ on $\mathcal{M}(\mathcal{K})$.

Lemma 2 (Necessary condition of existence) *Under Assumption 2, if the spatial distribution of agents λ minimises the potential functional $\mathcal{F}$ in the set $\mathcal{M}(\mathcal{K})$, then it is a spatial equilibrium of the economy $\mathcal{E}$.*

The proof of this result consists in deriving the optimality condition for the minimisation problem of functional $\mathcal{F}$. It turns out that the spatial indirect utility function U is a differential of $\mathcal{F}$ in the following sense. For any admissible spatial densities $(\lambda, \tilde{\lambda})$ in $\mathcal{M}(\mathcal{K})$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{F}[\lambda + \varepsilon(\tilde{\lambda} - \lambda)] - \mathcal{F}[\lambda]}{\varepsilon} = -\int_{\mathcal{K}} U(x)(\tilde{\lambda}(x) - \lambda(x))\,\mathrm{d}x.$$

If λ is a minimiser of $\mathcal{F}$, then the above limit is non negative, which implies that $\int_{\mathcal{K}} U(x)(\tilde{\lambda}(x) - \lambda(x))\,\mathrm{d}x \le 0$. As the above inequality holds for any arbitrary

[1] In the mathematics literature, these three integrals are referred to as the internal, the potential, and the interaction energies, see e.g. Villani [2003].

admissible density $\tilde{\lambda}$, the spatial indirect utility $U(x)$ achieves its maximum value $\bar{U}$ on the support of λ. A detailed proof of Lemma 2 is provided in Appendix A.1.

Lemma 2 relates the concept of spatial equilibrium of the economy $\mathcal{E}$ to the notion of minimiser of the potential functional $\mathcal{F}$. Yet, the global cost associated with $\mathcal{F}$ does not correspond to the aggregate cost of the spatial economy $\mathcal{E}$. Though the integral $\mathcal{A}$ is the spatial aggregate of the space-dependent amenities A, the integral $\mathcal{W}$ corresponds to half the aggregate accessing cost $\int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y$, and the integral $\mathcal{V}$ does not correspond to the aggregation of the residence cost $v(\lambda)$. As a consequence, the spatial equilibria of the economy $\mathcal{E}$ are not likely to minimise the total aggregate cost. This is hardly surprising given the presence of the spatial communication externality.

Given Lemma 2, a preliminary step for proving the existence of a spatial equilibrium is to address the existence of a minimiser of $\mathcal{F}$.

Assumption 3

- *The utility of land consumption u satisfies* $\lim_{s\to 0^+} u(s) = -\infty$,
- *The accessing cost W is continuous on $\mathcal{K}$,*
- *The spatial distribution of amenities A is continuous on $\mathcal{K}$ and bounded from above,*
- *If $\mathcal{K}$ is unbounded, either* $\lim_{|x|\to\infty} A(x) = -\infty$ *or A is constant and* $\lim_{|x|\to\infty} W(x) = +\infty$.

Examples of utility functions u satisfying Assumptions 1 and 3 are the logarithmic utility $u(s) = \beta(\log(s) + 1)$ and the hyperbolic utility $u(s) = -\beta/(2s)$, $\beta > 0$, used respectively by Beckmann [1976] and Mossay and Picard [2011].

Lemma 3 (Existence of a minimiser) *Under Assumptions 1 and 3, the potential functional $\mathcal{F}$ admits a minimiser in $\mathcal{M}(\mathcal{K})$.*

Proof: First note that $\mathcal{F}$ is bounded from below. Take a minimising sequence λ_k, where each λ_k is a spatial density over $\mathcal{K}$. We consider the weak convergence of λ_k to some limit measure λ and show that this limit measure actually corresponds to a spatial density. In order to exclude the possibility of minimisers given by singular measures (e.g. measures concentrated in a single point or in a thin set), we show the super-linearity of function V, which ensures that the measure is not concentrated. By using the expression of the residence cost v in terms of the utility function u, as given in Lemma 1, we have $V(\lambda) = -\lambda u\left(1/\lambda\right)$. This means that $\lim_{\lambda\to\infty} V(\lambda)/\lambda = -\lim_{s\to 0^+} u(s) = +\infty$, so that V is super-linear.

The weak convergence of the spatial densities $\lambda_k(x)$ to $\lambda(x)$ implies the weak convergence of $\lambda_k(x)\lambda_k(y)$, which are densities over $\mathcal{K}\times\mathcal{K}$, to the density $\lambda(x)\lambda(y)$. This weak convergence in $\mathcal{M}(\mathcal{K}\times\mathcal{K})$ ensures that functional $\mathcal{F}$ is lower-semicontinuous with respect to the weak-* topology. Finally, the third point of Assumption 3 ensures the tightness of the sequence λ_k, and hence there exists a sub-sequence of the sequence λ_k which converges weakly to the spatial density λ. *Q.E.D.*

The result on the equilibrium existence is summarised in the following Theorem.

THEOREM 1 (Existence of equilibrium) *Under Assumptions 1, 2 and 3, the spatial economy $\mathcal{E}$ admits a spatial equilibrium.*

PROOF: This is an immediate consequence of Lemmas 2 and 3. *Q.E.D.*

The convexity of the potential functional $\mathcal{F}$ would ensure the critical points of $\mathcal{F}$ to be minimisers of $\mathcal{F}$, and therefore spatial equilibria of $\mathcal{E}$. In addition, if the potential functional $\mathcal{F}$ were strictly convex, then it would not have more than one minimiser. This would provide the uniqueness of spatial equilibrium. Unfortunately, the potential functional $\mathcal{F}$ fails to be convex because of the bilinear form of the aggregate accessing cost $\mathcal{W}$. This term corresponds to the spatial externality associated with social interactions between agents located at different locations. The purpose of next section is to introduce another notion of convexity used in the theory of optimal transportation which will allow us to deal with this issue.

4. VARIATIONAL CHARACTERISATION AND UNIQUENESS OF EQUILIBRIUM

In this Section, in order to overcome the lack of standard convexity of the potential $\mathcal{F}$, we rely on a notion of convexity used for functionals defined over probability measures, referred to as *displacement convexity*. The concept has its origin in the theory of optimal transportation. We show that the functional $\mathcal{F}$ is displacement convex under mild assumptions on the primitives of the spatial economy $\mathcal{E}$ (*i.e.*, the spatial domain $\mathcal{K}$, the utility function u, the accessing cost W, and the space-dependent amenities A). As a consequence, for a wide class of spatial economies, there is an equivalence between the critical points and the minimisers of $\mathcal{F}$. This provides a variational characterisation of the spatial equilibria of $\mathcal{E}$. Moreover, if $\mathcal{F}$ is strictly displacement convex, the uniqueness of minimiser is ensured, and therefore that of equilibrium as well. Furthermore, the spatial symmetry of equilibrium is also derived depending on the geometry of the spatial domain $\mathcal{K}$, and the spatial properties of the accessing cost W and of the space-dependent amenities A.

In the sequel, we assume that $\mathcal{K} = \overline{\Omega}$ where Ω is some open bounded convex subset of $\mathbb{R}^2$.[2] We first introduce some basic concepts of the theory of optimal

[2]Note that the results of this Section also hold when the set Ω is unbounded provided that we restrict the set $\mathcal{M}(\mathcal{K})$ to spatial densities λ with a finite second moment $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y)\,\mathrm{d}y <$

transportation. For a detailed exposition of this subject, we refer the interested reader to Villani [2003], Ambrosio et al. [2005], Villani [2009], or Rachev and Rüschendorf [1998]. Let λ_0 and λ_1 be two spatial densities in $\mathcal{M}(\mathcal{K})$ and T a measurable map $\mathcal{K} \to \mathcal{K}$. The map T is said to transport the spatial density λ_0 onto λ_1 if, for any measurable set $B \subset \mathcal{K}$, we have

$$\int_B \lambda_1(x)\,\mathrm{d}x = \int_{T^{-1}(B)} \lambda_0(x)\,\mathrm{d}x.$$

This relation may also be expressed in terms of functions in the following way

$$\int_{\mathcal{K}} \zeta(y)\lambda_1(y)\,\mathrm{d}y = \int_{\mathcal{K}} \zeta[T(x)]\lambda_0(x)\,\mathrm{d}x, \quad \forall \zeta : \mathcal{K} \to \mathcal{K}\,. \tag{4.1}$$

The condition expressing that the map T transports λ_0 onto λ_1 is denoted by $T\#\lambda_0 = \lambda_1$, and T is referred to as the *transport map* between λ_0 and λ_1.

Transport maps can be used to define distances between probability measures. As we focus our analysis on spatial densities in $\mathcal{M}(\mathcal{K})$, the Monge-Kantorovich distance w_2 between λ_0 and λ_1 is defined by[3]

$$w_2(\lambda_0,\lambda_1) = \sqrt{\inf_{T:\ \lambda_1 = T\#\lambda_0} \int_{\mathcal{K}} |x - T(x)|^2 \lambda_0(x)\,\mathrm{d}x}\,.$$

In general, there is no reason for the infimum appearing in the above definition to be attained. Conditions ensuring the existence of a minimiser T are provided by Brenier [1991]: if λ_0 is absolutely continuous with respect to the Lebesgue measure, then there is a unique optimal map T from λ_0 onto λ_1, which is given by $T = \nabla\varphi$ for some convex function φ. As a consequence, the Monge-Kantorovich distance w_2 can be rewritten as

$$w_2(\lambda_0,\lambda_1) = \int_{\mathcal{K}} |x - \nabla\varphi(x)|^2\,\mathrm{d}\lambda_0(x)\,.$$

For any two spatial densities λ_0 and λ_1, we consider the optimal transport map T transporting λ_0 onto λ_1 and define

$$\lambda_t = [(1-t)\mathrm{id} + tT]\#\lambda_0 \quad \text{for } t \in [0,1]$$

$+\infty$.

[3]In the mathematics literature, the Monge-Kantorovich distance is often referred to as the Wasserstein distance. Note that the above expression does not apply to the case of atomic measures. Anyway, this latter case does not need to be considered here as we deal with spatial densities (i.e non-atomic measures).

Note that $\lambda_t|_{t=0} = \lambda_0$ and $\lambda_t|_{t=1} = \lambda_1$. The curve $\{\lambda_t\}_{t\in[0,1]}$ of spatial densities actually corresponds to the unique constant-speed geodesic connecting λ_0 to λ_1 under the metric w_2. For any $(t,s) \in [0,1]^2$, we have

$$w_2(\lambda_t, \lambda_s) = |t-s|\, w_2(\lambda_0, \lambda_1)\ .$$

Convexity in the space $\mathcal{M}(\mathcal{K})$ endowed with the metric w_2 has been studied first by McCann and is referred to as *displacement convexity* (or geodesic convexity), see McCann [1997].

Definition 2 (Displacement convexity) The functional $\mathcal{F}$ is said to be *displacement convex* (or *geodesically convex*) in $\mathcal{M}(\mathcal{K})$, if for all λ_0 and λ_1 in $\mathcal{M}(\mathcal{K})$,

$$\mathcal{F}[\lambda_t] \le (1-t)\mathcal{F}[\lambda_0] + t\mathcal{F}[\lambda_1] \quad \text{for } t \in [0,1]$$

When the above inequality is strict for $t \in (0,1)$ and $\lambda_0 \neq \lambda_1$, the functional $\mathcal{F}$ is said to be *strictly displacement convex.*

McCann [1997] provided some assumptions ensuring the displacement convexity of the functional $\mathcal{F}$ defined in Expression (3.1).

Assumption 4 (Displacement convexity) *Let $\mathcal{K} = \overline{\Omega}$ where Ω is an open bounded convex subset of $\mathbb{R}^d$, $d = 1, 2$,*

- *$V(0) = 0$ and the function $\lambda \mapsto \lambda^d V(\lambda^{-d})$ is convex and non-increasing in $(0, +\infty)$,*
- *The accessing cost W is convex,*
- *The spatial distribution of amenities A is concave.*

The logarithmic and the hyperbolic utility functions used in Beckmann [1976] and Mossay and Picard [2011] lead to functions V given respectively by $V(\lambda) = \beta\lambda^2/2$ and $V(\lambda) = \beta(\lambda \log \lambda - \lambda)$ with $\beta > 0$, both of which satisfy Assumption 4.

Theorem 2 (Variational characterisation) *Under Assumptions 2 and 4, the spatial distribution of agents λ is a spatial equilibrium of the economy $\mathcal{E}$ if and only if it is a minimiser of $\mathcal{F}$ in the set $\mathcal{M}(\mathcal{K})$.*

The necessary condition for spatial equilibrium was proved in Lemma 2 by using Assumption 2. The sufficiency proof consists in studying the Euler-Lagrange equation associated with the minimisation of $\mathcal{F}$. Under Assumption 4, the functional $\mathcal{F}$ is displacement convex. The proof makes use of displacement convexity and of optimal transportation arguments. In particular, the perturbations under consideration are to be understood in the optimal transportation sense. A detailed proof is provided in Appendix A.2.

To ensure the uniqueness of minimiser of functional $\mathcal{F}$, McCann [1997] also provided criteria so as to obtain the strict displacement convexity of $\mathcal{F}$. For instance, under Assumption 4, if W is strictly convex or if A is strictly concave, then the functional $\mathcal{F}$ is strictly displacement convex.

THEOREM 3 (Uniqueness of spatial equilibrium) *Under Assumption 4, if A is strictly concave (resp. if the accessing cost W or the function $\lambda \mapsto \lambda^d V(\lambda^{-d})$ is strictly convex), then any spatial equilibrium λ of the economy $\mathcal{E}$ is unique (resp. unique up to translation).*

PROOF OF THEOREM 3: By applying the criteria of McCann [1997], Assumption 4 and the strict concavity of A (resp. the strict convexity of W or of the function $\lambda \mapsto \lambda^d V(\lambda^{-d})$) ensure the strict displacement convexity of functional $\mathcal{F}$. Let λ_0 and λ_1 be two distinct minimisers of $\mathcal{F}$ and consider the optimal transport map T from λ_0 onto λ_1. By applying the strict displacement convexity in $\lambda_{1/2} = (\frac{1}{2}(\mathrm{id} + T))\#\lambda_0$, we obtain $\mathcal{F}[\lambda_{1/2}] < \frac{1}{2}(\mathcal{F}[\lambda_0] + \mathcal{F}[\lambda_1])$, which is in contradiction with λ_0 and λ_1 being minimisers of $\mathcal{F}$. This proves the uniqueness of minimiser, and hence that of equilibrium by the variational characterisation provided in Theorem 2. As the accessing cost W is invariant under translation, the uniqueness holds up to translation if only W or the function $\lambda \mapsto \lambda^d V(\lambda^{-d})$ is strictly convex. *Q.E.D.*

We now turn to the spatial properties of the spatial equilibrium.

ASSUMPTION 5 (Even symmetry) *Suppose that Assumption 2 holds. Moreover, the spatial distribution of amenities A is also even: $A(x) = A(-x)$ for all $x \in \mathcal{K}$.*

ASSUMPTION 6 (Radial symmetry) *Let $\mathcal{K}$ be $\mathbb{R}^2$ or a centred ball in $\mathbb{R}^2$. The accessing cost W and the spatial distribution of amenities A are radially symmetric: $A(x) = A(y)$ and $W(x) = W(y)$ for all $x, y \in \mathcal{K}$ with $|x| = |y|$.*

PROPOSITION 2 (Spatial symmetry of equilibria) *Suppose that Assumption 4 holds. Under Assumption 5 (resp. Assumption 6), any spatial equilibrium λ is even (resp. radially symmetric).*

PROOF: This is a direct consequence of the variational characterisation provided in Theorem 2 as Assumption 5 (resp. Assumption 6) implies that the minimizers of $\mathcal{F}$ are even (resp. radially symmetric). *Q.E.D.*

Proposition 2 goes beyond the approach traditionally used in the literature which systematically assumes the spatial symmetry of equilibria (e.g. the radial symmetry in Lucas [2001] and Lucas and Rossi-Hansberg [2002], or the even symmetry in Fujita and Ogawa [1982] and Berliant et al. [2002]).

5. EXAMPLES

In this Section, several examples illustrate the scope of the existence and uniqueness results obtained in the previous Sections. These examples extend existing models of the literature into many aspects: the dimension or the shape of the spatial domain, and the class of utility functions or accessing costs. In the sequel, we make use of the following notation $(W*\lambda)(x) := \int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y$.

5.1. *Linear accessing and residence costs*

This case has been studied by Mossay and Picard [2011]. The economy extends along the real line $\mathcal{K} \equiv \mathbb{R}$. Both the residence and the accessing costs are linear and there are no amenities. Their corresponding expressions are respectively given by $v(\lambda) = \beta\lambda$, $W(x) = \tau|x|$, and $A = 0$, with $\beta, \tau > 0$.

Assumption 3 is satisfied so that by Theorem 1, a spatial equilibrium exists. Also, function V is given by $V(\lambda) = (\beta/2)\lambda^2$, so that the function $\lambda V(1/\lambda) = \beta/(2\lambda)$ is strictly convex and strictly decreasing, which ensures the strict displacement convexity of functional $\mathcal{F}$. As a consequence, Theorem 3 provides the uniqueness of equilibrium up to translation.

By Proposition 1, we get the following equilibrium spatial distribution

$$\lambda(x) = \frac{1}{\beta}\left(Y + B - \bar{U} - W*\lambda(x)\right)_+ \ .$$

By inspection of the above expression, as both W and $x \mapsto x_+$ are Lipschitz, so is the spatial density λ. Moreover, given that $(W*\lambda)'' = W''*\lambda$, the convexity of the accessing cost W implies that of function $W*\lambda$. In particular, the lower level set of $W*\lambda$, $\{x : W*\lambda < c\}$, is an interval meaning that the spatial density λ will be positive on some interval and vanish everywhere else. Along this interval, the equilibrium spatial distribution is *uni-modal* and *concave* as it corresponds to the positive part of a concave function.

For an analytical expression of the spatial equilibrium, see Mossay and Picard [2011].

5.2. *A two-dimensional model*

We extend the previous example into several aspects by considering a two-dimensional geographical space, a residence cost given by a power function, and a general accessing cost. The economy $\mathcal{E}$ extends along $\mathcal{K} = \mathbb{R}^2$. The residence cost is given by $v(\lambda) = \beta\lambda^\gamma$, with $\beta,\ \gamma > 0$. The accessing cost W is Lipschitz continuous, strictly convex, and radially symmetric with $\lim_{|x|\to\infty} W(x) = +\infty$ (e.g. $W(x) = \tau|x|^2$ with $\tau > 0$).

Assumption 3 is satisfied so that a spatial equilibrium exists. As the accessing cost W is strictly convex, the strict displacement convexity of $\mathcal{F}$ ensures that *the equilibrium is unique up to translation.*

Moreover, by Proposition 2, *the equilibrium is radially symmetric around its barycentre.* Like in the previous example, the convexity of the accessing cost W implies that of function $W * \lambda$. As the support of λ is bounded, *it corresponds to a ball* given the radial symmetry of λ.

We now derive some regularity of the spatial equilibrium. By Proposition 1, we have

$$\lambda^\gamma \equiv \frac{1}{\beta}\left(Y + B - \overline{U} - W * \lambda\right)_+ \ .$$

Since the accessing cost W is Lipschitz continuous, both the function $\lambda * W$ and the spatial density λ are also Lipschitz continuous. As a consequence, the term $\nabla(\lambda * W) = \nabla\lambda * W$ corresponds to the convolution of a bounded function with a Lipschitz one, and hence, it is Lipschitz as well. This means that the function $\lambda * W \in C^{1,1}$ (*i.e.* it is differentiable at every point and its gradient is Lipschitz continuous), which implies that the spatial density λ is globally Lipschitz continuous on $\mathcal{K}$ and $C^{1,1}$ on the ball $\{x : \lambda(x) > 0\}$.

Finally, when the accessing cost is quadratic, $W(x) = |x|^2/2$, the equilibrium spatial density can be written as

$$\lambda(x) = \frac{1}{\beta^{1/\gamma}}\left(C - \frac{1}{2}|x - x_0|^2\right)_+^{1/\gamma}$$

with $C = Y + B - \overline{U} + x_0^2/2 - m_2/2$ where the barycentre x_0 and the second moment m_2 of the spatial distribution λ are given by $x_0 = \int_{\mathcal{K}} y\lambda(y)\,\mathrm{d}y$ and $m_2 = \int_{\mathcal{K}} |y|^2\lambda(y)\,\mathrm{d}y$ respectively. The result is obtained by plugging the expression of function $W * \lambda$

$$\begin{aligned}\int_{\mathcal{K}} |x-y|^2\lambda(y)\,\mathrm{d}y &= |x|^2\int_{\mathcal{K}}\lambda(y)\,\mathrm{d}y - 2x\cdot\int_{\mathcal{K}} y\lambda(y)\,\mathrm{d}y + \int_{\mathcal{K}}|y|^2\lambda(y)\,\mathrm{d}y \\ &= |x-x_0|^2 - x_0^2 + m_2\ .\end{aligned}$$

into Relation (2.4).

5.3. *A two-dimensional Beckmann model*

The model of Beckmann [1976], revisited by Fujita and Thisse [2002], is extended to the case of a two-dimensional geographical space. The economy extends along $\mathcal{K} = \mathbb{R}^2$. The residence cost is given by $v(\lambda) = \beta\log\lambda$, $\beta > 0$ and the accessing cost is quadratic, $W(x) = \tau|x|^2/2$, $\tau > 0$. The corresponding function V is given by

$$V(\lambda) = \begin{cases} \beta(\lambda \log \lambda - \lambda) & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

By inspection, the above function V satisfies the hypotheses ensuring the displacement convexity of $\mathcal{V}$ as the function $\lambda^2 V(\lambda^{-2}) = -\beta(2\log\lambda + 1)$ is both convex and decreasing.

Assumption 3 is satisfied so that a spatial equilibrium exists. Moreover, as the accessing cost W is strictly convex and radially symmetric, the strict displacement convexity of $\mathcal{F}$ ensures the *uniqueness up to translation and the radial symmetry of spatial equilibrium.*

The spatial equilibrium condition (2.3) can be written as

$$\begin{cases} -\beta \log(\lambda(x)) - W * \lambda(x)) \le \overline{U} - Y - B & \text{for almost every } x, \\ -\beta \log(\lambda(x)) - W * \lambda(x)) = \overline{U} - Y - B & \text{for almost every } x \text{ such that } \lambda(x) > 0. \end{cases}$$

Note that here, the equilibrium condition cannot be written as Relation (2.4) given that $v(0) \neq 0$. By inspection of the above equilibrium condition, there is no point x for which $\lambda(x) = 0$. Otherwise, the first condition would imply $\overline{U} = +\infty$ and the second one would not be satisfied. Hence, we can write

$$\beta \log(\lambda(x)) = Y + B - \overline{U} - W * \lambda(x)$$

which leads to

$$\lambda(x) = \mathrm{e}^{[Y+B-\overline{U}-W*\lambda(x)]/\beta} > 0.$$

By determining the function $W * \lambda$, we get the following equilibrium spatial distribution

$$\lambda(x) = C\mathrm{e}^{-\tau/(2\beta)|x-x_0|^2}$$

where $C = \mathrm{e}^{[Y+B-\overline{U}+\tau(x_0^2-m_2)/2]/\beta}$, and the barycenter x_0 and the second moment m_2 of the spatial density λ are given respectively by $x_0 = \int_{\mathcal{K}} y\lambda(y)\,\mathrm{d}y$ and $m_2 = \int_{\mathcal{K}} |y|^2\lambda(y)\,\mathrm{d}y$.

5.4. *A city centre model*

The economy extends along $\mathcal{K} = \mathbb{R}^2$ and amenities decrease with distance to the city centre $x = 0$, $A(x) = -\alpha|x|^2/2$, $\alpha > 0$. The residence cost is given by $v(\lambda) = \beta\lambda$, $\beta > 0$, and the accessing cost is quadratic, $W(x) = \tau|x|^2/2$, $\tau > 0$.

Assumption 3 holds so that a *spatial equilibrium exists*. As the spatial distribution of amenities A is strictly concave, the strict displacement convexity of $\mathcal{F}$ ensures the *uniqueness of equilibrium*. Moreover, the radial symmetry of A and W ensures that of equilibrium. By Proposition 1 and the expression of function $W * \lambda$, the equilibrium spatial distribution corresponds to the following truncated regular paraboloid centred in the city centre $x = 0$

$$\lambda(x) = \frac{1}{\beta}\left(Y + B - \overline{U} - \tau\frac{m_2}{2} - (\tau+\alpha)\frac{|x|^2}{2}\right)_+ .$$

where the second moment m_2 of the spatial density λ is given by $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y)\,\mathrm{d}y$.

5.5. *A linear city model*

We consider a linear city where amenities are distributed along a road and decrease with distance to the road. The economy extends along $\mathcal{K} = \mathbb{R}^2$. The residence cost is given by $v(\lambda) = \beta\lambda, \ \ \beta > 0$, and the accessing cost is quadratic, $W(x) = \tau|x|^2/2, \tau > 0$. Amenities are given by $A(x) = -\alpha|x\cdot e|^2/2$, with $\alpha > 0$ and $e = (1, 0)$. The larger the distance to the road $e^\perp = \{x \in \mathbb{R}^2 : x \cdot e = 0\}$, the lower the amenities.

Assumption 3 holds, so that a spatial equilibrium exists. As the accessing cost W is strictly convex, *the uniqueness of equilibrium is ensured up to translation parallel to* $e^\perp$.

By Proposition 1 and the computation of function $W * \lambda$, we get the following spatial equilibrium density

$$\lambda(x) = \frac{1}{\beta}\left(Y + B - \overline{U} + \tau\frac{x_0^2}{2} - \tau\frac{m_2}{2} - \tau\frac{(x-x_0)^2}{2} - \alpha\frac{|x\cdot e|^2}{2}\right)_+ .$$

where the barycenter x_0 and the second moment m_2 of the spatial distribution λ are given respectively by $x_0 = \int_{\mathcal{K}} y\lambda(y)\,\mathrm{d}y$ and $m_2 = \int_{\mathcal{K}} |y|^2\lambda(y)\,\mathrm{d}y$.

As the support of the equilibrium is an ellipse with a transverse axis corresponding to the road $e^\perp$ and a conjugate axis orthogonal to the road $e^\perp$, the equilibrium distribution corresponds to a truncated elliptic paraboloid.

5.6. *A sea-shore model*

We consider half a space in $\mathbb{R}^2$ representing a region on a sea-shore. The economy extends along the convex domain $\mathcal{K} = \{x \in \mathbb{R}^2 : x \cdot e \geq 0\}$, with $(0,0) \neq e \in \mathbb{R}^2$. The residence cost is $v(\lambda) = \beta\lambda^\gamma$, with $\beta,\ \gamma > 0$ and the accessing cost is quadratic $W(x) = \tau|x|^2/2$. The spatial distribution of amenities is given by $A : x \mapsto -x\cdot e$ so that -A stands for the distance from the boundary of $\mathcal{K}$, i.e. the hyperplane $e^\perp = \{x : x \cdot e = 0\}$.

Assumption 3 is satisfied, so that a spatial equilibrium exists. The strict convexity of the accessing cost W ensures *the uniqueness of equilibrium up to translation.*

By Proposition 1 and the computation of function $W * \lambda$, the equilibrium spatial distribution corresponds to the following truncated paraboloid centred in $y_0 = x_0 - e$

$$\begin{aligned}\lambda(x)^\gamma &= \frac{1}{\beta}\left(Y + B - \overline{U} - \tau\frac{|x-x_0|^2}{2} - \tau x\cdot e + \tau\frac{x_0^2}{2} - \tau\frac{m_2}{2}\right)_+ \\ &= \frac{1}{\beta}\left(C - \tau\frac{|x-(x_0-e)|^2}{2}\right)_+,\end{aligned}$$

where $C = Y + B - \overline{U} + \tau[e^2/2 + x_0 \cdot e + x_0^2/2 - m_2/2]$, and the barycenter x_0 and the second moment m_2 of the spatial density λ are given respectively by $x_0 = \int_{\mathcal{K}} y\lambda(y)\,\mathrm{d}y$ and $m_2 = \int_{\mathcal{K}} |y|^2\lambda(y)\,\mathrm{d}y$.

We still need to determine the admissible translations. The support of the spatial density λ corresponds to the intersection of a ball centred in y_0 and the spatial domain $\mathcal{K}$. Since the spatial density λ is unique up to translation, the shape of the support of any possible spatial equilibrium must be unique. In particular, this shape depends on the distance from y_0 to the boundary $e^\perp$ (see balls B_1 and B_2 in Figure 1), unless y_0 would be so far from that boundary that the ball would not intersect it. In this latter case, the support would be an entire ball (such as ball B_3 in Figure 1). However, this last scenario cannot arise because if the support were an entire ball, then x_0 would correspond to y_0, which is not possible. This means that the support of all possible spatial equilibria intersects the boundary $e^\perp$ and that the distance from y_0 to that boundary is constant (i.e. the same for all spatial equilibria).

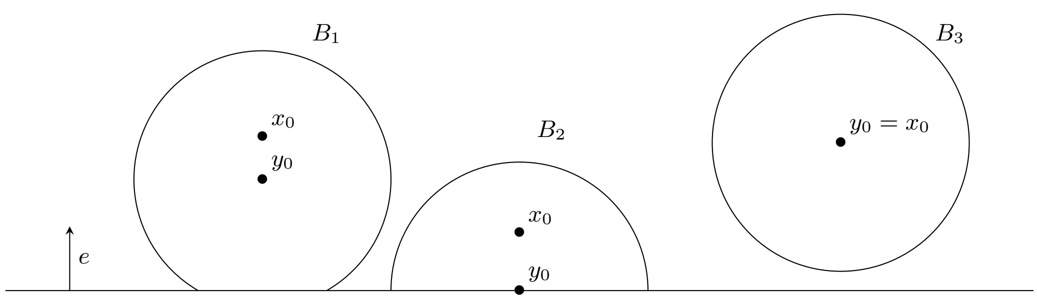

FIGURE 1.— Examples of equilibrium supports for the sea-shore model. Ball B_1 and B_2 are located at different distances from the boundary $e^\perp$. Ball B_3 is not admissible as $y_0 = x_0$.

6. A CIRCULAR ECONOMY: A NON-CONVEX EXAMPLE

In this Section, we revisit the model by Mossay and Picard [2011] along the unit circle $\mathcal{K} = \mathcal{C} = [0, 2\pi]$. In the light of Assumption 4 and Theorem 3, the emergence of multiple spatial equilibria can be explained by a lack of convexity of the spatial domain. As the problem along the circle is not convex, Theorem 3 does not apply. This is the reason why the model exhibits multiple equilibria along the circle while it admits a unique spatial equilibrium along the real line (see Example 5.1).

Studying spatial economies extending along a circle has a long tradition in economics, ranging from the circular Hotelling model in the industrial organization literature to the more recent racetrack economy used in the New Economic Geography literature. Here, we show that the circular model of spatial interactions cannot be interpreted as a simple variant of the corresponding model along the real line. As the spatial equilibria arising along the circle may involve disconnected cities, we find it useful to introduce the following Definition.

Definition 3 (City, city-centre and multiple cities) Let λ be a spatial density of agents. A *city* is defined as a connected component of the support of λ, and a *city-centre* (or centre) of a city as any point x which is a strict local maximum of λ. The spatial economy is said to be a *multiple-city* economy if it consists of several disjoint cities.

Following Mossay and Picard [2011], we consider a linear utility function, $u(s) = \beta s$ where β denotes the preference for land, and a linear accessing cost $W(x)$ equal to τx, for $x \in [0, \pi]$, and to $\tau(2\pi - x)$, for $x \in [\pi, 2\pi]$, where τ is the accessing cost.

Mossay and Picard used a constructive method to solve the model, making conjectures about candidates for equilibrium and, only then, determining which of these candidates do actually satisfy the equilibrium condition (2.3). In contrast to their approach, we propose a direct method which allows to determine all the spatial equilibria of the economy as solutions to a differential equation.

By spatial periodicity, we impose that $\lambda(x + 2\pi) = \lambda(x)$. Also, the point opposite to x along $\mathcal{C}$ is denoted by $\bar{x}$. By Proposition 1, any spatial equilibrium λ satisfies

$$\lambda(x) = \frac{1}{\beta}\left(Y + B - \overline{U} - \int_0^{2\pi} W(x-y)\lambda(y)\,\mathrm{d}y\right)_+ .$$

We make the following change of functions by defining the auxiliary function ϕ

$$\phi(x) = \frac{1}{\tau}\int_0^{2\pi} W(x-y)\lambda(y)\,\mathrm{d}y - \frac{\pi}{2} . \tag{6.1}$$

This allows to rewrite the spatial distribution λ as

$$\lambda(x) = \frac{1}{2}\left(C - \delta^2\phi(x)\right)_+ \tag{6.2}$$

where $\delta^2 = 2\tau/\beta$ and $C = 2[Y + B - \overline{U} - \tau\pi/2]/\beta$.

We now derive an equation for function ϕ.

PROPOSITION 3 (Differential equation for ϕ) *If λ is a spatial equilibrium along the geographical space $\mathcal{C}$, then the function ϕ defined in Expression (6.1) belongs to $C^2(\mathcal{C})$ and satisfies the following ordinary differential equation*

$$\phi'' = (C - \delta^2\phi)_+ - (C + \delta^2\phi)_+ \tag{6.3}$$

with the periodic condition

$$\phi(x) = -\phi(x \pm \pi), \quad \forall x \in [0, \pi) \tag{6.4}$$

PROOF: By using relation (6.1), function ϕ can be rewritten as

$$\phi(x) = \int_{x-\pi}^{x} (x-y)\lambda(y)\,\mathrm{d}y + \int_{x}^{x+\pi} (2\pi - x + y)\lambda(y)\,\mathrm{d}y - \pi\ .$$

By inspection of this expression, ϕ is differentiable. Its derivative is given by

$$\phi'(x) = \int_{x-\pi}^{x} \lambda(y)\,\mathrm{d}y - \int_{x}^{x+\pi} \lambda(y)\,\mathrm{d}y.$$

As ϕ is differentiable and thus continuous, λ is also continuous given Relation (6.2). The fundamental theorem of calculus allows to differentiate ϕ'. This leads to

$$\phi''(x) = \lambda(x) - \lambda(x-\pi) - \lambda(x+\pi) + \lambda(x) = 2[\lambda(x) - \lambda(\bar{x})]\ .$$

This implies that function $\phi \in C^2(\mathcal{C})$. By using Relation (6.2), we get $\phi''(x) = (C - \delta^2\phi(x))_+ - (C - \delta^2\phi(\bar{x}))_+$. We also have

$$\begin{aligned} \phi(x) + \phi(\overline{x}) &= 1/\tau \int W(x-y)\lambda(y)\,\mathrm{d}y - \frac{\pi}{2} + 1/\tau \int W(\overline{x}-y)\lambda(y)\,\mathrm{d}y - \frac{\pi}{2} \\ &= 1/\tau \int \left[W(x-y) + W(\overline{x}-y)\right]\lambda(y)\,\mathrm{d}y - \pi = 0 \end{aligned}$$

given the relation $W(x-y) + W(\overline{x}-y) = \tau\pi$ and the total population constraint $\int_{\mathcal{C}} \lambda(y)\,\mathrm{d}y = 1$. Finally, we get $\phi''(x) = (C - \delta^2\phi(x))_+ - (C + \delta^2\phi(x))_+$. *Q.E.D.*

Our resolution method consists in determining the solutions ϕ to Equation (6.3) with the periodic condition (6.4). Only then, the spatial equilibria λ will be obtained by Relation (6.2). Mossay and Picard identified spatial equilibria involving cities distributed according to a cosine function given by $\cos(\delta x)$. In what follows, these equilibria are referred to as one-frequency (δ) equilibria, as opposed to other solutions derived in this paper involving two frequencies (δ and $\sqrt{2}\delta$). All the details of the resolution are provided in Appendix B. We summarize them in the following Proposition.

Proposition 4 (Spatial equilibria along the circle) *The spatial equilibria arising in the circular economy $\mathcal{C}$ can be described as follows. Of course, the uniform spatial distribution is always an equilibrium. If $\sqrt{2}\delta$ happens to be an odd number, there exists a spatial equilibrium with full support exhibiting $\sqrt{2}\delta$ centres, see the illustration in Figure 2. When $\sqrt{2}\delta$ is not an odd number, for any odd number J such that $J \leq \delta$ (resp. such that $\delta < J \leq \sqrt{2}\delta$), there is a one-frequency (resp. two-frequency) spatial equilibrium with J identical and evenly spaced cities, see the illustration in Figure 3 (resp. Figure 4).*

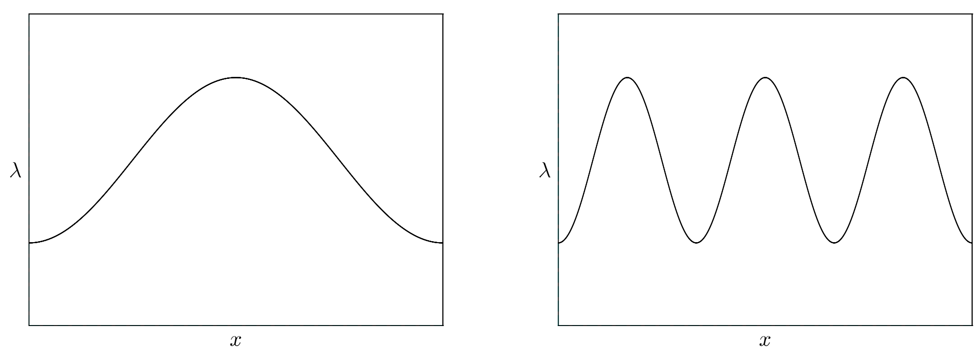

Figure 2.— Spatial equilibria with full support involving an odd number of centres. In the left panel, the spatial economy displays one centre for $\delta = \sqrt{2}/2$. In the right panel, the spatial economy displays three centres for $\delta = 3\sqrt{2}/2$.

Our direct resolution method has allowed us to determine all the spatial equilibria of the circular economy. This completes the analysis initiated by Mossay and Picard and reemphasizes the emergence of multiple equilibria, which has been interpreted here as a lack of convexity arising in the circular model.

7. CONCLUSION

We have studied a spatial model of social interactions for a large class of preferences for land, accessing costs and space-dependent amenities in a one- or two-dimensional geographical space. By showing that spatial equilibria derive from a potential and by providing their variational characterisation, we have proved their existence and uniqueness under mild conditions on the primitives of the economy. Various examples from the existing literature as well as some new ones have been used to illustrate the scope of our results. In particular, the role of strict displacement convexity has been shown to be crucial for the uniqueness of equilibrium.

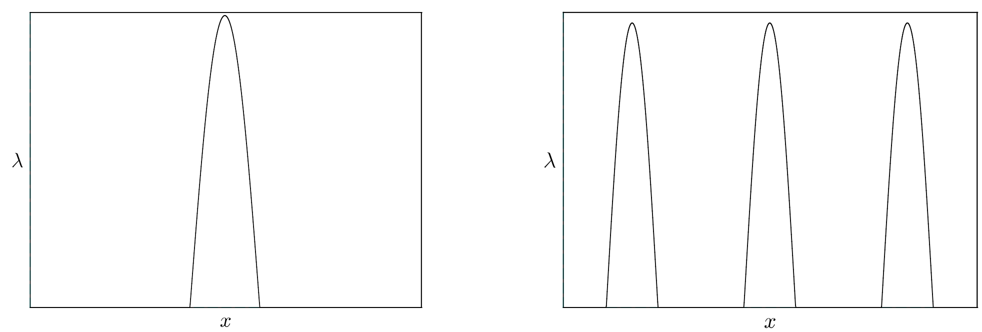

FIGURE 3.— One-frequency spatial equilibria involving an odd number of cities. In the left panel, the spatial economy displays $J = 1$ city for $\delta = 3$. In the right panel, the spatial economy displays $J = 3$ cities for $\delta = 4$.

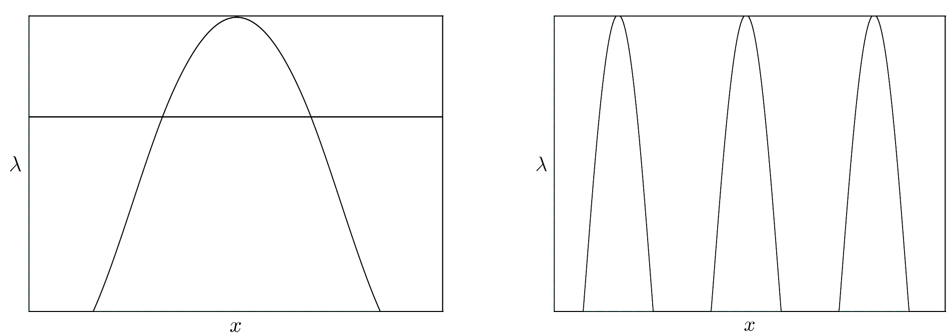

FIGURE 4.— Two-frequency spatial equilibria involving an odd number of cities. In the left panel, for $\delta = 3/4$, the equilibrium displays $J = 1$ city where the frequency is $\sqrt{2}\delta$ for the portion of the curve above the line and δ for the portion of the curve below that line. In the right panel, for $\delta = 2.8$, the equilibrium displays $J = 3$ cities.

Moreover, the emergence of multiple equilibria arising along the circular economy has been explained by a lack of convexity of the problem.

Several extensions are of interest for future research. Here are some suggestions. First, considering heterogeneous populations of agents should allow to study intra- and inter-group social interactions, and therefore to tackle spatial segregation and integration issues. Second, the extension of the model along a sphere seems natural.

However, dealing with spatial symmetries in our economic environment is far from obvious. Third, a further analysis of the multiple equilibria arising along a circle could study whether some dynamics induced by the spatial mobility of agents could be used as a device to select equilibria.

APPENDIX A: VARIATIONAL CHARACTERISATION

A.1. *Proof of Lemma 2*

Let λ minimise $\mathcal{F}$ in $\mathcal{M}(\mathcal{K})$. We consider some admissible spatial density $\widetilde{\lambda} \in \mathcal{M}(\mathcal{K})$ and a family of perturbations $\lambda_\varepsilon = (1-\varepsilon)\lambda + \varepsilon\widetilde{\lambda}$, indexed by $0 \leq \varepsilon \leq 1$.

Given that λ minimises $\mathcal{F}$, we have

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}\left[\lambda_\varepsilon\right]_{|\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{V}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0} \tag{A.1}$$

As $V' = v$, the first derivative in Relation (A.1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{V}[\lambda_\varepsilon]_{|\varepsilon=0} = \int V'(\lambda(x))\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda_\varepsilon(x)\,\mathrm{d}x_{|\varepsilon=0} = \int v(\lambda(x))[\widetilde{\lambda}(x) - \lambda(x)]\,\mathrm{d}x\;.$$

The second derivative in Relation (A.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}[\lambda_\varepsilon]_{|\varepsilon=0} = -\int A(x)(\widetilde{\lambda}(x) - \lambda(x))\,\mathrm{d}x.$$

Under Assumption 2, the accessing cost W is even, so that the third derivative in Relation (A.1) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0} &= \frac{1}{2}\iint W(x-y)\Big(\lambda(x)[\widetilde{\lambda}(y) - \lambda(y)] + [\widetilde{\lambda}(x) - \lambda(x)]\lambda(y)\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint W(x-y)\lambda(y)[\widetilde{\lambda}(x) - \lambda(x)]\,\mathrm{d}x\,\mathrm{d}y \\ &= \int W{*}\lambda(x)(\widetilde{\lambda}(x) - \lambda(x))\,\mathrm{d}x\;.\end{aligned}$$

where $(W{*}\lambda)(x)$ denotes $\int_{\mathcal{K}} W(x-y)\lambda(y)\,\mathrm{d}y$.

By plugging the expressions of these three derivatives into Relation (A.1), we obtain

$$\int [A(x) - v(\lambda(x)) - W{*}\lambda(x)]\,\widetilde{\lambda}(x)\,\mathrm{d}x \leq \int [A(x) - v(\lambda(x)) - W{*}\lambda(x)]\,\lambda(x)\,\mathrm{d}x\;.$$

As this inequality holds for any admissible density $\widetilde{\lambda}$, the spatial density λ is concentrated on the set where the function $U(x)$ realises its maximum value $\bar{U}$. Hence, λ is a spatial equilibrium of the economy $\mathcal{E}$.

A.2. *Sufficiency Proof of Theorem 2*

Under Assumption 4, the functional $\mathcal{F}$ is displacement convex. Let λ be a spatial equilibrium of the economy $\mathcal{E}$, $\widetilde{\lambda}$ some admissible spatial density, and T the optimal transport map from λ onto $\widetilde{\lambda}$. At this stage, we assume that T is C^1. However, the changes to be made if $T \notin C^1$

will be discussed later on. We define the maps $T_\varepsilon := (1-\varepsilon)\mathrm{id} + \varepsilon T$ and consider the family of perturbations $\lambda_\varepsilon = T_\varepsilon \# \lambda$, indexed by $0 \le \varepsilon \le 1$.[4]

As the curves $\{\varepsilon \mapsto \lambda_\varepsilon\}_{\varepsilon\in[0,1]}$ are geodesics in $\mathcal{M}(\mathcal{K})$ and $\mathcal{F}$ is displacement convex (i.e. geodesically convex), the function $\varepsilon \mapsto \mathcal{F}[\lambda_\varepsilon]$ is convex. In what follows, we show that the derivative of that function in $\varepsilon = 0$ is positive. This will prove that $\mathcal{F}[\tilde\lambda] \ge \mathcal{F}[\lambda]$.

First we derive the equation for the perturbation λ_ε. By Expression (4.1), as the map T_ε transports λ onto λ_ε, we have

$$\int_{\mathcal{K}} \zeta(y)\lambda_\varepsilon(y)\,\mathrm{d}y = \int_{\mathcal{K}} \zeta[T_\varepsilon(x)]\lambda(x)\,\mathrm{d}x \quad \forall \zeta : \mathcal{K} \to \mathcal{K}\ .$$

By performing the change of variable $y = T_\varepsilon(x)$ in the left-hand side term, we obtain

$$\int_{\mathcal{K}} \zeta(T_\varepsilon(x))\lambda_\varepsilon(T_\varepsilon(x))|\mathrm{J}_{T_\varepsilon}(x)|\,\mathrm{d}x = \int_{\mathcal{K}} \zeta[T_\varepsilon(x)]\lambda(x)\,\mathrm{d}x \quad \forall \zeta : \mathcal{K} \to \mathcal{K}\ ,$$

where $\mathrm{J}_{T_\varepsilon}$ denotes the Jacobian determinant of the map T_ε,

$$|\mathrm{J}_{T_\varepsilon}| = \det\left((1-\varepsilon)I + \varepsilon DT\right) = \det\left(I + \varepsilon(DT - I)\right) \tag{A.2}$$

and DT denotes the Jacobian matrix of map T. By equating the expressions of the two integrands appearing in the above relation, we obtain the following equation[5]

$$\lambda_\varepsilon(T_\varepsilon(x)) = \frac{\lambda(x)}{|\mathrm{J}_{T_\varepsilon}(x)|} \quad \text{or equivalently} \quad \lambda_\varepsilon(y) = \frac{\lambda(T_\varepsilon^{-1}(y))}{|\mathrm{J}_{T_\varepsilon}(T_\varepsilon^{-1}(y))|}\ . \tag{A.3}$$

Let us now evaluate the derivative of $\mathcal{F}$ in $\varepsilon = 0$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}\left[\lambda_\varepsilon\right]_{|\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{V}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0}\ . \tag{A.4}$$

By Equation (A.3), the first derivative in Relation (A.4) can be rewritten as

$$\int_{\mathcal{K}} V(\lambda_\varepsilon(x))\,\mathrm{d}x = \int_{\mathcal{K}} V\left(\frac{\lambda(T_\varepsilon^{-1}(x))}{|\mathrm{J}_{T_\varepsilon}(T_\varepsilon^{-1}(x)|}\right)\,\mathrm{d}x\ .$$

By performing the change of variable $y = T_\varepsilon^{-1}(x)$, we obtain

$$\int_{\mathcal{K}} V\left(\frac{\lambda(T_\varepsilon^{-1}(x))}{|\mathrm{J}_{T_\varepsilon}(T_\varepsilon^{-1}(x)|}\right)\,\mathrm{d}x = \int_{\mathcal{K}} V\left(\frac{\lambda(y)}{|\mathrm{J}_{T_\varepsilon}(y)|}\right)|\mathrm{J}_{T_\varepsilon}(y)|\,\mathrm{d}y\ .$$

So as to differentiate this expression, we need to compute the derivative of the Jacobian term $\mathrm{J}_{T_\varepsilon}$. As $\det(I+H) = 1 + \mathrm{tr}(H) + o(\|H\|)$, using the Jacobian determinant (A.2) leads to

$$|\mathrm{J}_{T_\varepsilon}| = 1 + \varepsilon\,\mathrm{tr}(DT - I) + o(\varepsilon) = 1 + \varepsilon((\mathrm{div}\,T) - d) + o(\varepsilon)\ ,$$

where $\mathrm{div}\,T$ denotes the divergence of T, that is the trace of the Jacobian determinant $\mathrm{J}_{T_\varepsilon}$. As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|\mathrm{J}_{T_\varepsilon}|_{|\varepsilon=0} = (\mathrm{div}\,T) - d \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{1}{|\mathrm{J}_{T_\varepsilon}|}_{|\varepsilon=0} = -\frac{1}{|\mathrm{J}_{T_\varepsilon}|^2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|\mathrm{J}_{T_\varepsilon}|_{|\varepsilon=0} = -(\mathrm{div}\,T) + d.$$

[4] Note that we rely on a family of perturbations which is distinct from that of an additive type $(1-\varepsilon)\lambda + \varepsilon\tilde\lambda$ used in the necessity part (Lemma 2), see Appendix A.1. These two different types of perturbations are equally used in the theory of optimal transportation, see for instance Santambrogio [2012].

[5] In the mathematics literature, the condition relating the density of the transported density to the Jacobian determinant of the transport map is referred to as the Monge-Ampère equation.

Hence, by integration by parts, the first derivative in Relation (A.4) can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathcal{K}} V(\lambda_\varepsilon(x))\,\mathrm{d}x_{|\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathcal{K}} V\left(\frac{\lambda(y)}{|\mathrm{J}_{T_\varepsilon}(y)|}\right)|\mathrm{J}_{T_\varepsilon}(y)|\,\mathrm{d}y_{|\varepsilon=0}\\
&= -\int_{\mathcal{K}} \lambda(y)((\operatorname{div} T)(y)-d)V'(\lambda(y))\,\mathrm{d}y\\
&\quad + \int_{\mathcal{K}} V(\lambda(y))((\operatorname{div} T)(y)-d)\,\mathrm{d}y\\
&= \int_{\mathcal{K}} \left[V(\lambda(y))-\lambda V'(\lambda(y))\right]((\operatorname{div} T)(y)-d)\,\mathrm{d}y\\
&= -\int_{\mathcal{K}} \nabla\left[V(\lambda(y))-\lambda V'(\lambda(y))\right]\cdot(T(y)-y)\;\mathrm{d}y\\
&\quad + \int_{\partial\mathcal{K}} \left[V(\lambda(y))-\lambda(y)V'(\lambda(y))\right](T(y)-y)\cdot n\,\mathrm{d}\sigma\;,
\end{aligned}$$

where n is the normal outward vector. By convexity of $\mathcal{K}$, $(T(y)-y)\cdot n\le 0$. Also, by convexity of V and $V(0)=0$, $V(\lambda(x))-\lambda V'(\lambda(x))$ is negative. Hence the boundary integral is positive. Moreover, $\nabla\left[V(\lambda)-\lambda V'(\lambda)\right]=V'(\lambda)\nabla\lambda-V'(\lambda)\nabla\lambda-V''(\lambda)\lambda\nabla\lambda=-\lambda\nabla(V'(\lambda))=\lambda\nabla(v(\lambda))$. This allows to write

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathcal{K}} V(\lambda_\varepsilon(x))\,\mathrm{d}x_{|\varepsilon=0}\ge\int_{\mathcal{K}}\lambda(y)\nabla[v(\lambda(y))]\cdot(T(y)-y)\,\mathrm{d}y$$

By the push-forward Definition (4.1), the second derivative in Relation (A.4) can be written as

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathcal{K}} A(x)\lambda_\varepsilon(x)\,\mathrm{d}x_{|\varepsilon=0} &= -\int_{\mathcal{K}} A(T_\varepsilon(x))\lambda(x)\,\mathrm{d}x_{|\varepsilon=0}\\
&= -\int_{\mathcal{K}} \nabla A(x)\cdot(T(x)-x)\lambda(x)\,\mathrm{d}x\;.
\end{aligned}$$

Similarly, the last derivative in Relation (A.4) is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{1}{2}\iint_{\mathcal{K}^2} W(T_\varepsilon(x)-T_\varepsilon(y))\lambda(x)\lambda(y))\,\mathrm{d}x\,\mathrm{d}y_{|\varepsilon=0}\\
&= \frac{1}{2}\iint_{\mathcal{K}^2}\nabla W(x-y)\cdot\left[(T(x)-x)-(T(y)-y)\right]\lambda(x)\lambda(y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{\mathcal{K}^2}\nabla W(x-y)\cdot(T(x)-x)\,\lambda(x)\lambda(y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathcal{K}}\nabla W{*}\lambda(x)\cdot(T(x)-x)\,\lambda(x)\,\mathrm{d}x\;.
\end{aligned}$$

Thus, by summing up the expressions of the three derivatives in Relation (A.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}\left[\lambda_\varepsilon\right]_{|\varepsilon=0}\ge-\int_{\mathcal{K}}\nabla\left[v(\lambda(x))-A(x)+W{*}\lambda(x)\right]\cdot\lambda(x)(T(x)-x)\,\mathrm{d}x=0$$

Because $v(\lambda)-A+W*\lambda$ is constant on the set where $\lambda(x)>0$ for almost every $x\in\mathcal{K}$, this last integral vanishes and the derivative of $\mathcal{F}$ at $\varepsilon=0$ is positive. This means that the spatial equilibrium λ is a minimiser of $\mathcal{F}$.

We now comment on the case where the optimal transport map T is not C^1. This may often arise depending on the spatial density λ. The main problem is the distinction between the divergence

$\operatorname{div}(T - \mathrm{id})$, which appears when computing the first derivative in Relation (A.4) and which is computed pointwise, and the divergence that we need to perform the integration by parts, which is the divergence in the distributional sense. For non-regular maps, these two notions may differ. However, the formal computations to be made in the case $T \notin C^1$ can be rigorously justified in the framework of non-smooth analysis, see [Villani, 2003, Theorem 5.30]. As T is the gradient of a convex function φ, we have $(\operatorname{div} T) = \Delta_A \varphi$ almost everywhere, where $\Delta_A \varphi$ denotes the Alexandroff Laplacian of φ, which is also the absolutely continuous part of the distributional Laplacian $\Delta\varphi$. By convexity, $\Delta\varphi$ is a positive measure and $\Delta_A\varphi \le \Delta\varphi$. This shows that the pointwise divergence $(\operatorname{div} T)$ is smaller than the distributional divergence $\operatorname{div}_{\mathrm{dist}} T$. This implies that the first derivative in Relation (A.4) is smaller than $\int_{\mathcal{K}} [V(\lambda) - \lambda V'(\lambda)]\, \nabla_{\mathrm{dist}} \cdot (T - \mathrm{id})\, \mathrm{d}x$. This leads to the same result as that obtained when assuming $T \in C^1$.

APPENDIX B: SPATIAL EQUILIBRIA ARISING IN THE CIRCULAR SPATIAL ECONOMY

In this Appendix, the explicit solutions ϕ to the differential equation (6.3) with the periodic condition (6.4) are determined. Then, the spatial equilibria λ are obtained by using Relation (6.2). For notational convenience, we will denote the maximum value of ϕ along $\mathcal{C}$ by Φ. Without loss of generality, we will assume that this maximum value Φ is attained in $x = 0$. It is convenient to rewrite the problem (6.3)- (6.4) as $\phi'' = f(\phi)$ with $\phi(0) = \Phi$ and $\phi'(0) = 0$, where the function f is defined by

$$f(t) = (C - \delta^2 t)_+ - (C + \delta^2 t)_+. \tag{B.1}$$

We distinguish three families of solutions: one-frequency equilibria ($C \le 0$), two-frequency equilibria ($C > 0$ and $\Phi > C/\delta^2$), and equilibria with full support ($C > 0$ and $\Phi \le C/\delta^2$). Note that unlike parameters β and τ, the values of C and Φ have to be determined in equilibrium.

B.1. *Case 1: $C \le 0$ (One-frequency spatial equilibria)*

When $C \le 0$, the function f defined in Expression (B.1) can be rewritten as

$$f(t) = \begin{cases} C - \delta^2 t & \text{if } t < \dfrac{C}{\delta^2}, \\ 0 & \text{if } \dfrac{C}{\delta^2} \le t \le -\dfrac{C}{\delta^2}, \\ -C - \delta^2 t & \text{if } t > -\dfrac{C}{\delta^2}. \end{cases}$$

The graph of f is illustrated in Figure 5.

First of all, the case $\Phi \le -C/\delta^2$ can be discarded for the following reason. As the function f vanishes in $[C/\delta^2, -C/\delta^2]$, the solution to Equation (6.3) is linear. Hence, no linear periodic function ϕ with $\phi(x) + \phi(\bar{x}) = 0$ can be expected, except $\phi = 0$. However, in this latter case, $\lambda = C_+/2 = 0$ since $C < 0$, which is not an equilibrium as the total population constraint cannot be satisfied.

We now consider the case $\Phi > |C|/\delta^2$. In the neighbourhood of $x = 0$, we have to solve the Cauchy problem associated to the following second order linear differential equation $\phi'' = -C - \delta^2\phi$ with $\phi(0) = \Phi$ and $\phi'(0) = 0$. This equation has the following unique solution

$$\phi_1 : x \mapsto \left(\Phi + \frac{C}{\delta^2}\right)\cos(\delta x) - \frac{C}{\delta^2}\ .$$

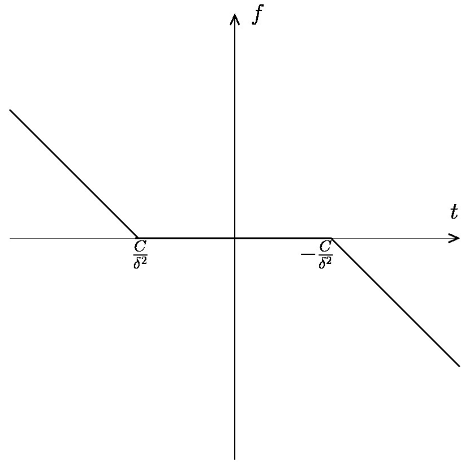

FIGURE 5.— Graph of f in the case $C \leq 0$.

This expression is valid as long as $\phi_1(x) > -C/\delta^2$. Let a be the first value of x for which $\phi_1(x) = -C/\delta^2$, so that $\phi_1(x) > -C/\delta^2$ in the interval $(-a, a)$. Note that this interval is symmetric as ϕ_1 is even. It follows that $a = \pi/(2\delta)$. In the neighbourhood at the right of $x = a$, we have to solve the equation $\phi'' = 0$ with $\phi(a) = -C/\delta^2$. By Proposition 3, ϕ is regular so that $\phi'(a) = \phi_1'(a) = -\delta(\Phi + C/\delta^2)$. The solution to the differential equation at the right of $x = a$ is given by

$$\phi_2 : x \mapsto -\delta\left(\Phi + \frac{C}{\delta^2}\right)x + \frac{\pi}{2}\left(\Phi + \frac{C}{\delta^2}\right) - \frac{C}{\delta^2} .$$

This expression is valid in $(a, a + 2b)$ where $a + b$ denotes the first zero of ϕ_2, *i.e.*

$$a + b = \frac{|C|}{\delta(\delta^2\Phi + C)} + \frac{\pi}{2\delta} . \tag{B.2}$$

The construction of solution ϕ can be extended to obtain a solution of period $T = (4a + 4b)$

$$\phi(x) = \begin{cases} \phi_1(x) = \left(\Phi + \frac{C}{\delta^2}\right)\cos(\delta x) - \frac{C}{\delta^2} & \text{if } -a \leq x \leq a, \\ \phi_2(x) = -\delta\left(\Phi + \frac{C}{\delta^2}\right)x + \frac{\pi}{2}\left(\Phi + \frac{C}{\delta^2}\right) - \frac{C}{\delta^2} & \text{if } a \leq x \leq a + 2b, \\ -\phi_1(x - 2a - 2b) & \text{if } a + 2b \leq x \leq 3a + 2b, \\ -\phi_2(x - 2a - 2b) & \text{if } 3a + 2b \leq x \leq 3a + 4b. \end{cases} \tag{B.3}$$

The period T of function ϕ has to satisfy the periodic condition (6.4), which can be written as $(2j + 1)(4a + 4b) = 2\pi$, for $j \in \mathbb{N}$. We still need to determine the admissible values of period T

$$T = 4(a + b) = \frac{2\pi}{\delta} + \frac{4|C|}{\delta(\delta^2\Phi + C)}.$$

By inspection of the above expression, when $C \neq 0$, the period T is a monotone function of the ratio Φ/C. As $\Phi/|C| > 1/\delta^2$, the admissible values of period T are the interval $(2\pi/\delta, +\infty)$. Hence, for a given value of δ and for any $j \in \mathbb{N}$ such that $2\pi/(2j+1) > 2\pi/\delta \quad \Leftrightarrow \quad 2j+1 < \delta$, a unique value of Φ/C can be determined so that the above solution ϕ is of period $T = 2\pi/(2j+1)$. The values of C and Φ are determined by imposing the total population constraint

$$1 = (2j+1) \int_{a+2b}^{3a+2b} \left(\delta^2 \Phi + C\right) \cos(\delta(x - 2a - 2b))\, \mathrm{d}x$$
$$= (2j+1)\left(\delta^2\Phi + C\right) \int_{-a}^{a} \cos(\delta x)\, \mathrm{d}x = 2(2j+1)\left(\delta\Phi + \frac{C}{\delta}\right) .$$

which leads to $\delta^2\Phi + C = \delta/(2(2j+1))$. The positive part of the equilibrium spatial distribution λ in the interval $(-a, 3a+4b)$ is obtained by using Relation (6.2)

$$\lambda(x) = \frac{\delta}{2(2j+1)} \cos(\delta(x - 2a - 2b)) \quad \text{ if } a + 2b \le x \le 3a + 2b$$

Note that by using Expression (B.2), the values of Φ and $\bar{U}$ can also be obtained. The solution ϕ is illustrated in Figure 6. The corresponding spatial equilibrium λ is represented in Figure 3.

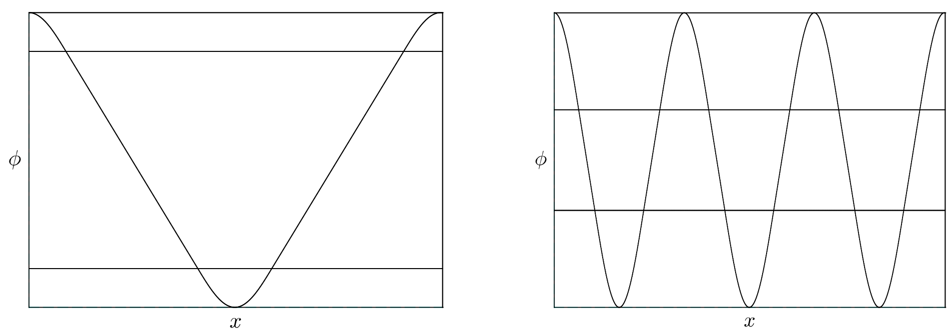

FIGURE 6.— Solution ϕ for $C = -1$ and $\delta = 3$ in the left panel and for $C = -1$ and $\delta = 4$ in the right panel. The horizontal lines correspond to the values $\pm C/\delta^2$.

B.2. *Case 2:* $C > 0$

When $C > 0$, the function f defined in Expression (B.1) can be rewritten as

$$f(t) = \begin{cases} C - \delta^2 t & \text{if } t < -\dfrac{C}{\delta^2}, \\ -2\delta^2 t & \text{if } -\dfrac{C}{\delta^2} \le t \le \dfrac{C}{\delta^2}, \\ -C - \delta^2 t & \text{if } t > \dfrac{C}{\delta^2}. \end{cases}$$

The graph of f is illustrated in Figure 7.

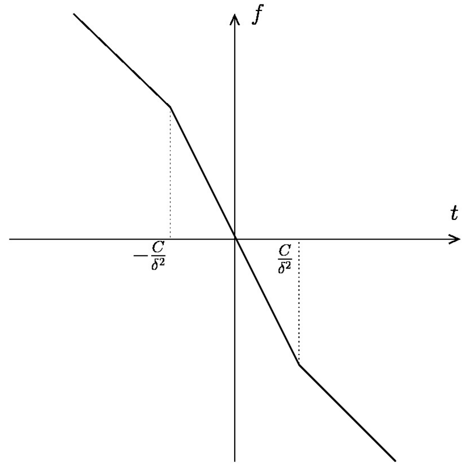

FIGURE 7.— Graph of f in the case $C > 0$.

B.2.1. *Case 2.1:* $\Phi \le C/\delta^2$ *(Spatial equilibria with full support)*

In this case, we have to solve the following differential equation $\phi''(x) = -2\delta^2\phi$ with $\phi(0) = \Phi$ and $\phi'(0) = 0$. The unique solution to this equation is given by $\phi(x) = \Phi\cos(\sqrt{2}\delta x)$. The periodic condition $\phi(x) = -\phi(\bar{x})$ leads to two cases, either $\Phi = 0$ or $cos(\sqrt{2}\delta x) = -\cos(\sqrt{2}\delta(x+\pi))$. When $\Phi = 0$, $\phi = 0$, and $\lambda = C/2$. By using the total population constraint, we get the uniform spatial equilibrium $\lambda = 1/(2\pi)$. The other case corresponds to $\sqrt{2}\delta$ being an odd number J, i.e. $\sqrt{2}\delta = J = (2j+1)$ for some $j \in \mathbb{N}$. By making use of Relation (6.2) and of the total population constraint, the spatial distribution λ is then given by $\lambda(x) = 1/(2\pi) - m\cos(\sqrt{2}\delta x))$, $\forall m \in [-1/(2\pi), 1/(2\pi)]$. Examples of such equilibria are drawn in Figure 2.

B.2.2. *Case 2.2:* $\Phi > C/\delta^2$ *(Two-frequency spatial equilibria)*

In the neighbourhood of $x = 0$, we have to solve the following second order linear differential equation $\phi'' = -C - \delta^2\phi$ with $\phi(0) = \Phi$ and $\phi'(0) = 0$. The unique solution to the equation is given by $\phi_1(x) = (\Phi + C/\delta^2)\cos(\delta x) - C/\delta^2$. This expression is valid for any $x \in (-a, a)$ where a is the first value of x for which $\phi_1(a) = C/\delta^2$, *i.e.* $a = (1/\delta)\arccos\left[2C/(\delta^2\Phi + C)\right]$. In $x = a$, the function ϕ_1 satisfies

$$\phi_1(a) = \frac{C}{\delta^2} \quad \text{and} \quad \phi_1'(a) = -\delta\left(\Phi + \frac{C}{\delta^2}\right)\sin(\delta a) = -\frac{1}{\delta}\sqrt{(\delta^2\Phi + C)^2 - 4C^2}.$$

Since the solution ϕ is C^1, at the right of $x = a$, we have to solve the following second order linear differential equation $\phi'' = -2\delta^2\phi$ with $\phi(a) = \phi_1(a)$ and $\phi'(a) = \phi_1'(a) < 0$. There is a unique solution ϕ_2 to this equation in the interval $(a, a+b)$ where $a+b$ is the first root of ϕ_2. The solution ϕ_2 is given by

$$\phi_2(x) = \frac{C}{\delta^2}\cos(\sqrt{2}\delta(x-a)) - \frac{1}{\delta^2\sqrt{2}}\sqrt{(\delta^2\Phi + C)^2 - 4C^2}\sin(\sqrt{2}\delta(x-a)).$$

This expression of ϕ_2 remains valid in the interval $(a+b, a+2b)$. So, we have constructed a solution ϕ of period $T = (4a+4b)$ with $(2j+1)(2a+2b) = \pi$

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } -a \le x \le a, \\ \phi_2(x) & \text{if } a \le x \le a+2b, \\ -\phi_1(x-2a-2b) & \text{if } a+2b \le x \le 3a+2b, \\ -\phi_2(x-2a-2b) & \text{if } 3a+2b \le x \le 3a+4b. \end{cases}$$

We now need to determine the value of b by imposing that $\phi_2(a+b)=0$, *i.e.*

$$\frac{C}{\delta^2}\cos(\sqrt{2}\delta b) - \frac{1}{\delta^2\sqrt{2}}\sqrt{(\delta^2\Phi+C)^2-4C^2}\,\sin(\sqrt{2}\delta b) = 0,$$

which leads to

$$b = \frac{1}{\delta\sqrt{2}}\arctan\left(\frac{C\sqrt{2}}{\sqrt{(\delta^2\Phi+C)^2-4C^2}}\right).$$

So, the period T of solution ϕ can be written as

$$T = 4(a+b) = 4\frac{1}{\delta}\arccos\left(\frac{2\alpha}{\delta^2\Phi+C}\right) + 4\frac{1}{\delta\sqrt{2}}\arctan\left(\frac{C\sqrt{2}}{\sqrt{(\delta^2\Phi+C)^2-4C^2}}\right).$$

We still need to determine C and Φ in equilibrium. Let us define $r=(\delta^2\Phi+C)/C = 1+\delta^2\Phi/C$. For $\Phi \ge C/\delta^2$, the value of r ranges from 2 to $+\infty$. We now study the monotonicity of the following function

$$r \mapsto \arccos\left(\frac{2}{r}\right) + \frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}}{\sqrt{r^2-4}}\right), \quad r \in [2,+\infty[.$$

By computing the derivative of the above function, it can be readily checked that the above function is strictly decreasing. The image of this function on $[2,+\infty)$ is given by $[\pi/(2\sqrt{2}),\pi/2[$. This means that, for a given value of δ, any period $T \in [\sqrt{2}\pi/\delta, 2\pi/\delta[$ may be obtained for a unique value of the ratio $\Phi/C \ge 1/\delta^2$. In particular, given the value of δ and any $j\in\mathbb{N}$ satisfying

$$\frac{2\pi}{2j+1} \in \left[\frac{\sqrt{2}\pi}{\delta}, \frac{2\pi}{\delta}\right[\quad \Leftrightarrow \quad \delta < 2j+1 \le \sqrt{2}\delta,$$

we can determine a unique value of Φ/C such that the solution ϕ that we have constructed above is of period $T=2\pi/(2j+1)$. Note that the limit case $2j+1=\sqrt{2}\delta$ actually corresponds to the case (B.2.1) when $\Phi = C/\delta^2$.

There is still one degree of freedom left as only the ratio Φ/C has been determined. The values of C and Φ can be determined by using the total population constraint. As the spatial equilibrium λ is obtained from solution ϕ by Relation (6.2), if both C and Φ are multiplied by some constant K, so will be the spatial density λ. This allows to tune the values of C and Φ so as to get the total population of agents equal to 1. The solution ϕ is illustrated in Figure 8. The corresponding spatial equilibrium λ is represented in Figure 4.

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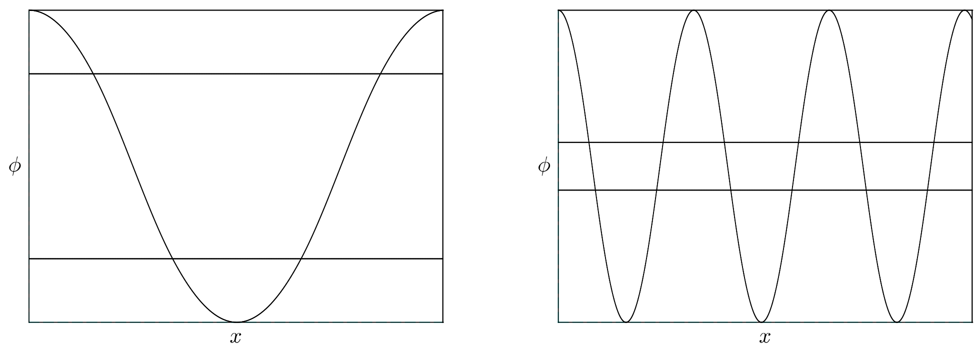

FIGURE 8.— Solution ϕ for $C = 1$, $\delta = 3/4$ and $\Phi = 3$ in the left panel, and for $C = 1$, $\delta = 2.8$ and $\Phi = 1$ in the right panel. The horizontal lines represent the values $\pm C/\delta^2$ so that the frequency changes when crossing these lines.

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