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Scaling Laws in Labor Productivity

FUJIWARA Yoshi
University of Hyogo

AOYAMA Hideaki
RIETI

Mauro GALLEGATI
Polytechnic University of Marche, Italy

Research Institute of Economy, Trade & Industry, IAA
The Research Institute of Economy, Trade and Industry
http://www.rieti.go.jp/en/

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FUJIWARA Yoshi
University of Hyogo

AOYAMA Hideaki
RIETI and Kyoto University

Mauro GALLEGATI
Polytechnic University of Marche, Italy

Abstract

Empirical study of firms' growth and fluctuation requires the understanding of the dynamics of labor and productivity among firms by using large-scale data including that of small and medium-sized enterprises (SMEs). Specifically, the key to such understanding is the findings in statistical properties in equilibrium distributions of output, labor, and productivity.

We uncover a set of scaling laws of conditional probability distributions, which are sufficient for characterizing joint distributions by employing an updated database covering one million firms including domestic SMEs. These scaling laws show the existence of lognormal joint distributions for sales and labor, and the existence of a scaling law for labor productivity, both of which are confirmed empirically. This framework offers characterization of equilibrium distributions with a small number of scaling indices, which determine macroscopic quantities, thus opening a new perspective of bridging microeconomics and macroeconomics.

Keywords: Labor productivity; Dispersion; Scaling laws
JEL classification: C02, E20, O40

I. Introduction

Labor and productivity play central roles at the core of economic growth in any developed country. Policy making necessarily involves utilization of bounded labor forces and maintaining productivity at a certain level and even higher. Empirical analysis of labor allocation among different productivity levels is required to understand how workers are allocated and reallocated to industrial sectors with varying levels of productivity.

The dynamical processes of allocation of labor across firms of differing productivity are essentially of stochastic nature. The large body of literature in labor economics is consistent with this viewpoint. Notably, Mortensen (2003) concluded as saying that the problem of wage dispersion, namely why are similar workers paid differently, reflects differences in employer productivity. The factors of exogenous labor turnover, job destruction and search friction prevent the labor market from converging into a state of equilibrium in which all workers are employed by the firms of highest productivity. Instead of such a static equilibrium, a steady-state allocation of labor across firms of differing productivity is generated stochastically by a process of reallocation of workers from less to more productive firms, including transitions to unemployment by job destruction and other reasons for labor turnover. See also the literature on the analysis of job flow, e.g. Davis et al. (1996); Albaek and Sorensen (1998).

In addition, because the productivity dispersion relates to the notion of equilibrium in economics, it concerns the foundation of economic theory as pointed out lucidly in the works (Yoshikawa, 2000a,b; Aoki and Yoshikawa, 2007). Although there exists a mainstream approach (Kydland and Prescott, 1982; Galí, 2008), its internal coherence and ability in explaining empirical evidences are increasingly questioned (Yoshikawa, 2000a,b; Delli Gatti et al., 2008; Aoki and Yoshikawa, 2007; Aoyama et al., 2010b). Indeed, in many economic models, an equilibrium is described as a state in which labor productivity is merely an equilibrium point, not a distribution. Empirical studies have shown that productivity has dispersion among firms and sectors. See, for example, Shinohara (1955); Salter (1960); Yoshikawa (2000a). The notion of statistical equilibrium, in which the aggregate equilibrium is compatible with stochastic behavior of the constituents, is outside the box of tools of mainstream economists. Statistical physics teaches us that the equilibrium of a system does not require that every single element be in equilibrium by itself, but rather that the statistical distributions describing macroscopic aggregate phenomena be stable. See also Foley (1994); Blume and Durlauf, eds (2006).

In this paper we will show that a system populated by many heterogeneous interacting agents generates equilibrium distribution in the form of scaling laws. In particular, we have shown with our collaborators the existence of large and persistent differences in labour productivity across industries and countries Aoyama et al. (2010a); Ikeda and Souma (2009); Aoyama et al. (2009). Productivity is often measured in terms of the ratio between firms' revenues and the number of employees. It can be expected to be a unique value only within a very straightjacket hypothesis, such as the Representative Agent. If agents are heterogeneous and interacting, then scaling laws emerge, and dispersion is nothing but a consequence of it. Using Japanese data we empirically demonstrate it. To be specific, the existence of a scaling law is demonstrated empirically for the output and labor of a million Japanese firms. Theoretical study of the scaling laws suggests lognormal joint distributions, and a scaling law for labor productivity, both of which are confirmed empirically.

In Section II, we employ the Credit Risk Database (CRD) as the largest database, which covers small and medium firms in Japan, in order to analyze the distributions of labor input L, total output Y, their joint and conditional distributions. We find a significant property in the distributions, what we call *double scaling law* in Section III. Specifically, in Section III.A, we formulate scaling relations in terms of conditional distributions for L and Y. Then, in Section III.B we show mathematical consequences of the scaling relations for joint and marginal distributions of L and Y, and also their consequence for labor productivity in Section III.C. We check the validity of these results in the CRD data. Section IV gives the summary.

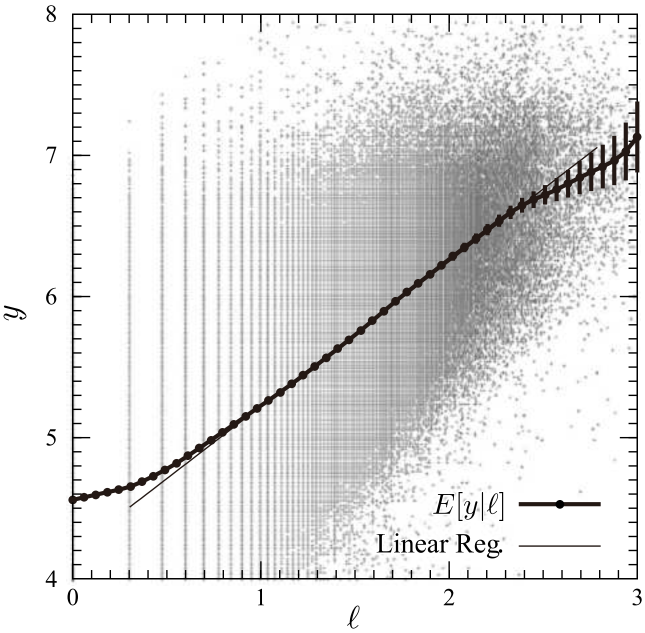

Figure 1: Scatter plot for (L, Y) (gray dots). The curve with error bars is the nonparametric estimation for $E(y|\ell) := E(\ln Y|L)$, while the thin straight line is the linear regression (both axes are in units of $\ln 10$.)

II. Data and Analysis

For the purpose to study properties in probability distributions, it is essential to observe a large portion of the entire population of firms and workers. A database of only listed firms, for example, is insufficient to analyze properties of distributions. We employ the largest database of Credit Risk Database (CRD) in Japan (years 1995 to 2009), which includes a million firms and fifteen million workers in the year 2006, covering the large portion of the whole domestic population. Below we give our analysis for the year 2006, but note that the qualitative results are valid for other years as well.

We measure the value added Y and the labor L of each firm to have the information of output and input in the production at the individual level. We use simply the business sales/profits as a proximity to the value added, and the end-of-year number of workers (excluding managers) as the labor in order to calculate distributions for Y and L and to uncover their properties.

To understand how workers are distributed among different levels of output and productivity, we shall study the distributions of Y and L using the following probability density functions (PDFs). The joint PDF, P_{YL}, the conditional PDFs, $P_{Y|L}$ and $P_{L|Y}$, and the marginal PDFs, P_L and P_Y, are defined by

$$P_{YL}(Y, L) = P_{Y|L}(Y|L)P_L(L) = P_{L|Y}(L|Y)P_Y(Y). \tag{1}$$

The conditional average of $f(Y)$ is defined by

$$E(f(Y)|L) := \int_0^\infty f(Y)P_{Y|L}(Y|L)dY, \tag{2}$$

for an arbitrary function $f(\cdot)$, and similarly for $E(f(L)|Y)$.

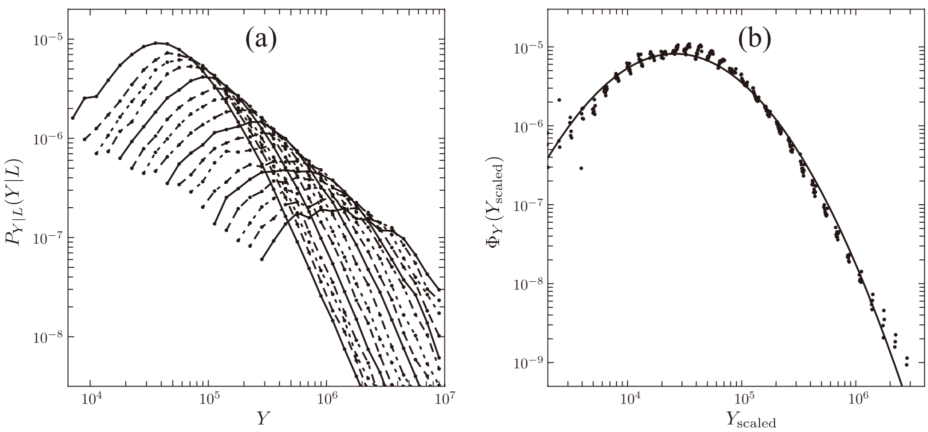

Figure 2: (a) The conditional PDF $P_{Y|L}(Y|L)$ for $L \in [5, 200]$, where the conditioning values of L are chosen at a logarithmically equal interval. (b) The scaled conditional PDF $\Phi_Y(Y_{\text{scaled}})$ defined by Eq. (6). Dots are the scaled data points and the curve is the lognormal function given by Eq. (16).

As we shall see, since the PDFs are heavy-tailed for Y and L, it is convenient for the purpose of statistical analysis to take the logarithms of variables:

$$y := \ln \frac{Y}{Y_0}, \quad \ell := \ln \frac{L}{L_0}, \tag{3}$$

where Y_0 and L_0 are arbitrary scales.

Fig. 1 shows the scatter plot for (ℓ, y). To reveal statistical structure in data, which can be easily missed by parametric methods, we employ a kernel-based nonparametric methods Li and Racine (2007). Fig. 1 depicts a nonparametric regression curve with error bars (95% significance level) for $E(y|\ell) = E(\ln Y|L)$. We can observe that there exists a range $10^{0.7} < L < 10^{2.5}$ for which the relation:

$$E(y|\ell) = \alpha\ell + \text{const.}, \tag{4}$$

holds where α is a constant. In fact, the goodness of fit for nonparametric regression ($R^2 = 44.04\%$; see Hayfield and Racine (2008) for the definition) has a same level as that for linear regression ($R^2 = 44.03\%$) for the range, the estimation of which gives the estimation, $\alpha = 1.037(\pm 0.003)$ (shown by a straight line in Fig. 1).

Similarly, for the range of $10^{4.5} < Y < 10^{7.0}$, we have another relation, namely

$$E(\ell|y) = \beta\, y + \text{const.}, \tag{5}$$

with a constant β. The validity for this relation is checked by the nonparametric ($R^2 = 47.18\%$) and linear ($R^2 = 47.09\%$) regressions, the latter of which gives the estimation, $\beta = 0.655(\pm 0.002)$.

III. Double Scaling Law

A. *Scaling Relations*

We find that these relations are simple consequences from two *scaling relations* for the conditional PDFs, $P_{Y|L}(Y|L)$ and $P_{L|Y}(L|Y)$. Fig. 2 (a) depicts the conditional PDF, $P_{Y|L}(Y|L)$, with the conditioning values of L are chosen at a logarithmically equal interval corresponding to the range

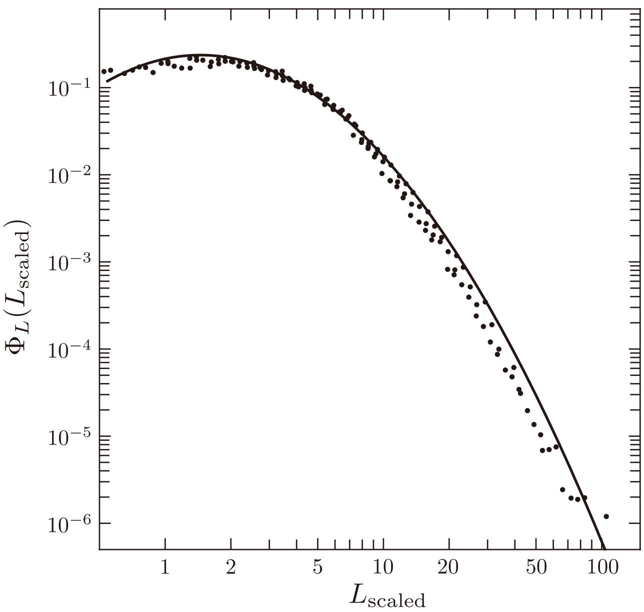

Figure 3: The scaled conditional PDF $\Phi_L(L_{\text{scaled}})$ defined by Eq. (7). The curve is the lognormal function given by Eq. (18).

$10^{0.7} < L < 10^{2.0}$ in terms of histograms. By using the values of α estimated above, we find that the conditional PDF obeys a scaling relation:

$$P_{Y|L}(Y|L) = \left(\frac{L}{L_0}\right)^{-\alpha} \Phi_Y(Y_{\text{scaled}})\,, \tag{6}$$

where $Y_{\text{scaled}} := (L/L_0)^{-\alpha}Y$ and $\Phi_Y(\cdot)$ is a scaling function, as shown by the fact that the PDFs $P_{Y|L}(Y|L)$ for different values of L fall onto a curve depicted in Fig. 2 (b). It is straightforward to show that Eq. (4) follows from Eq. (6).

Similarly, we have another scaling relation for

$$P_{L|Y}(L|Y) = \left(\frac{Y}{Y_0}\right)^{-\beta} \Phi_L(L_{\text{scaled}})\,, \tag{7}$$

where $L_{\text{scaled}} := (Y/Y_0)^{-\beta}L$ and $\Phi_L(\cdot)$ is a scaling function, as shown by Fig. 3. And also Eq. (5) immediately follows from Eq. (7).

The two scaling laws, Eqs. (6) and (7), which we collectively call *the double scaling law* have strong consequences to the function form of the joint PDF. Let us briefly describe them in the following.

B. *Consequence for Joint and Marginal PDFs*

Let us choose the reference scales Y_0 and L_0 to be within the region of the (Y, L)-plane where the double scaling law is valid. Then by substituting $Y = Y_0$, $L = L_0$ into Eqs. (1), (6) and (7), We obtain the marginal PDFs, P_Y and P_L in terms of the invariant functions, Φ_Y and Φ_L:

$$P_L(L) = \left(\frac{L}{L_0}\right)^{\alpha} \frac{\Phi_L(L)}{\Phi_L(L_0)} \frac{\Phi_Y(Y_0)}{\Phi_Y\left((L/L_0)^{-\alpha} Y_0\right)} P_L(L_0), \tag{8}$$

$$P_Y(Y) = \left(\frac{Y}{Y_0}\right)^{\beta} \frac{\Phi_Y(Y)}{\Phi_Y(Y_0)} \frac{\Phi_L(L_0)}{\Phi_L\left((Y/Y_0)^{-\beta} L_0\right)} P_Y(Y_0). \tag{9}$$

From Eqs. (6) and (7) and the above, we arrive at the following equation for the Φs:

$$\frac{\Phi_L\left((Y/Y_0)^{-\beta} L\right)}{\Phi_L\left((Y/Y_0)^{-\beta} L_0\right)} \frac{\Phi_L(L_0)}{\Phi_L(L)} = \frac{\Phi_Y\left((L/L_0)^{-\alpha} Y\right)}{\Phi_Y\left((L/L_0)^{-\alpha} Y_0\right)} \frac{\Phi_Y(Y_0)}{\Phi_Y(Y)}. \tag{10}$$

This equation puts strong constraints the form of Φ's. We have converted the above to differential equations and have derived *complete solutions* of Eq. (10).[1] Depending on whether $\alpha\beta = 1$ or not, the solution is qualitatively different, which we shall explain below.

When $\alpha\beta = 1$, we find the following relation between Φ's is necessary and sufficient for Eq. (10):

$$\Phi_L(L) = \Phi_Y\left((L/L_0)^{-\alpha} Y_0\right) \left(\frac{L}{L_0}\right)^{a} \frac{\Phi_L(L_0)}{\Phi_Y(Y_0)}. \tag{11}$$

In other words, we have one arbitrary function in the solution. In this case, Eqs. (8) and (9) implies that

$$P_L(L) = \left(\frac{L}{L_0}\right)^{-\mu_L - 1} P_L(L_0), \tag{12}$$

$$P_Y(Y) = \left(\frac{Y}{Y_0}\right)^{-\mu_Y - 1} P_Y(Y_0), \tag{13}$$

with

$$\alpha = \frac{\mu_L}{\mu_Y}, \quad \beta = \frac{\mu_Y}{\mu_L}, \quad a = -\frac{\mu_L + \mu_Y + \mu_L \mu_Y}{\mu_Y}. \tag{14}$$

This result is straightforward to understand: Due to $\alpha\beta = 1$, we have

$$L_{\text{scaled}}^{-\alpha} \propto Y L^{-\alpha} \propto Y_{\text{scaled}}. \tag{15}$$

Therefore, an arbitrary function of Y_{scaled} is a function of L_{scaled} as far as dependence on Y and L is concerned. This is why an arbitrary function is left in the solution. Also the marginal PDFs in Eqs. (12) and (13) results from the relation (1).

For $\alpha\beta \neq 1$, we obtain,

$$\Phi_Y(Y) = e^{-\beta p y^2 + q y} \Phi_Y(Y_0), \tag{16}$$

$$\Phi_L(L) = e^{-\alpha p \ell^2 + s \ell} \Phi_L(L_0), \tag{17}$$

$$P_L(L) = e^{-\alpha(1-\alpha\beta) p \ell^2 + (s + (q+1)\alpha)\ell} P_L(L_0). \tag{18}$$

$$P_Y(Y) = e^{-\beta(1-\alpha\beta) p y^2 + (q + (s+1)\beta) y} P_Y(Y_0). \tag{19}$$

The joint PDF is given by the following:

$$P_{YL}(Y, L) = e^{-\alpha p \ell^2 + 2\alpha\beta p \ell y - \beta p y^2 + s\ell + q y} P_{YL}(Y_0, L_0). \tag{20}$$

We find that in the limit $\alpha\beta \to 1$ we obtain the power laws for $P_L(L)$ and $P_Y(Y)$, which is consistent with the results (12) and (13) above.

[1] Since the proof is too lengthy for this letter, it will be published elsewhere in near future.

The parameter of p is estimated in two ways: The best fit of the theoretical expression (16) in Fig. 2 (b) yields $p = 0.692(\pm 0.027)$, while Eq. (16) in Fig. 3 yields $p = 0.704(\pm 0.016)$. These measured values of p agree with each other very well, with combined value $p = 0.698(\pm 0.025)$, assuming equal weight and no correlation. The marginal PDF $P_L(L)$ and $P_Y(Y)$ in Eqs. (18) and (19) agrees with empirical data very well with these values of p. These analysis show that our theoretical results above are in good agreement with data.

We stress that the above results are proved in the *local region* of the (Y, L)-plane where the double scaling law is valid. On the other hand, if the lognormal PDF (20) is valid *everywhere* on the (Y, L) plane, one may obtain the marginal PDFs and Φs as given in Eq. (17)–(19), as $P_L(L)$ can be obtained by integrating Eq. (20) over $Y \in [0, \infty)$, and then obtain $\Phi_L(L)$ in Eq. (17) from Eqs. (6) and (7), and so forth, which constitute easy checks of the relation between various functions.

C. *Consequence for Labor productivity*

Let us now study the labor productivity $C := Y/L$ in case of $\alpha\beta \neq 1$, as such is the reality as shown empirically. The joint PDF of (C, L), $P_{CL}(C, L)$ is given by the following:

$$P_{CL}(C, L) = L P_{YL}(CL, L). \tag{21}$$

Substituting the expression (20) into the above, we find that is express, we obtain that $P_{CL}(C, L)$ too is of lognormal form like the r.h.s. of (20) with α, β, p replaced by

$$\tilde{\alpha} := \alpha - 1, \quad \tilde{\beta} := \beta \frac{\alpha - 1}{\alpha + \beta - 2\alpha\beta}, \tag{22}$$

$$\tilde{p} := \frac{\alpha + \beta - 2\alpha\beta}{\alpha - 1} p, \tag{23}$$

respectively. By substituting empirical values found for α, β and p (the average of the two central values of p found above), we find that they are $\tilde{\alpha} = 0.037(\pm 0.003)$, $\tilde{\beta} = 0.072(\pm 0.007)$, $\tilde{p} = 6.353(\pm 0.680)$. By comparing this with Eq. (19), we find the following marginal PDF for the productivity C:

$$P_C(C) \propto e^{-\gamma_1 c^2 + \gamma_2 c}, \tag{24}$$

$$\gamma_1 := \frac{\alpha\beta(1 - \alpha\beta)}{\alpha + \beta - 2\alpha\beta} p, \tag{25}$$

where $c := \ln(C/C_0)$ with $C_0 := Y_0/L_0$ and the measured values of α, β, p yield $\gamma_1 = 0.456(\pm 0.017)$. Also, the conditional PDF $P_{L|C}(C, L)$ satisfies the scaling law;

$$P_{L|C}(C, L) = \left(\frac{C}{C_0}\right)^{-\tilde{\beta}} \Psi_L\left(\left(\frac{C}{C_0}\right)^{-\tilde{\beta}} L\right), \tag{26}$$

$$\Psi_L \propto e^{-\tilde{\beta}\tilde{p}c^2 + \tilde{q}c}, \tag{27}$$

while the average of the labor L for a given productivity C is given by the following;

$$E[L|C] \propto \left(\frac{C}{C_0}\right)^{\tilde{\beta}}. \tag{28}$$

Fig. 4 checks the result of Eq. (28) with the parameter $\tilde{\beta}$ estimated from the relation in Eq. (27), where the thick straight line is the theoretical calculation. Fig. 5 depicts the scaling function $\Psi(\cdot)$, where the curve is given by the fitting in Eq. (27). Both of these results confirm our results in the region where the scaling relations are valid.

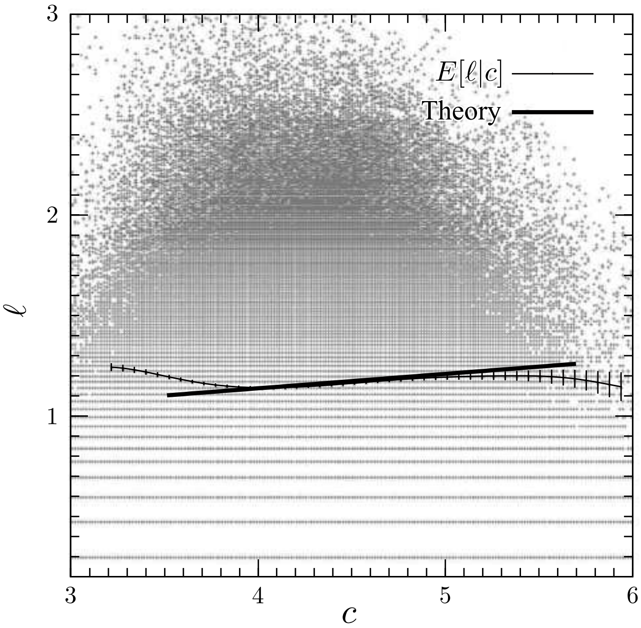

Figure 4: Scatter plot for (C, L) (gray dots). The curve with error bars is the nonparametric estimation for $E(\ell|c) := E(\ln L|C)$, while the thick straight line is the theoretical prediction (28) with $\tilde{\beta}$ given by Eq. (23) (in units of $\ln 10$).

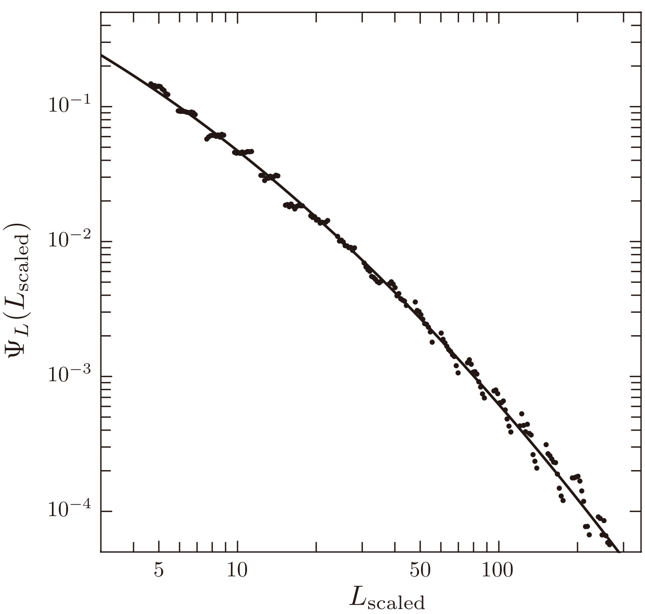

Figure 5: The scaled conditional PDF $\Psi_L(L_{\rm scaled})$ defined by Eq. (27). The curve is the lognormal function given in the same equation.

IV. Summary and Discussion

In this paper, we have shown that the Japanese data for some one million firms show that the firm distribution in (Y, L) plane satisfy a double scaling law. We have shown that the double scaling law leads to either power-law for the marginal PDF of Y and L, or the lognormal PDF for the joint PDF of Y and L. Although we have concentrated on these specific two variables because of their importance for the labour productivity, we believe that other financial quantities obey double scaling law. Furthermore, joint PDF of more than two variables are expected to obey the similar scaling laws, which yield extensions of the results explained above, including multi-variate lognormal distributions, yielding simple relations such as Eq. (25) between scaling exponents.

Our empirical finding of the double scaling law can be considered as a generalization of *production function* in the following sense. The concept of production function is a functional relation between output Y, labor input L and capital input K, where one usually assumes homogeneous production function, i.e. assumption of how output scales under a scaling of K and L. Our discovery is a *distributional* generalization of production function, which takes into account conditional distributions rather than conditional averages. This insight can potentially lead to a new perspective to look at distributions of productivity, labor, capital and output.

In addition, our findings offer straightforward guidance to the method of aggregation: the macroscopic quantities are now expressed in terms of just a few parameters in the PDFs, conditioned by double scaling law. Thus scaling laws and the resulting lognormal distributions should be the basic ingredient of economics of heterogeneous interacting agents, which is an equivalence of the statistical physics, bridging micro and macro economy in a new perspective.

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References

Albaek, K. and B. E. Sorensen, "Worker and Job Flows in Danish Manufacturing," *Economic Journal*, 1998, *108*, 1750–1771.

Aoki, M. and H. Yoshikawa, *Reconstructing Macroeconomics — A perspective from statistical physics and combinatorial stochastic processes*, Cambridge University Press, 2007.

Aoyama, H., H. Yoshikawa, H. Iyetomi, and Y. Fujiwara, "Labour Productivity Superstatistics," *Progress of Theoretical Physics Supplement*, 2009, *179*, 80–92.

– , – , – , and – , "Productivity dispersion: facts, theory, and implications," *Journal of Economic Interaction and Coordination*, 2010, *5*, 27–54.

– , Y. Fujiwara, Y. Ikeda, H. Iyetomi, and W. Souma, *Econophysics and Companies: Statistical Life and Death in Complex Business Networks*, Cambridge University Press, 2010.

Blume, L. and S.N. Durlauf, eds, *Economy as an evolving complex system III* Current perspectives and future directions Oxford University Press 2006.

Davis, S. J., J. C. Haltiwanger, and S. Schuh, *Job Creation and Destruction*, MIT Press, 1996.

Delli Gatti, D., E. Gaffeo, M. Gallegati, G. Giulioni, and A. Palestrini, *Emergent Macroeconomics: An Agent-based Approach to Business Fluctuations*, Springer-Verlag, 2008.

Foley, D. K., "A Statistical Equilibrium Theory of Markets," *Journal of Economic Theory*, 1994, *62* (2), 321–345.

Galí, J., *Monetary Policy, Inflation, and the Business Cycle*, Princeton University Press, 2008.

Hayfield, T. and J. S. Racine, "Nonparametric Econometics: The np package," *Journal of Statistical Software*, 2008, *27* (5).

Ikeda, Y. and W. Souma, "International Comparison of Labor Productivity Distribution for Manufacturing and Non-Manufacturing Firms," *Progress of Theoretical Physics Supplement*, 2009, *179*, 93–102.

Kydland, F.E. and E.C. Prescott, "Time to build and aggregate fluctuations," *Econometrica*, 1982, pp. 1345–1370.

Li, Q. and J. S. Racine, *Nonparametric Econometrics: Theory and Practice*, Princeton University Press, 2007.

Mortensen, D. T., *Wage Dispersion : Why are similar workers paid differently?*, MIT Press, 2003.

Salter, W.E.G., *Productivity and technical change*, Cambridge University Press, 1960.

Shinohara, M., *Income distribution and wage distribution*, Iwanami Shoten, Tokyo, 1955. in Japanese.

Yoshikawa, H., *Macroeconomics*, Sobunsha Publishing, 2000.

– , "Productivity dispersion and macroeconomics," 2000. in Ed. H. Yoshikawa and M. Otaki, *Macroeconomics — cycles and growth*, University of Tokyo.