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A diagrammatic analysis

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Abstract

In this paper, we show that the main results of dynamic Heckscher-Ohlin models (with non-homothetic preferences) can be derived from diagrams which represent the basic functions in static models such as the Rybczynski line, income expansion paths, and excess demand functions at steady states. Results include not only the existence and the multiplicity of steady states in autarky and under free trade, but also their stabilities and the static and dynamic Heckscher-Ohlin theorems.

Key words: Two-country model, Heckscher-Ohlin, Rybczynski line, Diagrammatic analysis
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1 Introduction

It is frequently assumed in dynamic versions of the Heckscher-Ohlin (H-O) model that countries have identical and homothetic preferences with a constant intertemporal elasticity of substitution (CIES), which has the effect of making the growth rate of the consumption expenditure in each country independent of the distribution of wealth across countries. While these assumptions about preferences simplify the analysis of steady states and transitional dynamics, they are not consistent with the empirical evidence. The assumption of homotheticity of preferences is suspect because studies of consumer demand have found significant departures from unit income elasticity of demand for some goods, even when considering highly aggregated categories of goods. For example, wealthy countries tend to have lower budget shares for food and higher budget shares for services than poor countries. In addition, there is evidence that the level of the intertemporal elasticity of substitution varies systematically with the level of wealth. In light of the relatively poor performance of the H-O model in explaining trade patterns, it is of interest to know the extent to which taste differences across countries may account for these results by influencing the patterns of trade and capital accumulation.

The benchmark model with homothetic preferences and CIES, versions of which have been studied by Chen [6] and Ventura [14], yields three main results regarding comparative advantage and the pattern of trade. The first is that there is a continuum of steady state capital stocks for the two countries consistent with a free trade equilibrium for the world economy. Each of these potential steady states is a saddle point characterized by factor price equalization (in efficiency units) and incomplete specialization in production, and each yields the same world capital stock. Which of these steady state distributions of capital the economy converges to is determined by the initial distribution of capital across countries, so that initial positions for the countries will have permanent effects on their capital labor ratios. The second feature is that a steady state H-O theorem holds, in the sense that the country that is capital abundant in the steady state will export the capital intensive good. The third result is that the country that is relatively capital abundant at the initial position will be relatively capital abundant in the steady state, and will export the capital intensive good on the path to the steady state. This represents a form of dynamic H-O theorem, in that the future trade patterns are predicted from the initial relative factor endowments.

In our recent studies, Bond et al. [2], [3], and [4], we investigate the properties of a dynamic H-O model with non-homothetic preferences, where there are two goods one of which is a pure consumption good and the other is a consumable capital good.[1] We showed that if labor productivities and discount factors are the same across countries and the labor intensive good[2] is normal in consumption, then the main results of the benchmark H-O model will hold. The primary difference

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[1] With respect to the production side, our analysis is based on the fundamental structure of the H-O model, which was first noted in Jones [8], also see his recent work Jones [9].

[2] As shown in Bond et al. [4], the factor intensity ranking between two sectors does not affect the results in our studies.

introduced in this case is that the world capital stock in the steady state will depend on the distri-
bution of income across countries. We have also provided an example to show that the results may
differ dramatically if the labor intensive good is inferior. These differences include the possibility
that there are multiple steady state equilibria in autarky, that the static H-O theorem is violated
in the steady state, and that local indeterminacy arises around some steady states when discount
factors differ across countries.[3]

Our aim in this paper is to show that main results in dynamic H-O models (with non-homothetic
preferences) can be derived and/or examined from diagrams using (i) the steady state Rybczynski
line and (ii) the income expansion path evaluated at the steady state prices. These curves, combined
with the steady state resource constraints, yield steady state excess demand functions that specify
the country's excess demand as a function of its capital stock. Using these excess demand functions,
we can derive the locus of home and foreign capital stocks that are consistent with a steady state
equilibrium with free trade. Also, we can see the stability of steady states and the steady state
trade pattern only from their shapes.[4] Our diagrammatic characterization of the steady state is
possible because the assumptions of the dynamic H-O model yield a unique price associated with
the steady state equilibrium, so the level of capital stocks in each country must be consistent with
market clearing at these prices.

This paper is organized as follows. Section 2 presents the dynamic two country H-O model and
review basic properties of H-O models. By a diagrammatic analysis, Section 3 derives the steady
state equilibria in autarky and under free trade, and examines their properties. Section 4 considers
the case where there are three steady states in autarky due to inferiority in consumption. Section 5
offers some concluding remarks.

2   The Dynamic Two Country Heckscher-Ohlin Model

In this section we formulate the continuous-time version dynamic optimization problem for a repre-
sentative country in a dynamic H-O model. By dynamic H-O model, we mean that each country has
access to the same technology for producing two goods using a fixed factor (labor) and a reproducible
factor (capital) under conditions of perfect competition and constant returns to scale. Good 1 is a
pure consumption good, and the second good is a consumable capital good. Factors of production
are assumed to be mobile between sectors within a country, but immobile internationally, and there
are no markets for international borrowing and lending. We refer to the representative country as
the home country: the corresponding behavioral relations for the other (foreign) country will be

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[3]Bond et al. [3] showed that the multiplicity of autarkic steady states leads to the occurrence of poverty trap, and
that poverty trap feature of the autarky equilibrium also applies to the world economy.

[4]With normality in consumption, the stability can be determined only from the slope of the excess demand as
the Walrasian stability in static models. Without it, however, there may be some types of steady states the stability
analysis of which requires more cumbersome calculations.

denoted by a "*."

We normalize the population in the home country to be one, and assume that each household has an endowment of labor, L, and a concave utility function u defined over consumption of goods 1 and 2, C_1 and C_2.[5]

2.1 The Production Side

On the production side, we will assume that

Assumption 1: The production function in each sector is quasi-concave and linearly homogeneous. Both factors are indispensable for producing and pure consumption good 1 is labor intensive.

The results of the static Heckscher-Ohlin model are well known, so here we provide only a brief review of properties that will be important to the dynamic model. Since the assumptions to be made below will ensure that the economy is incompletely specialized in both the autarkic and free trade steady states, limit our presentation of the production side to the case of incomplete specialization. Our assumption on the factor intensity ranking of sectors is chosen for convenience due to emphasis on a diagrammatic presentation. We indicate below in cases where the factor intensity rankings play a role.

Letting w denote the wage rate and r the rental on capital, the technology in sector i can be characterized by the unit cost function $a_i(w,r)$, $i = 1, 2$. Under incomplete specialization, the competitive profit conditions require that

$$a_1(w,r) = p, \tag{1}$$

$$a_2(w,r) = 1, \tag{2}$$

where good 2 is chosen as numeraire. Let $w(p)$ and $r(p)$ be the solution to the system of equations, (1) and (2). Totally differentiating these conditions yields the Stolper-Samuelson theorem:

$$\frac{pw'(p)}{w(p)} = \frac{a_{2r}}{\Delta} \cdot \left(a_{1w} + \frac{r}{w} a_{1r} \right) > 1,$$

$$r'(p) = -\frac{a_{2w}}{\Delta} < 0.$$

where a_{iw} and a_{ir} are the labor and capital input coefficients in sector i, respectively, and Assumption 1 implies that they satisfy

$$\Delta \equiv a_{1w}a_{2r} - a_{2w}a_{1r} > 0.$$

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[5] As shown in Bond et al. [2], labor productivity differences are easily incorporated into the production side of the H-O model by assuming that a unit of labor in the home country represents $\mu \geq 1$ efficiency units of labor, with the productivity of foreign labor normalized to 1.

Factor market equilibrium requires that

$$L = a_{1w}Y_1 + a_{2w}Y_2, \tag{3}$$

$$K = a_{1r}Y_1 + a_{2r}Y_2, \tag{4}$$

where K is the stock of capital and Y_i is the output of good i. From the equations above, we see

$$Y_1(p, K, L) = \frac{a_{2r}L - a_{2w}K}{\Delta} \tag{5}$$

$$= w'(p)L + r'(p)K, \tag{6}$$

$$Y_2(p, K, L) = \frac{-a_{1r}L + a_{1w}K}{\Delta} \tag{7}$$

$$= w(p)L + r(p)K - p[w'(p)L + r'(p)K] \tag{8}$$

and the Rybczynski theorem holds:

$$\frac{\partial Y_1}{\partial K} = r' < 0,$$

$$\frac{\partial Y_2}{\partial K} \cdot \frac{K}{Y_2} = \frac{(r - pr')K}{(w - pw')L + (r - pr')K} > 1.$$

Notice that (1), (2), (3), and (4) yield

$$w(p)L + r(p)K = pY_1 + Y_2, \tag{9}$$

and hence we can use (3) and (9), instead of (3) and (4), to find outputs (Y_1, Y_2).

Let

$$p_0 \equiv \inf\{p | w(p) > 0\},$$

$$p_\infty \equiv \sup\{p | r(p) > 0\}.$$

Then, for $p \in (p_0, p_\infty)$ and $k \equiv K/L \in (k_1(p), k_2(p))$, where $k_i(p) \equiv a_{ir}(w(p), r(p))/a_{iw}(w(p), r(p))$ is the capital labor ratio in sector i, the factor prices and the outputs of both goods are all positive and the argument above applies for such p and k.

2.2   The Consumption Side

We analyze the optimization problem for a representative household that owns L units of labor. We will impose the following restrictions on this utility function:[6]

Assumption 2: The utility function is strictly concave, with $u_{11} < 0$ and $D \equiv u_{11}u_{22} - u_{12}u_{21} > 0$ for any $(C_1, C_2) \in \{(C_1, C_2) \in \mathbb{R}^2_+ | u_i(C_1, C_2) > 0, \ i = 1, 2\}$, and satisfies $\lim_{C_i \to 0} u_i(C_1, C_2) = \infty$ $(i = 1, 2)$ for any C_j $(j \neq i)$.

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[6]This assumption allows the utility function to be non-homothetic.

The representative household is assumed to maximize the discounted sum of its utilities

$$\max \int_0^\infty u(C_1, C_2) \exp(-\rho t) dt, \tag{10}$$

subject to its flow budget constraint

$$wL + rK = pC_1 + C_2 + \dot{K} + \delta K, \qquad K_0 \text{ given}, \tag{11}$$

where δ is the rate of depreciation on capital and ρ is the discount rate. The budget constraint reflects the assumed absence of an international capital market, since it requires that $pZ_1 + Z_2 = 0$, where $Z_1 = C_1 - Y_1$ ($Z_2 = C_2 + \dot{K} + \delta K - Y_2$) is the excess demand for good 1 (2).

Solving the current value Hamiltonian for this problem yields the necessary conditions for the choice of consumption levels, the differential equation describing the evolution of the costate variable, λ, and the transversality conditions:

$$u_1(C_1, C_2) = \lambda p, \quad u_2(C_1, C_2) = \lambda, \tag{12}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r), \tag{13}$$

$$\lim_{t \to \infty} K(t)\lambda(t) \exp(-\rho t) = 0. \tag{14}$$

3 Steady States in Autarky and under Free trade

A steady state will be characterized by the existence of a price $\tilde{p}$ and capital stock $\tilde{K}$ such that $\dot{K} = 0$, $\dot{\lambda} = 0$ and markets clear. From (13), a steady state with incomplete specialization will require that there is some $\tilde{p} > 0$ such that $r(\tilde{p}) = \rho + \delta$ holds. We will impose the following condition, which ensures the existence of a price $\tilde{p}$ that is consistent with incomplete specialization:

Assumption 3: $\inf\{r|a_2(w, r) = 1\} < \rho + \delta < \sup\{r|a_2(w, r) = 1\}$.

The Stolper-Samuelson theorem guarantees that the solution for the steady state price with incomplete specialization will be unique.[7]

3.1 Determination of a steady state in autarky

An autarkic steady state requires that $Z_1 = 0$ and $Z_2 = 0$. The former condition requires production of good 1 in the steady state as a result of Assumption 2, and the latter condition requires production of good 2 in order to sustain the steady state capital stock. Therefore, the autarkic steady state price must be the consistent with incomplete specialization.

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[7]This assumption plays a role analogous to the Inada conditions in the one sector growth model, since it ensures that the marginal product of capital is high enough to ensure replacement for K sufficiently low, and that the marginal product is low enough to shut of accumulation for K sufficiently high.

The market clearing conditions in the autarkic steady state are

$$Y_1 = w'(\tilde{p})L + r'(\tilde{p})K = C_1 \text{ and } Y_2 = C_2 + \delta K. \tag{15}$$

Substituting (15) into the labor market equilibrium condition, (3), we define the *steady state* Rybczynski line as follows:

$$\left[\frac{\tilde{a}_{1w}}{\tilde{a}_{2w}} + \frac{\delta}{r'(\tilde{p})}\right] C_1 + C_2 = \left[\frac{1}{\tilde{a}_{2w}} + \frac{\delta w'(\tilde{p})}{r'(\tilde{p})}\right] L, \quad \text{for } C_1 \geq 0 \text{ and } C_2 \geq -\delta k_1(\tilde{p})L, \tag{16}$$

where $\tilde{a}_{iw} \equiv a_{iw}(w(\tilde{p}), r(\tilde{p}))$, $i = 1, 2$. The steady state Rybczynski line is the locus of steady state consumption levels that are attainable as the stock of capital is varied, given the stock of labor and relative prices, and is illustrated by the negatively sloped line in Figure 1. An increase in the capital stock reduces the output of labor intensive good 1 and raises the output of capital intensive good 2 more than proportionally by the Rybczynski theorem, so the net output of good 2 must increase. The steady state Rybczynski line coincides with the Rybczynski line from the static trade model if $\delta = 0$.

We obtain the following Lemma.

Lemma 1 *Let K be the steady state capital stock. Then, the outputs of two goods at the steady state, $(Y_1(K), Y_2(K))$, are derived from the intersection between the steady state Rybczynski line,*

$$\left[\tilde{p} - \frac{\rho}{r'(\tilde{p})}\right] C_1 + C_2 = [w(\tilde{p}) + \rho k_2(\tilde{p})] L, \tag{17}$$

and the steady state resource constraint,

$$\tilde{p} C_1 + C_2 = w(\tilde{p})L + \rho K, \tag{18}$$

as $(Y_1(K), Y_2(K)) = (C_1, C_2 + \delta K)$.

Proof. From (5)–(8), we have

$$a_{1w} = (r - pr')\Delta, \ a_{1r} = (pw' - w)\Delta, \ a_{2w} = -r'\Delta, \ a_{2r} = w'\Delta, \text{ and } \Delta = \frac{1}{w'r - wr'}. \tag{19}$$

Then, it is easy to see that (16) is identical to (17). On the other hand, (18) is easily derived from (9). ∎

Since $r(\tilde{p}) - \delta = \rho$, the *steady state* resource constraint (18) also represents the budget constraint for households with capital stock K and investment δK. If the level of capital stock is K in a steady state, then consumption bundles at the steady state correspond to the intersection between the income expansion path with $\tilde{p}$ and (18). Notice that $C_1(K) - Y_1(K)$ in Figure 1 is an excess demand for good 1.

Therefore, at the intersection, (C_1^A, C_2^A), between the income expansion path and the steady state Rybczynski line, goods market will clear with $\dot{K} = \dot{\lambda} = 0$. The intersection corresponds to the steady state in autarky. Here, the steady state values of K and λ are given by

$$K^A = \frac{C_1^A - w'(\tilde{p})L}{r'(\tilde{p})} \text{ and } \lambda^A = u_2(C_1^A, C_2^A).$$

Hence, we have

Proposition 1 *An intersection between the steady state Rybczynski line and the income expansion path with the steady state price of good 1 corresponds to an autarkic steady state. Therefore, it uniquely exists as long as labor intensive good 1 is normal and preferences exhibit neither a satiation level nor a minimum subsistence level.*

Notice that the intersection must be unique when good 2 is inferior at some income levels, because the slope of the steady state Rybczynski line is steeper than that of the budget constraint (18). If good 1 is capital intensive, the Rybczynski line will be negatively sloped but flatter than the budget line. In that case a sufficient condition for uniqueness is that labor intensive good 2 be normal.[8]

3.2   Excess demand for good 1

We define the steady state excess demand function as follows.

$$Z_1(K) \equiv C_1(K) - Y_1(K) \text{ for } K \in [k_1(\tilde{p})L, k_2(\tilde{p})L],$$

For given K, excess demand for good 1 corresponds to the horizontal difference between the intersection of the income consumption curve with the steady state budget constraint and the intersection of the steady state Rybczynski line with the steady state resource constraint in Figure 1. Excess demand will be strictly increasing in K with normality in consumption, Since $Y_1(K)$ is linear in K, the shape of Z_1 reflects exactly that of the income expansion path. In the case of homothetic preferences the slope of the function is constant (the solid line in Figure 2), while the function is concave (convex) in K, when good 1 is a necessity (luxury) (the dashed (dotted) curve in Figure 2).

When the steady state excess demand is upward sloping, we have

Lemma 2 *With normality in consumption, the steady state in autarky is a saddle point.*

We omit the proof of Lemma 2 because it may be considered as a special case of Proposition 3 in Section 4. Lemma 2 can be interpreted as indicating that if an increase in the capital stock above the autarkic steady state creates an excess demand for the labor intensive good, then there will be an

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[8]Brock [5] and Benhabib and Nishimura [1] prove that a steady state is unique in the multi-sector optimal growth model by assuming all consumption goods to be normal. In the two-sector model our assumption is weaker as we require only one of goods (the labor intensive good) to be normal.

increase in the price and a decrease in the rental on capital in the economy.[9] A similar result applies if good 1 is capital intensive, in which case stabiltiy is obtained with normality in consumption because the excess demand schedule for good 1 is negatively sloped.

3.3 The Foreign Country and World Market Equilibrium

In order to focus on the role of relative factor supplies on the pattern of trade, as in the static Heckscher-Ohlin model, we will impose the following restrictions on preferences and technologies:

Assumption 4: The home and foreign countries have identical utility functions, $u(C_1, C_2)$, identical technologies, $a_i(w, r)$, $\rho = \rho^*$, and $\delta = \delta^*$.

These assumptions ensure that the autarkic steady state prices are the same in each country, and will be the same as the free trade steady state prices.[10] The foreign country excess demand function is given by $H^* Z_1^*(K^*)$, where H^* is the number of households in the foreign country and $Z_1^*(K^*)$ is derived in a similar fashion to the home excesss demand. If we impose the further restriction that $L = L^*$, then $Z_1^*(.) = Z_1(.)$ and the autarkic capital stocks in the two countries will be identical. Figure 3 illustrates a case where $L > L^*$, so the foreign country has a lower autarkic steady state.

A steady state equilibrium with trade is a pair (K, K^*) such that

$$Z_1(K) + H^* Z_1^*(K^*) = 0 \ \ \text{with } K \in [k_1(\tilde{p})L, k_2(\tilde{p})L] \text{ and } K^* \in [k_1(\tilde{p})L^*, k_2(\tilde{p})L^*].$$

The pairs (K_1, K_1^*) and (K_2, K_2^*) in Figure 3 illustrate two possible steady state equilibria. Clearly, there is a continuum of steady states under free trade, where both countries are incompletely specialized. Letting (K^T, K^{T*}) be one of the steady state free trade pairs, the values of λ and λ^* at the steady state are given by

$$\lambda^T = u_2(C_1(K^T), C_2(K^T)) \ \text{ and } \ \lambda^{T*} = u_2(C_1^*(K^{T*}), C_2^*(K^{T*})).$$

Remark 1 *For given technologies, preferences, and a labor endowment in each country, we can draw, for each country, (i) the steady state Rybczynski line, (ii) the income expansion path with $\tilde{p}$, and (iii) the steady state resource constraints at the highest and the lowest capital stocks consistent with incomplete specialization. They yield the steady state excess demand function for each country, from which we can precisely derive the locus of (K^T, K^{T*}), the pair of capital stocks in a free trade equilibrium.*

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[9]It was verified in Bond et al. [3], where they derive the phase diagram of a two-sector model with non-homothetic preferences and examine the global stability of the steady states.

[10]Indeed, if $\rho + \delta \neq \rho^* + \delta^*$, one country must be specialized in any steady state under free trade (see Stiglitz [13]). Chen et al. [7] consider a dynamic H-O model with endogenous time preference and show that there exists a unique steady state under free trade and each country is incompletely specialized at the steady state, because the discount rates are endogenously determined to satisfy $\rho + \delta = \rho^* + \delta^*$.

If $L = L^*$ and $H^* = 1$, the steady state excess demand function for the foreign country coincides with that for the home country, and the locus of the steady state pairs (K^T, K^{T*}) can be easily derived from its graph as shown by Figures 2 and 4. In the case of homothetic preferences the locus is given by a solid line with $K^T + K^{T*} = 2K^A$, corresponding to the one in Figure 2, while the locus becomes convex (concave) to the origin, when good 1 is a necessity (luxury). Notice that the locus must be bounded by $K^T, K^{T*} > k_1(\tilde{p})L$ or $K^T, K^{T*} < k_2(\tilde{p})L$, and the former constraint will bind when $Z_1(k_1(\tilde{p})L) + Z_1(k_2(\tilde{p})L) > 0$, and vice versa.

We have the following Lemma on the stability of a free trade equilibrium as a special case of Proposition 3 below.

Lemma 3 *With normality in consumption, all the free trade steady states are saddle points.*

The intuition for this result is the same as for Lemma 2.

When $L = L^*$, we easily examine the relationship between the capital/labor ratio ranking and the steady state trade pattern. See Figure 2. It is apparent from the monotonicity of Z_1 that in free trade steady states, the excess demand for labor intensive good 1 is positive (negative) in the capital abundant (scarce) country, i.e. in any steady state, the capital abundant country exports the capital intensive good. Moreover, it can be proved that the steady state trade pattern depends on their initial capital endowments as follows.

Proposition 2 *(Heckscher-Ohlin theorem) Let goods be normal and $L = L^*$. Then, the initially capital abundant country remains capital abundant along the dynamic general equilibrium path to the steady state, and the capital abundant country exports the capital intensive good at the steady state.*

Proof. First, notice that if the initial capital stock in each country is the same, $K_0 = K_0^*$, then the world economy converges to the steady state where the capital stock in each country is the same as K^A. Suppose that for some initial pair of capital stocks, the economy converges to one of free trade steady states where the capital labor ratio ranking between two countries is different from the initial one. Then, there is some time $t > 0$, possibly infinite, such that at time t the capital stock in each country is the same, say $\hat{K}$. However, the dynamic general equilibrium path with $K_0 = K_0^* = \hat{K}$ will not converge to any asymmetric steady state, and the path with $K_0 \neq K_0^*$ will not converge to the symmetric steady state due to its saddle-point stability. So, the capital labor ratio ranking does not change along any dynamic general equilibrium path. ∎

The former result may be violated when indeterminacy occurs due to factor generated externality (Nishimura and Shimomura [10])[11] or inferiority in consumption (Nishimura and Shimomura [11] and Bond et al. [2]). The latter can be violated when $L \neq L^*$ or $\rho \neq \rho^*$ (see Proposition 3 in Bond et al. [2]).

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[11] See Nishimura et al. [12] for indeterminacy in a discrete-time version of a two-country model with externality.

4 An Example with Inferior Goods

In this section, we consider the case where there are three steady states in autarky as shown in Figure 5.[12] A multiplicity of steady states in autarky, the number of which is generally odd, is possible when labor intensive good 1 is inferior at some range of income and $\tilde{a}_{2w}$, the labor input coefficient in capital intensive sector 2, is sufficiently small (see equation (16)). In this case, the steady state excess demand function is not monotone and has the shape as shown in Figure 6, where K_j^A, $j = L, M, H$, corresponds to each of autarkic steady states and $\hat{K}_1$ and $\hat{K}_2$ is defined as the values where the slope of Z_1 is equal to zero.

With a non-monotonic excess demand, the stability of steady states is determined as follows.

Proposition 3 *(Stability condition) If the steady state demand function in each country is upward sloping at the value of capital stock in a free trade steady state, then the steady state is a saddle point. If it is downward and the discount factor in each country is the same, then the steady state is unstable.*

Proof. Bond et al. [2] derived the steady state excess demand for good 1 as a function of λ (the shadow value of capital stock), and showed that if it is decreasing in each country at a free trade steady state, then the steady state is a saddle point. On the other hand, if it is upward sloping and $\rho = \rho^*$, then the steady state is unstable. It can be easily verified that there is a negative relationship between the steady state values of K and λ. Indeed, under the strict concavity of u (Assumption 2), as the steady state value of K increases, which implies an increase in households income: $w(\tilde{p})L + r(\tilde{p})K$, the marginal utility of income decreases, and hence the steady state value of λ falls (see (12)). ∎

In what follows, we assume $L = L^*$ and $H^* = 1$. With this assumption, the loci of the steady state pairs of (K^T, K^{T*}) that correspond to the case of Figure 6 are drawn as in Figure 7.[13] Since

$$\frac{dZ_1(K)}{dK} > 0 \ \text{ for } K \in (k_1(\tilde{p})L, \hat{K}_1) \cup (\hat{K}_2, k_2(\tilde{p})L),$$

$$\frac{dZ_1(K)}{dK} < 0 \ \text{ for } K \in (\hat{K}_1, \hat{K}_2),$$

each pair that exists on loci AA', EE', CbD, and $C'b'D'$ is a saddle point, while those on BB' are unstable.[14]

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[12] Bond et al. [3] examine the case and discuss the implication of the multiplicty in a closed economy (poverty trap) as well as under free trade (effects of opening trade on the steady state welfare in each country).

[13] Their shapes reflect the fact that $Z_1(\hat{K}_1) < -Z_1(\hat{K}_2) < Z_1(k_2(\tilde{p})L) < -Z_1(k_1(\tilde{p})L)$ hold in Figure 6. However, it does not matter in the following argument whether these inequalities are satisfied or not.

[14] After opening an international goods market, steady state equilibria will emerge around each of the pairs of autarkic equilibria in Home and Foreign. Notice that the stability of free trade equilibria that sufficiently close to one of the pairs is the same as that for the autarkic equilibrium in each country (see Weller and Yano [15]). However, the stability of the pairs on positively sloped regions is ambiguous, even if they are sufficiently close to autarkic free trade

Notice that for any pair one of which is in the range (K_L^A, K_M^A) and the other is in (K_M^A, K_H^A), the capital abundant country exports labor intensive good 1 in the steady state: the static H-O theorem is violated. Such steady states exist on loci $aBB'a'$, bDc, and $b'D'c'$ in Figure 7. Therefore, some steady states are characterized by the saddle-point stability and the violation of the H-O theorem.

If the initial capital stock in each country is the same, opening trade between the countries has no effect on their transitional paths due to the symmetry of the countries and the economy will converge to an autarkic free trade equilibrium, $(K^T, K^{T*}) = (K_L^A, K_L^A)$ or (K_H^A, K_H^A) as indicated in Figure 7.[15] So, the initial ranking of factor endowment ratios among countries is maintained along any dynamic equilibrium path. On the dynamic trade pattern, however, we see that there is some path along which the steady state trade pattern will change from the initial one. One example is a path such that it initially starts from sufficiently close to locus BB', where the static H-O theorem is violated, and converge to the steady state on locus EE', where it holds (see Figure 7).

Remark 2 *Under the symmetry on countries' fundamentals except for their initial capital stocks, the capital abundant country remains capital abundant along the trajectory to the steady state. However, it is possible that the trade pattern varies along the path due to inferiority in consumption.*

5 Concluding Remarks

We have shown that main results in dynamic H-O models (with non-homothetic preferences) can be derived and/or examined from some diagrams which represent the basic functions in static models such as the Rybczynski line, an income expansion path, and an excess demand function. For given technologies, preferences, and a labor endowment in each country, we have derived the diagrams and shown that they can clarify not only the existence and the multiplicity of steady states in autarky and under free trade, but also their stabilities and the static and the dynamic Heckscher-Ohlin theorems.

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steady states. To clarify their stability, we have to derive the loci of the steady state pairs of (λ, λ^*) as in Bond et al. [2].

[15] The global stability of these steady states come from the phase diagram analysis in Bond et al. [3].

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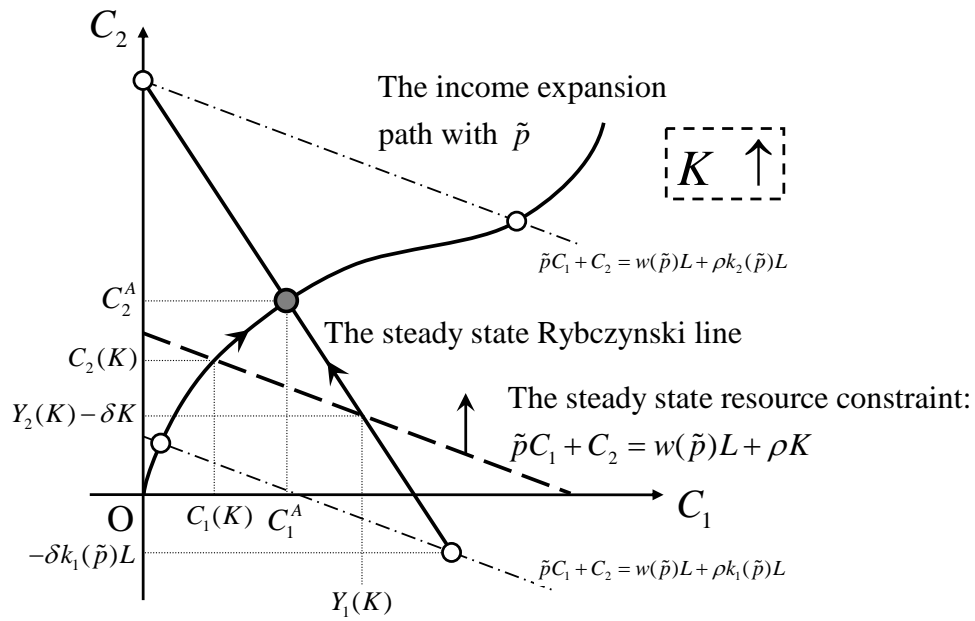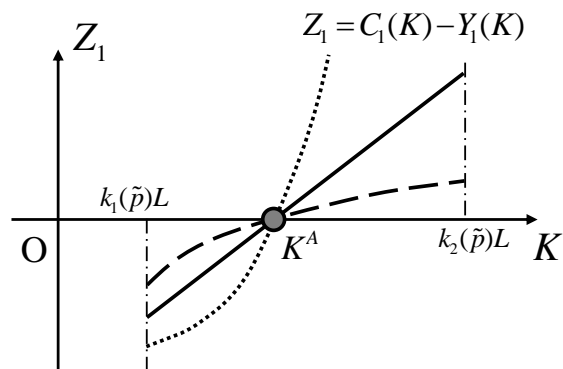
Figure 1

The income expansion path with $\tilde{p}$

$K \uparrow$

$\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho k_2(\tilde{p})L$

C_2^A

$C_2(K)$

$Y_2(K) - \delta K$

The steady state Rybczynski line

The steady state resource constraint:
$\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho K$

C_1

O $\quad C_1(K) \quad C_1^A$

$-\delta k_1(\tilde{p})L$

$\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho k_1(\tilde{p})L$

$Y_1(K)$

Figure 2

$Z_1 = C_1(K) - Y_1(K)$

Z_1

$k_1(\tilde{p})L$

O $\quad K^A \quad k_2(\tilde{p})L \quad K$

Figure 3

$Z_1, H^*Z_1^*$

$H^*Z_1^* = H^*[C_1^*(K) - Y_1^*(K)]$

$Z_1 = C_1(K) - Y_1(K)$

K_2^* K_1^* K^{A*}

O

K^A K_1 K_2

K

Figure 4

K^{T*}

$k_2(\tilde{p})L$

K^A

$k_1(\tilde{p})L$

$K^T + K^{T*} = 2K^A$

O $k_1(\tilde{p})L$ K^A $k_2(\tilde{p})L$ K^T

Figure 5

C_2

The income expansion path with $\tilde{p}$

C_2^H

$\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho k_2(\tilde{p})L$

The steady state Rybczynski line

C_2^M

C_2^L

O C_1^H C_1^M C_1^L C_1

$-\delta k_1(\tilde{p})L$ $\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho k_1(\tilde{p})L$

Figure 6

Z_1

$Z_1 = C_1(K) - Y_1(K)$

$\hat{K}_2$

$k_1(\tilde{p})L$ K_L^A K_M^A K^T

O K^{T*} K_H^A $k_2(\tilde{p})L$ K

$\hat{K}_1$

Figure 7