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KONNO Tomohiko
the University of Tokyo

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Knowledge Spillover on Complex Networks

Tomohiko Konno
Graduate School of Economics, The University of Tokyo
Research Fellow of the Japan Society for the Promotion of Science
tomo.konno@gmail.com

Abstract

Most growth theories have focused on R&D activities. Although R&D significantly influences economic growth, the spillover effect also has a considerable influence. In this paper, we study knowledge spillover among agents by representing it as network structures. The objective of this study is to construct a framework to treat knowledge spillover as a network. We introduce a knowledge spillover equation, solve it analytically to find a workable solution. It has mainly three properties: (1) the growth rate is common for all the agents only if they are linked to the entire network regardless of degrees, (2) the TFP level is proportional to degree, and (3) the growth rate is determined by the underlying network structure. We compare growth rate among representative networks: regular, random, and scale-free networks, and find the growth rate is the greatest in scale-free network. We apply this framework, i.e., knowledge spill over equation, to the problem of firms forming a network endogenously and show how distance and region size affect the economic growth. We also apply the framework to network formation mechanism. The aim of our paper is not just showing results, but in constructing a framework to study spillover by network.

Keywords: Networks; Spillover; Economic Growth

JEL Classification: O4; R11; R12

*T.K. thanks to Takatoshi Tabuchi and Hiroshi Yoshikawa for helpful comments.

1 Introduction

Growth theories (Solow et al. (1957); Romer (1990); Aghion and Howitt (1998); Grossman and Helpman (1991)) show that the progress of technology determines the long-term growth. Most growth theories have focused on R&D activities. Although R&D significantly influences economic growth, the spillover effect also has a considerable influence. In this paper, we study knowledge spillover among agents by representing it as network structures and use complex network theories.

Although it is occasionally assumed, mainly for simplicity, that once new technology is invented, it spreads worldwide immediately at no cost, technology diffusion takes time and incurs various costs beyond any doubt. The following are just a few examples:

- It took a millennium for the water mill to be widely adopted in Europe; it is felt that the main reason for this slow pace of diffusion was the absence of significant mobility during pre-medieval and medieval times.
- The spread of new hybrid seed has been central to the increase in agricultural productivity over the past century. The classical work by Ryan and Gross (1943) documents that hybrid corn seed were adopted over a period of several years in the early twentieth century, in the United States. Moreover, diffusion of these seed displayed clear spatial patterns; initially, a small group of farmers adopted the seed, followed by their neighbors adopting it, and this was followed by the neighbors of the neighbors adopting it, and so on.

The examples above are taken from Goyal (2007). The classic paper by Griliches (1957) shows that even more productive hybrid corn diffused only slowly in the U.S and the diffusion process was affected by the local economic conditions. He also found that technology diffusion can be described by the logistic curve, occasionally referred to as the S-shaped curve. Initially, it spreads only slowly, but once adoption reaches the critical point, it begins to spread very rapidly; finally after a large fraction adopts, the rate of adoption declines. Griliches shows that the diffusion process takes time. Recent reviews by Asheim and Gertler (2005) also shows that geographical proximity is an important factor for spillovers and Konno (2008) shows that the transaction between firms decreases as the distance increases.

There is no doubt that technology difference exists across countries, furthermore the difference is not only across countries but also within a single country. We can observe significant differences across firms in even a narrowly defined industry, Bartelsman et al. (2000) shows that technology is local rather than global. Many studies find correlations between productivity and firm size, various measures of technology (e.g, IT technology level), skill level of the employee, management practices, and so on. However, the question why there exists a significant productivity difference among firms within a single country and within even a narrowly defined industry is not yet answered. Therefore, it is not surprising that we still lack a consensus on what determines cross-country productivity differences. These evidences show that new technology does not spread instantaneously and technology difference exists, and in particular it suggests that knowledge spillover structure surely exist, which we will express by network structures. Unless such networks existed meaning that technology diffuses world wide instantaneously, it would be difficult to explain why such a significant technology difference exists.

The following recent papers study knowledge spillover. Keller (2004) discussing spillovers and geographical relation; Eaton and Kortum (2001) showing convergence, spillovers and trade; Eaton and Kortum (1999); Acemoglu et al. (2006); and Vandenbussche et al. (2006). Knowledge spillover must be related to geography; in this respect Fujita and Thisse (2002), Spatial Economics, may also be relevant.

Our study also focuses on the network structure, placing it among other studies on networks. Goyal and Moraga-Gonzalez (2001) studied the R&D formation mechanism by which coalition diminishes marginal cost, while our model directly deals with knowledge spillover which increases TFP and focuses on the processes of technology diffusion by network structure of that process. For the Economics of networks, please refer Goyal (2007); Jackson (2008).

Before introducing our model, we briefly explain the standard model that analyzes knowledge spillover. For convenience, we explain it with the model of the international world technology frontier by following Acemoglu (2009). The world consists of J countries indexed by $j = 1, 2, \cdots, J$. Each country has the following production function:

$$Y_j(t) = F(K_j(t), A_j(t)L_j(t)) \tag{1}$$

We define growth rate of the country j, g_j, by

$$g_j(t) \equiv \frac{\dot{A}_j(t)}{A_j(t)} \tag{2}$$

Let us assume that world technology frontier, which is denoted by $A(t)$, grows exogenously at the constant rate

$$g \equiv \frac{\dot{A}(t)}{A(t)} \tag{3}$$

The population growth is ignored, then the utility function is

$$U_j = \int_0^\infty e^{-\rho t} \left[\frac{\tilde{c}_j(t)^{1-\theta} - 1}{1-\theta} \right] dt \tag{4}$$

where $\tilde{c}_j \equiv C_j(t)/L_j(t)$ is the per capita consumption in country j at time t. We assume that ρ is the same across all the countries[1].

As in the neoclassical growth model, the flow of capital is described by

$$\dot{k}_j(t) = f(k_j(t)) - c_j(t) - (\delta + g_j(t))k_j(t) \tag{5}$$

where $c_j(t) \equiv \tilde{c}_j(t) \equiv C_j(t)/A_j(t)L_j(t)$ is the consumption normalized by effective units of labor.

In this model, knowledge spillover is described by the following equation:

$$\dot{A}_j(t) = \sigma_j \left(A(t) - A_j(t) \right) + \lambda_j A_j(t) \tag{6}$$

Eq.(6) states that each country absorbs world technology at the exogenous constant rate σ_j. If the country j is far behind the world technology frontier, then $A_j(t)$ grows faster. In contrast, if $A_j(t) = A(t)$, the country j has nothing to learn from the world technology frontier.

We also define $a_j(t)$ as

$$a_j(t) \equiv \frac{A_j(t)}{A(t)} \tag{7}$$

Then we can re-write Eq.(6) as

$$\dot{a_j}(t) = \sigma_j - \left(\sigma_j + g - \lambda_j \right) a_j(t) \tag{8}$$

$g_j(t)$ becomes

$$g_j(t) = \frac{\dot{a_j}(t)}{a_j} + g \tag{9}$$

There exists a unique steady state such that $\dot{a_j}(t) = 0 \; \forall j$.

$$a_j^\star = \frac{\sigma_j}{\sigma_j + g - \lambda_j} \tag{10}$$

$$f'(k_j^\star) = \rho + \delta + \theta g \tag{11}$$

[1] Do not confuse this ρ with the ρ which will be introduced into our model later and means self evolution rate.

and consumption per capita in each country grows at the constant rate g, which is the growth rate of world technology.

In this type of models, the network structure of knowledge spillover is not explicitly considered. There exist only two kinds of countries an ordinary country and one that is not actually a country; but a world technology frontier. Interactions among agents where knowledge diffuses from one agent to other agents are not explicitly considered. Instead of introducing such interactions among many countries, ordinary countries are affected only by the world technology frontier. However, in this model, these countries do not affect other countries nor they are affected by them. This is probably because it is difficult to find an analytical solution where there is an asymmetric network structure of knowledge spillover in which degree distribution is heterogenous. It is also probably because this type of model was invented before "Complex Networks" emerged around 2000, when people realized that explicitly considering the network structure in the model has significant importance. Studies on "Complex Networks" small-world and scale-free networks are widely recognized and strongly suggest that underlying network structures, especially scale-free networks, determine the outcome of models. However, most of the network models considered in Economics are based on random networks. In our view, a regular network is not a true network because it is just a lattice in which all the vertices have the same degree; it is a symmetric network without heterogeneity in degree distribution[2]. In some respect, a random network is almost the same as regular network, because the mean degree of nearest neighbors $\langle \xi_{nn} \rangle$ of a random network is almost the same as that of a regular network. Of course, from another view, random networks are different from regular networks. Unfortunately, scale-free networks have not received enough attention in Economics so far, except for network formation mechanism studies. Scale-free networks have not been used in models with network structures, and our study is possible because of the discovery of the scale-free network around 2000 and subsequent complex network studies. We need to mention it has been already shown that scale-free networks are in reality ubiquitous in the reality rather than exceptional, see Konno (2009).

In contrast to the existing models, our model considers the explicit network structure of knowledge spillover. Because we know that network structures determine outcomes of models, we need to have a framework dealing with such phenomena. Unlike preceding models, our model has no world technology frontier, no country, no firm, no agent playing that role , all the agents are affected by and affect other agents. Our model is not limited to the analysis of spillover among countries, but among any agents like firms,

[2]Of course, from another respect random regular network has significant importance.

regions, people, and so on. We show that the growth rate depends on the underlying network structure. In scale-free networks, the growth rate is greater than that of regular and random networks. We find that the long-term growth rate does not depend on degree, but on the global network characteristic, $\langle \xi_{nn} \rangle$. The growth rate of all the agents are the same only if they are connected to the network regardless of degree. However, TFP itself depends on degree and is proportional to it.

1.1 Outline of the Paper

The aim of the present paper is to provide a fundamental framework dealing with knowledge spillover. For this purpose, we demonstrate that it is indeed useful; it is simple and workable enough to analyze problems, and especially solvable. First, we explain spillover and spillover network, then briefly explain some types of representative networks. We introduce knowledge spillover equation, solve it analytically, and show other different kinds of spillover equations: degree dependent network, CES spillover, directed network, multi-technology network, hierarchical network. We also show the relationship to the existing model and solve it by different method. We compare growth rates among representative three kinds of networks. We use the equation to show the relationship among spillover, growth rate, and distance. We show that network formation mechanism is studied by our knowledge spillover equation. Finally, we state conclusion.

1.2 What is Knowledge Spillover on Networks?

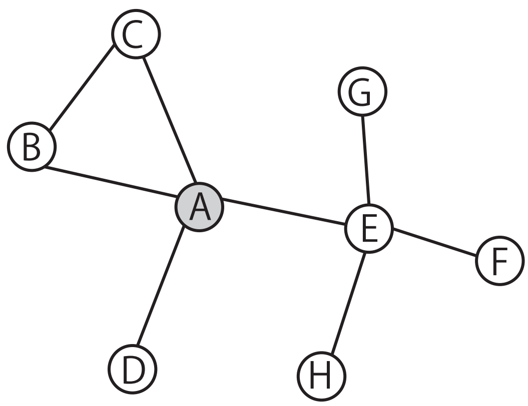

Figure 1: Knowledge Spillover Through a Network

Around 2000, the"Complex Networks Theory" arose and has stimulated a variety of scientific fields including Physics, Social Sciences, Biology and other sciences. However, there seem to be many areas in Economics where complex network theory can be applied and Economics has not yet conducted enough analysis using complex networks. In this study, we introduce complex networks into Economics and show how network structure, particularly "Scale-Free Networks" affects the outcome of the model. We believe economic growth theory is an excellent candidate for applying complex network analysis because the externality of TFP is significant and knowledge spillover can be regarded as a network structure.

First, we briefly explain knowledge spillover on a network. In the modern industrialized society, countries, firms, and people communicate with each other to acquire new information and enhance their knowledge. For example, why are many firms built in famous research centers such as Silicon Valley ? Why do many firms exist in big cities? Despite the cost and congestion, firms in these locales can more easily acquire information from other firms to improve their productivity, a phenomena called "knowledge spillover". In this paper, we study this effect with explicit network structures. For instance, some firms obtain spillover from other firms and other firm do not obtain spillover, we describe these relations as network structures.

Fig.1 illustrates an example of this "knowledge spillover through a Network". The vertices represent, for example, firms[3], (here eight firms). The edges represent the knowledge spillover relationship. Firm-A enhances its TFP by receiving spillover from adjacent firms, C, B, D ,and E. For example, if firm-B develops a new technology, then firm-A acquires the information and also enhances its TFP. However, a firm far away from firm-B, in terms of physical or informational distance, is not able to enhance its TFP instantaneously from firm-B. The distant firm has to wait for a time until the information comes through the network.

1.3 Brief Introduction to Complex Networks

In this section, we explain three type of representative networks. We explain the only minimum information on "Complex Networks" to understand this paper.

[3]The agents on vertices can represent not only firms but also people, countries, cities, and so on. Here, for simplicity, we regard them as firms.

1.3.1 Regular Network

Fig.3 illustrates a regular network. Regular network is the network in which all the vertices have the same degree. Degree is the number of edges the vertex has. In this example, the degrees of all the vertices are the same, four. Fig.3 is the degree distribution, which is the delta function, $P(\xi) = \delta(\xi - 4)$. Here, ξ stands for degree.

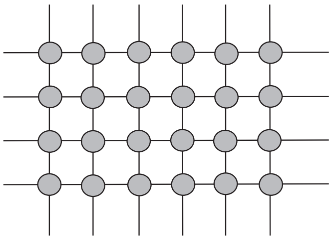

Figure 2: Regular Network

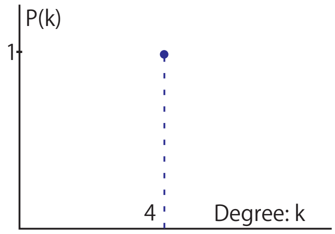

Figure 3: Degree Distribution

1.3.2 Complete Network

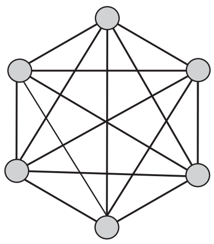

Figure 4: Complete Network

Fig.4 illustrates a complete network, we will discuss in Section 3.1. The definition is simple, every vertex is connected to all the other vertices. The complete network is a regular network.

1.3.3 Random Network

A random network (P.Erdos and A.Renyi (1959)) is constructed as follows. Choose two vertices, then connect them in probability p and do not connect them in probability $1 - p$. After doing this procedure for all the pairs of the vertices, we do this $\frac{V(V-1)}{2}$ times and V is the number of vertices on the entire network, then we have a random network. Taking $p \to 0$ with keeping

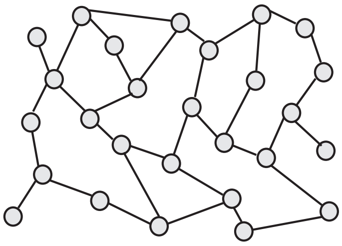

Figure 5: Random Network

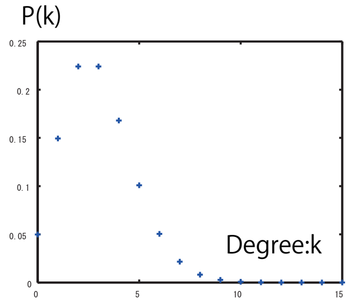

Figure 6: Degree Distribution

$np = \lambda$ so that mean degree is kept constant, the degree distribution becomes a Poisson distribution such as

$$P(k) \sim \frac{e^{-\lambda}\lambda^k}{k!} \tag{12}$$

which is illustrated in Fig.6. In other words random network is the network in which any pair of vertices is connected in the constant probability. The random networks explained here might be called Poisson random network, because there are other classes of random networks. However, in the present paper, Poisson random network is referred to as random network. Many network literatures call Poisson random networks as random networks. If we needed to discriminate Poisson random networks from other classes of random networks, we would call it as Poisson random, however, in the present paper it is not necessary. Remember that if you see "Random networks", in most cases it means Poisson random networks as the present paper does.

1.3.4 Scale-free Network

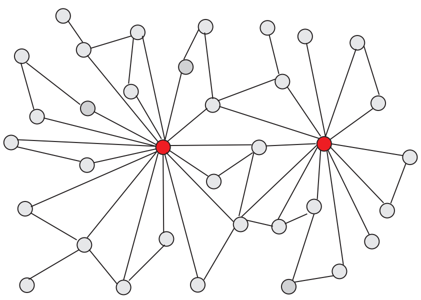

Figure 7: Scale-free Network

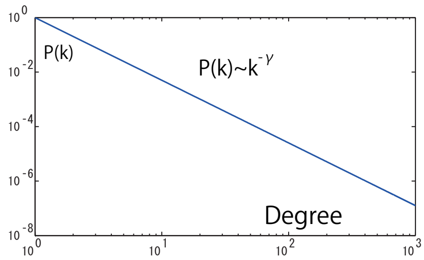

Figure 8: Degree Distribution

The networks we have explained thus far are "classical networks", in contrast, the "scale-free network" is a member of "Complex Network". It was discovered that, contrary to our assumption, many real networks are not random networks, but scale-free networks. It was also found that underlying network structure, in particular scale-free network, drastically changes the outcomes of the models. This is the reason why complex networks have attracted considerable interest. It is not too much to say that the explosion of Complex Networks literature begun with the discovery of scale-free networks and small world networks[4].

The scale-free network and its degree distribution in log log plots are illustrated in Fig.7 and Fig.8. The scale-free network is defined as a network with the following degree distribution

$$P(\xi) \sim \xi^{-\gamma} \tag{13}$$

In random networks vertices have almost the same degree however, in a scale-free network, there exists a very high degree. In this paper, we also show how the underlying network structure changes the outcome of the economy. We are going to discuss in Section 3.4 that inter-firm transaction network is also a scale-free network, Konno (2009). Generally speaking, scale-free network is constructed when links are formed by preferential attachment. Preferential attachment is such a mechanism that the more degree a vertex has, the more likely the vertex attract new link. It is something like winner takes all. It is yet recognized that social networks like friendship networks have scale-free structures.

1.3.5 The difference between $\langle\xi\rangle$ and $\langle\xi_{nn}\rangle$

We will use the important fact that the mean degree $\langle\xi\rangle$ is different from the mean degree of nearest neighbors $\langle\xi_{nn}\rangle$ repeatedly. It is this difference which brings many interesting phenomenon in complex networks. A good example is epidemic spread, Pastor-Satorras and Vespignani (2001) show that epidemic threshold does not exist in scale-free networks with $\gamma \leq 3$, which is typical parameter of real scale-free networks. They show that epidemic explosion always breaks out in scale-free networks. Then, what do $\langle\xi\rangle$ and $\langle\xi_{nn}\rangle$ mean ? First, we will explain it by words, then by figures, and finally demonstrate they actually differ contrary to our naive assumption with an example. First, chose a vertex randomly. The mean degree of randomly chosen vertices is, by definition, $\langle\xi\rangle$. What about the mean degree of the vertices linked to the randomly chosen vertices? It is denoted by $\langle\xi_{nn}\rangle$; the

[4]Small world networks have the following two properties: high clustering coefficient and small average path length. Clustering coefficient is a measure of the local density of relationships.

"nn" stands for nearest neighbors. You may think that it must be the same as $\langle \xi \rangle$; however, it is not the case. Only when the network is a regular network, $\langle \xi \rangle = \langle \xi_{nn} \rangle$ holds true. The mean degree, $\langle \xi \rangle$, and the mean degree of nearest neighbors, $\langle \xi_{nn} \rangle$, are illustrated in Fig.9.

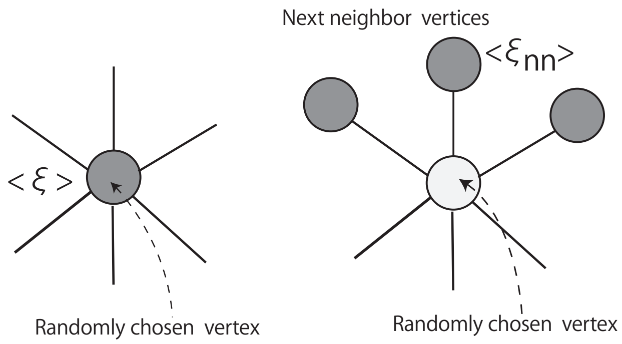

Figure 9: Explanation for $\langle \xi \rangle$ and $\langle \xi_{nn} \rangle$

We would like to demonstrate that $\langle \xi \rangle$ and $\langle \xi_{nn} \rangle$ are really different using Fig.10. In the figure, the degrees of the vertices are $\xi_A = 1$, $\xi_B = 2$, and $\xi_C = 1$, respectively. The mean degree is

$$\langle \xi \rangle = \frac{1}{3}(1+2+1) = \frac{4}{3} \tag{14}$$

On the other hand, mean degree of nearest neighbors, $\langle \xi_{nn} \rangle$, is given by

$$\begin{aligned} \langle \xi_{nn} \rangle =& \frac{1}{3}\left(\xi_B + \frac{\xi_A + \xi_C}{2} + \xi_B\right) \\ =& \frac{1}{3}\left(2 + \frac{1+1}{2} + 2\right) = \frac{5}{3} \end{aligned} \tag{15}$$

Actually, $\langle \xi \rangle \neq \langle \xi_{nn} \rangle$ holds true in the even simple network illustrated in Fig.10. For uncorrelated networks in which degree-degree correlation is absent $\langle \xi_{nn} \rangle = \frac{\langle \xi^2 \rangle}{\langle \xi \rangle}$.

1.3.6 Review Papers

We raise some reviews as to complex networks for the interested readers, Vega-Redondo (2007)
,S.N.Dorogovtesev and J.F.F.Mendes (2003). Albert and Barabási (2002), Newman (2003), Jackson (2008), and Goyal (2007).

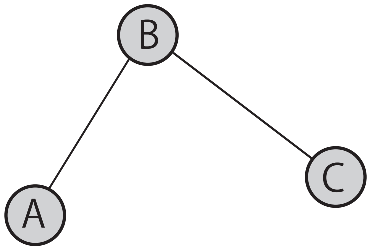

Figure 10: $\langle \xi \rangle \neq \langle \xi_{nn} \rangle$

2 Knowledge Spillover and TFP Growth Rate

2.1 Mathematical Expression for Technology Diffusion in Networks

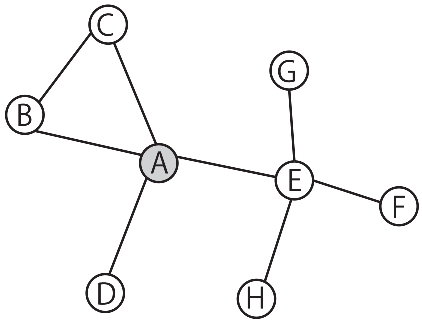

Figure 11: Knowledge Spill over through Network

$$A_j(\xi_j, t + \triangle t) = \underbrace{(1 + \rho \triangle t) A_j(\xi_j, t)}_{\text{Self}-\text{Evolution}} + \underbrace{\sum_{i \in \partial j} w_{ji} \delta_N A_i(\xi_i, t) \triangle t}_{\text{Spill Over Effect}} \tag{16}$$

The equation above is the starting point, where knowledge spillover is expressed. The TFP of the firm-j, $A_j(\xi_j, t)$, evolves according to this equation. Note that although we write $A_j(\xi_j)$, TFP is determined by degree ξ, thus it might be better to write it as $A(\xi_j)$. However, we write it as $A_j(\xi_j)$, to show clearly whose TFP we are discussing. The first term $(1 + \rho \triangle t)$ is the self evolution effect. Without any spillover from other firms, firms increase their TFP by $(1+\rho \triangle t)$ after $\triangle t$. Before explaining the network effect of knowledge spillover, we need to explain some terms and conventions. $i \in \partial j$ means all the vertices adjacent to vertex-j. For example, in Fig.1, $\{B, C, D, E\} \in \partial A$ means that firm-j receives spillover from all the adjacent firms. ξ_j is the degree of vertex j. The degree stands for the number of edges the vertex has; for example $\xi_A = 4$. δ_N is the depreciation factor of the network spillover effect. w_{ji} is the weight, meaning how strongly agent-i and agent-j are connected, in the present context in terms of information flow. For a while we will assume that all the weights, w_{ij} are equal to w. The equation (16) has the characteristics of the level effect, such that the agent with low TFP receives great deal of spillover from the agent with high TFP, and conversely the agent with high TFP receives little spillover from the agent with low TFP.

After transforming Eq.(16) into continuous form, we have,

$$\dot{A}_j(\xi_j, t) = \rho A_j(\xi_j, t) + \delta_N w \sum_{i \in \partial j} A_i(\xi_i, t) \tag{17}$$

Then, we are going to show the method of solving TFP evolution equation (17) analytically using the method called "mean field approximation" which replaces other elements with the ensemble average.

Fig.13 schematically shows mean field approximation. We see the problem of firm-A in Fig.1. In mean field approximation, we replace the degrees of firms adjacent to firm-A by its mean, $\langle \xi_{nn} \rangle$. The "nn" stands for next neighbor.

The notation $\langle \cdots \rangle$ means ensemble average, $\langle x \rangle \equiv E(x)$. In this respect, the problem of the firm with degree ξ becomes as follows. (We label the vertex with degree ξ as j for convenience.)

$$\begin{aligned}\dot{A}_j(\xi, t) &= \rho A_j(\xi, t) + \delta_N w \sum_{i \in \partial j} A_i(\xi_i, t) \\ &= \rho A_j(\xi, t) + \delta_N w \xi A(\langle \xi_{nn} \rangle, t)\end{aligned} \tag{18}$$

To solve above Eq.(18), we need to know $A(\langle \xi_{nn} \rangle, t)$. We also use mean field approximation to obtain this. The vertex with degree $\langle \xi_{nn} \rangle$ is also surrounded by the vertices with degree $\langle \xi_{nn} \rangle$ as illustrated by Fig.14

Thus, in a similar fashion, we have the equation for the vertex with degree $\langle \xi_{nn} \rangle$ as,

$$\dot{A}(\langle \xi_{nn} \rangle, t) = \rho A(\langle \xi_{nn} \rangle, t) + \delta_N w \langle \xi_{nn} \rangle A(\langle \xi_{nn} \rangle, t) \tag{19}$$

Then we have the solution for the firm with degree $\langle \xi_{nn} \rangle$ as follows.

$$A(\langle \xi_{nn} \rangle, t) = A_{nn}(0) \exp\left(\delta_N w \langle \xi_{nn} \rangle + \rho\right) t \tag{20}$$

Figure 12: Mean Field Approximation

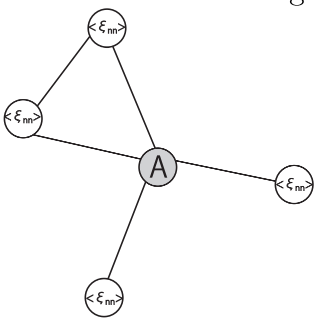

Figure 13: Around "A"

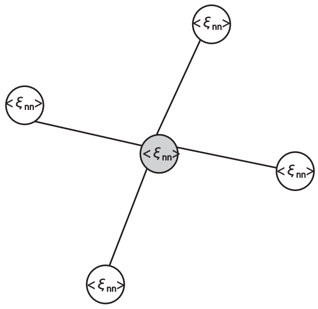

Figure 14: Around $\langle\xi_{nn}\rangle$

where, $A_{nn} \equiv A(\langle\xi_{nn}\rangle, 0)$.

Next, substitute Eq.(20) into Eq.(18) and solve the differential equation to find

$$A(\xi, t) = \frac{A_{nn}(0)}{\langle\xi_{nn}\rangle}\xi e^{(\delta_N w\langle\xi_{nn}\rangle+\rho)t} + \{A(\xi, 0) - \frac{A_{nn}(0)}{\langle\xi_{nn}\rangle}\xi\}e^{\rho t} \tag{21}$$

The asymptotic solution of Eq.(17), as $t \to \infty$, becomes

$$A(\xi, t) \sim \xi \; e^{(\rho+\delta_N w\langle\xi_{nn}\rangle)t} \quad (t \sim \infty) \tag{22}$$

This is the method by which we derive the solution. The point is that the growth rate of TFP for each vertex is common across all the distinct agents because $\langle\xi_{nn}\rangle$ is the global value for the entire network.

$$g_A = \frac{\dot{A}_j(\xi_j)}{A_j(\xi_j)} = \frac{\dot{A}}{A} = \delta_N w\langle\xi_{nn}\rangle + \rho \tag{23}$$

The growth rate of $A(\xi_i)$ is common across all the firms, regardless of degree ξ_i, whereas the TFP of individual firm depends on degree ξ_i. Therefore, if we regard degree as a result of investment, and a vertex as a country, we can interpret the properties of the model to demonstrate that the growth rate does not depend on investment and is common across all the linked countries; however, the TFP level depends on investment.

Proposition 1 (Knowledge Spillover)**.**
There are three points as follows:

1. *The TFP spillover equation Eq.(16) is solved analytically.*

$$A(\xi, t) \sim \xi \; e^{(\delta_N w\langle\xi_{nn}\rangle+\rho)t} \tag{24}$$

2. *The growth rates are the same across all the agents linked to the network and independent of their degree.*

$$g_A \equiv \frac{\dot{A}_j(\xi_j)}{A_j(\xi_j)} = \frac{\dot{A}}{A} = \delta_N w \langle \xi_{nn} \rangle + \rho \tag{25}$$

3. *The level of TFP is proportional to the degree.*

$$\frac{A(\xi_j, t)}{A(\xi_i, t)} = \frac{\xi_j}{\xi_i} \tag{26}$$

2.2 Degree Dependent Network Depreciation

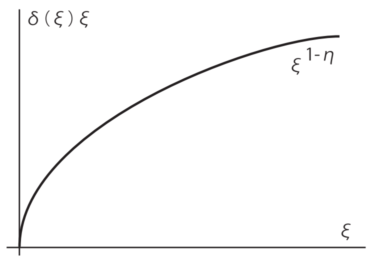

Figure 15: $\delta(\xi) \cdot \xi = \delta_N \xi^{1-\eta}$

We assumed that network depreciation, δ_N, is constant so far. However, it is natural to assume that it depends on the degree. If the degree is small, an agent receives much spillover from one agent. On the other hand, if the degree is large an agent receives much spillover from all the adjacent agents, but little from one agent. Mathematically, this relation is described as

$$\begin{aligned} \frac{\partial \delta_N(\xi)\xi}{\partial \xi} &> 0 \\ \frac{\partial^2 \delta_N(\xi)\xi}{\partial \xi^2} &\leq 0 \end{aligned} \tag{27}$$

For the time being, we employ a special case where $\delta_N(\xi) = \delta_N \xi^{-\eta}$. From the condition, Eq.(27), we have $0 \leq \eta < 1$. The knowledge spillover equation becomes

$$A_j(\xi_j, t + \triangle t) = (1 + \rho \triangle t) A_j(\xi_j, t) + \delta_N(\xi_j) \sum_{i \in \partial j} w_{ji} A_i(\xi_i, t) \triangle t \tag{28}$$

The solution, by mean field approximation, is then

$$A(\xi, t) \sim \xi^{1-\eta} \exp\left(\rho + w\delta_N \langle \xi_{nn}^{1-\eta} \rangle\right) t \tag{29}$$

where, $0 \leq \eta < 1$. In general, if $\delta_N \equiv \delta_N(\xi)$, the solution becomes

$$A(\xi, t) \sim \xi \delta_N(\xi) \exp\left[\rho + w\langle \delta_N\left(\xi_{nn}\right) \cdot \xi_{nn} \rangle\right] t \tag{30}$$

2.3 CES Type Spillover

Eq.(16) describes only the linear spillover effect only. We describe more general spillover equation which incorporates the linear spillover equation. We introduce CES spillover equation represented by

$$\triangle A(\xi_j, t) = \left[\alpha_I \left(\rho A(\xi_j, t)\triangle t\right)^{\frac{\theta-1}{\theta}} + \alpha_N \left\{\sum_{i\in\partial J} \delta_N w_{ji} A(\xi_j, t)\triangle t\right\}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{31}$$

In Eq.(31), the effect of self-evolution ρ and spillover effect $\sum_{i\in\partial j} A(\xi_j, t)$ are mixed. Even in this case, we have asymptotic solution of Eq.(31) with the properties stated in the following proposition 2

$$A(\xi_j, t) \sim \xi_j \exp\left[\alpha_I \rho^{\frac{\theta-1}{\theta}} + \alpha_N (\langle \xi_{nn} \rangle \delta_N w)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} t \tag{32}$$

Proposition 2 (CES Spillover)**.**
The solution of knowledge spillover of CES form, Eq.(31), also has the two characteristics.

- *Growth rate, g_A, is common only if an agent is connected to the entire network.*

$$g_A = \left[\alpha_I \rho^{\frac{\theta-1}{\theta}} + \alpha_N (\langle \xi_{nn} \rangle \delta_N w)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{33}$$

- *TFP level is proportional to degree ξ*

When $\theta \to 1$, the solution of spillover equation of CES form, Eq.(32), becomes Cobb-Douglas form,

$$A(\xi_j, t) \sim \xi_j \exp\left[\rho^{\alpha_I} \cdot (\langle \xi_{nn} \rangle \delta_N w)^{\alpha_N})\right] t \tag{34}$$

This CES form spillover equation contains linear spillover equation as a special case. When $\theta \to \infty$, then Eq.(32) becomes linear type

$$A(\xi_j, t) \sim \xi_j \exp\left[\alpha_I \rho + \alpha_N \langle \xi_{nn} \rangle \delta_N w\right] t \tag{35}$$

which is equivalent to Eq.(22)

2.4 Directed Network

Roughly speaking, there are two kinds of networks: undirected and directed. We have dealt with undirected networks only so far. We now introduce knowledge spillover with a directed network. For undirected networks, if two agents are connected, then the both agents can receive knowledge spillover from each other. With directed networks, however, the directions of links are introduced. In our model, the direction of link is the direction of knowledge flow of spillover. In Fig.16, there are three agents, say firm-A, firm-B, and firm-C. The arrow between firm-A and firm-B means that knowledge flows from firm-A to firm-B only, it does not flow from firm-B to firm-A. Between firm-A and firm-C, knowledge flows in both directions as in undirected networks. We need to introduce another notion of the degree, that is in-degree ξ^{in} and out-degree ξ^{out}. The names describe what they are. $\xi^{\text{in}}_A = 2,\ \xi^{\text{out}}_A = 3,$ $\xi^{\text{in}}_B = 4,\ \xi^{\text{out}}_B = 1$.

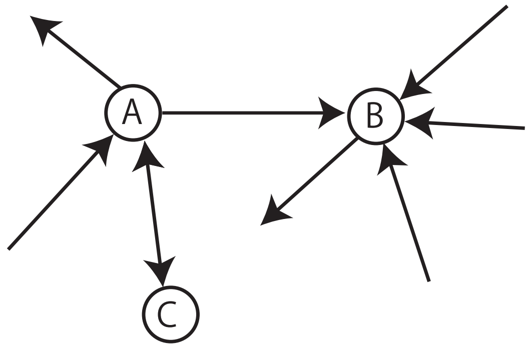

Figure 16: Directed Network

For directed network, the knowledge spillover equation becomes

$$\frac{d}{dt}A_j(\xi^{\text{in}}_j, \xi^{\text{out}}_j, t) = \rho A_j(\xi^{\text{in}}_j, \xi^{\text{out}}_j, t) + \delta_N \sum_{i \in \partial j} w_{ji} A_i(\xi^{\text{in}}_i, \xi^{\text{out}}_i, t) \tag{36}$$

For directed networks, $i \in \partial j$ has a different meaning. In Fig.16, $B \in \partial A$ holds true, however, $A \notin \partial B$, because the link points from A to B only. We express the mean field solution of Eq.(36) as

$$A(\xi^{\text{in}}, \xi^{\text{out}}, t) \sim \xi^{\text{in}} \exp\left(\rho + \delta_N w \langle \xi^{\text{in}}_{nn} \rangle\right) t \tag{37}$$

Here it is worth noting that the condition, $\langle \xi^{\text{in}} \rangle = \langle \xi^{\text{out}} \rangle$, must be satisfied because the total number of in-degrees for the entire network must be equal to the total number of out-degrees for the entire network. Out-degree, as well as in-degree plays a role in determining knowledge spillover. Eq.(37) has the similar characteristics.

Proposition 3 (Knowledge Spill Over for Directed Networks)**.**
Knowledge spillover on directed networks has the following properties.

- *Knowledge spillover for directed networks has the solution:* $A(\xi^{in}, \xi^{out}, t) \sim \xi^{in} \exp\left(\rho + \delta_N w \langle \xi^{in}_{nn} \rangle\right) t$
- *The growth rate,* g_A*, is common only if the agent is linked to the entire network.* $g_A = \rho + \delta_N w \langle \xi^{in}_{nn} \rangle$
- *The TFP level of each agent is proportional to in-degree.* $A_j \propto \xi^{in}$

2.5 Relationship with Existing Models

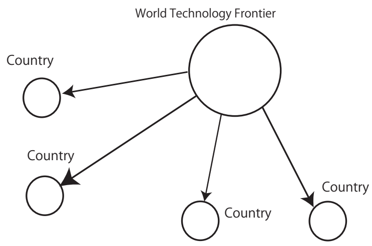

Figure 17: Countries and the World Technology Frontier

The existing model we have described in Section 1 can be taken as a special case of our model. Eq.(6) becomes

$$\dot{A}_j(t) = (\lambda_j - \sigma_j)\, A_j(t) + \sigma_j A_{W.F.}(t) \tag{38}$$

$$\dot{A}_{W.F}(t) = g A_{W.F}(t) \tag{39}$$

We compare Eq.(38) with our knowledge spillover Eq.(16) of directed networks, and for convenience we provide our equation again,

$$\dot{A}_j(\xi^{\mathrm{in}}_j, t) = \rho A_j(\xi^{\mathrm{in}}_j, t) + \delta_N w \sum_{i \in \partial j} A_i(\xi^{\mathrm{in}}_i, t) \tag{40}$$

If we consider such a network structure as illustrated in Fig.17, and set

$$\rho = \lambda_j - \sigma_j \tag{41}$$

$$\delta_N w = \delta_j \tag{42}$$

The countries improve their knowledge, A_j, by themselves without spillover at the rate of $\rho = \lambda_j - \delta_j$. The countries receive knowledge spillover from the world technology frontier with a depreciation rate, $\delta_N w = \delta_j$. The network is directed in that only ordinary countries receive spillover from the world technology frontier, while the world technology frontier does not receive any spillover. In Fig.17, all the countries are the same and thus symmetric. It is known that the existing model has the level effect that the low TFP agent receives a high amount of knowledge spillover from the high TFP agent, and conversely the high TFP agent receives little from the low TFP agent. Because the existing model can be taken as a special case of our model, it is confirmed that our model also has the level effect.

2.6 Multi Technology

So far, for the sake of simplicity, we have assumed that there is only one kind of knowledge A in the economy. However, the reality is that there are many kinds of technologies, and firms use several kinds of technologies. Each kind of technology diffuses and this process can be regarded as a network structure. The technology of firm-j, A_j, consists of M kinds of different technologies that can be described as

$$A_j \equiv A^{\phi_1^j}(\xi_{j,1};1) \cdot A^{\phi_2^j}(\xi_{j,2};2) \cdots A^{\phi_{M-1}^j}(\xi_{j,M-1};M-1) \cdot A^{\phi_M^j}(\xi_{j,M};M) \quad (43)$$

$$\phi_1^j + \phi_2^j + \cdots + \phi_M^j = 1 \quad (44)$$

We assume that the weight is always 1, $w_{ji} = 1 \; \forall i, j$. The knowledge diffusion equation for multi-technology becomes

$$\begin{aligned}
A(\xi_{j,1}, t + \triangle t; 1) &= (1 + \rho_1 \triangle t) A(\xi_{j,1}, t; 1) + \sum_{i \in \partial j} \delta_{N,1} A(\xi_{i,1}, t; 1) \triangle t \\
A(\xi_{j,2}, t + \triangle t; 2) &= (1 + \rho_2 \triangle t) A(\xi_{j,2}, t; 2) + \sum_{i \in \partial j} \delta_{N,2} A(\xi_{i,2}, t; 2) \triangle t \\
&\vdots \\
A(\xi_{j,M}, t + \triangle t; M) &= (1 + \rho_M \triangle t) A(\xi_{j,M}, t; M) + \sum_{i \in \partial j} \delta_{N,M} A(\xi_{i,M}, t; M) \triangle t
\end{aligned} \quad (45)$$

The solutions for each technology are

$$
\begin{aligned}
A(\xi_{j,1}, t; 1) &= \xi_{j,1} \exp\left(\rho_1 + \delta_{N,1}\langle \xi_{nn,1}\rangle\right) t \\
A(\xi_{j,2}, t; 2) &= \xi_{j,2} \exp\left(\rho_2 + \delta_{N,2}\langle \xi_{nn,2}\rangle\right) t \\
&\vdots \\
A(\xi_{j,M}, t; M) &= \xi_{j,M} \exp\left(\rho_M + \delta_{N,M}\langle \xi_{nn,M}\rangle\right) t
\end{aligned}
\tag{46}
$$

The total technology of firm-j is represented by Eq.(43). Substituting Eq.(46) into Eq.(43), omitting subscript j, we have

$$
\begin{aligned}
&A_j(\{\xi_{j,1}, \cdots, \xi_{j,M}\}) \\
&= \xi_{j,1}^{\phi_1^j} \times \xi_{j,2}^{\phi_2^j} \times \cdots \times \xi_{j,M}^{\phi_M^j} \exp\left[\phi_1\rho_1 + \cdots + \phi_M\rho_M + \phi_1\delta_{N,1}\langle \xi_{nn,1}\rangle + \cdots + \phi_M\delta_{N,M}\langle \xi_{nn,M}\rangle\right] t \\
&= \prod_i^M \xi_i^{\phi_i^j} \exp\left[\sum_l^M \phi_l^j \rho_l + \sum_s^M \phi_s^j \delta_{N,s}\langle \xi_{nn,s}\rangle\right] t
\end{aligned}
\tag{47}
$$

We let g_A^m denote the growth rate of technology m that is common to all the connected agents represented by

$$
g_A^m \equiv \rho_m + \delta_{N,m}\langle \xi_{nn,m}\rangle \tag{48}
$$

The growth rate for firm-j is a linear combination of the growth rate of each technology represented by

$$
g_A(j) = \frac{\dot{A}_j(\{\xi_{j,1}, \cdots, \xi_{j,M}\})}{A_j(\{\xi_{j,1}, \cdots, \xi_{j,M}\})} = \sum_{m=1}^M \phi_m^j g_A^m \tag{49}
$$

If ϕ_m^j does not depend on firm-j, $\phi_m^j = \phi_m$, then the growth rate for any firm is the same only if it is connected to the entire network, regardless of degree. The growth rate is represented by $g_A = \sum_{m=1}^M \phi_m g_A$. However, the TFP level of firm-j is proportional to a function of degree of the firm-j, represented by

$$
A_j(\{\xi_{j,1}, \cdots, \xi_{j,M}\}) \propto \xi_{j,1}^{\phi_1} \times \xi_{j,2}^{\phi_2} \times \cdots \times \xi_{j,M}^{\phi_M} \tag{50}
$$

Example $(M = 2)$

Suppose there are two kinds of technologies; thus, $M = 2$. ϕ_m^j does not depend on firm-j, and so is written as ϕ_m. The network depreciation rate is independent of the kind of technology, $\delta_{N,m} = \delta_N$. Each technology, $m = 1, 2$, spillovers through different networks. Knowledge spillover for technology-1 occurs through the network illustrated in Fig.18 and that for technology-2 goes through the network illustrated in Fig.19.

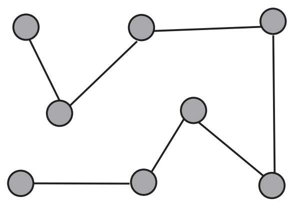

Figure 18: Technology-1

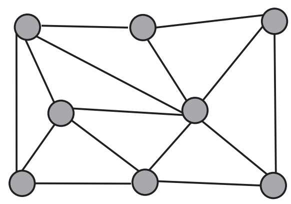

Figure 19: Technology-2

The technology of firm-j is represented by

$$A_j(\xi_{j,1}, \xi_{j,2}) = \xi_{j,1}^{\phi_1} \cdot \xi_{j,2}^{\phi_2} \cdot \exp\left(\phi_1\rho_1 + \phi_2\rho_2 + \delta_N(\phi_1\langle\xi_{nn,1}\rangle + \phi_2\langle\xi_{nn,2}\rangle)\right) t \quad (51)$$

Because we assumed that ϕ is independent of firm, the growth rate of the technology for any firm is common across all the firms

$$g_A = \phi_1\rho_1 + \phi_2\rho_2 + \delta_N(\phi_1\langle\xi_{nn,1}\rangle + \phi_2\langle\xi_{nn,2}\rangle) \quad (52)$$

On the other hand, the level of TFP for firm-j is proportional to a function of the degree $\xi_{j,1}, \xi_{j,2}$

$$A_j(\xi_{j,1}, \xi_{j,2}) \propto \xi_{j,1}^{\phi_1}\xi_{j,2}^{\phi_2} \quad (53)$$

Proposition 4 (Multi Technology Networks)**.**
Knowledge spillover with multi technologies has the following properties.

- *The solution is*

$$A_j(\{\xi_{j,1}, \cdots, \xi_{j,M}\}) = \prod_i^M \xi_i^{\phi_i^j} \exp\left[\sum_l^M \phi_l^j\rho_l + \sum_s^M \phi_s^j\delta_{N,s}\langle\xi_{nn,s}\rangle\right] t \quad (54)$$

- *The growth rate is given by* $\sum_{m=1}^M \phi_m^j g_A^m$

- *The TFP level of each agent is proportional to the product of the degree of each network* $\xi_{j,1}^{\phi_1} \times \xi_{j,2}^{\phi_2} \times \cdots \times \xi_{j,M}^{\phi_M}$

2.7 Hierarchical structure

We study knowledge spillover for the network with a hierarchical structure or community structure. Konno (2009) show that the clustering coefficient of inter-firm transaction network implies a hierarchical structure.

First, we have the solution for level-H network. In level-H network, each vertex that actually consists of small components is regarded as one component and grows at the rate of ρ_H without spillover effect.

$$A_H(\xi_H, t) \sim \xi_H \exp\left(\rho_H + \delta_{H,N} w_H \langle\xi_{H,nn}\rangle\right) t \quad (55)$$

And level-(H-1) network has the following solution

$$A_{H-1}(\xi_{H-1}, t) \sim \xi_{H-1} \exp\left(\rho_{H-1} + \delta_{H-1,N} w_{H-1} \langle \xi_{H-1,nn} \rangle\right) t \tag{56}$$

ρ_H can be represented by the growth rate of level-(H-1), $g_{A,H-1}$. In this example, $\rho_H = \rho_{H-1} + \delta_{H-1,N} w_{H-1} \langle \xi_{H-1,nn} \rangle$.

In general, we have

$$\rho_h = \rho_{h-1} + \delta_{h-1,N} w_{h-1} \langle \xi_{h-1,nn} \rangle \tag{57}$$

$$\begin{aligned} g_{A,h} &= \rho_h + \delta_{h,N} w_h \langle \xi_{h,nn} \rangle \\ &= \rho_{h-1} + \delta_{h-1,N} w_{h-1} \langle \xi_{h-1,nn} \rangle + \delta_{h,N} w_h \langle \xi_{h,nn} \rangle \end{aligned} \tag{58}$$

where we denote hierarchy level by h. More generally, we have

$$g_{A,h} = \rho_1 + \sum_{j=1}^{h} \delta_{j,N} w_j \langle \xi_{j,nn} \rangle \tag{59}$$

where $h = 1$ is the lowest level of the hierarchy.

An Example of Hierarchical Network

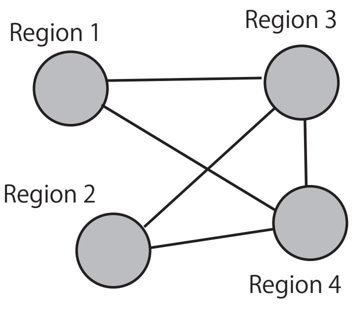

Figure 20: Region Network 1

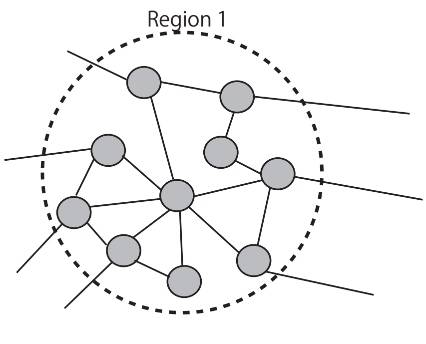

Figure 21: Firms in one Region

We explain a hierarchical structure with an example. There are four regions affecting each other as illustrated in Fig.20. In each region, there are firms forming the spillover network within one region as illustrated in Fig.21. Regions and firms constitute the hierarchical network; thus it is 2-level as illustrated in Fig.22. Suppose that self evolution rate of the firms is ρ_1, and the mean degree of nearest neighbors of the firms' network within one region is $\langle \xi_{1,nn} \rangle$, where 1 denote hierarchy level. Without spillover from other regions, the firms, in other words, and regions grow at the rate of

$$\rho_2 = \rho_1 + \delta_{1,N} w_1 \langle \xi_{1,nn} \rangle \tag{60}$$

As illustrated in Fig.20 regions form spillover network. Let $\langle \xi_{2,nn} \rangle$ denote the mean degree of nearest neighbors of the network. With the spillover from other regions, the regions and the firms grow at the rate of

$$\begin{aligned} &\rho_2 + \delta_{2,N} w_2 \langle \xi_{2,nn} \rangle \\ =&\rho_1 + \delta_{1,N} w_1 \langle \xi_{1,nn} \rangle + \delta_{2,N} w_2 \langle \xi_{2,nn} \rangle \end{aligned} \tag{61}$$

If all of the spillovers are considered, the growth rate of each firm in the network is given by Eq.(61). Although we described as if regions formed the network, actually firms form the network as illustrated in Fig.22. The network among firms across different regions can be viewed as the regions' network; it is also mean field picture, and this prescription makes it possible for us to calculate the growth rate as we did. Note that we do not exclude any picture in which not only firms but also regions like cities or countries form a network. What we did here is that network of firms across different regions can be regarded as a network formed by regions.

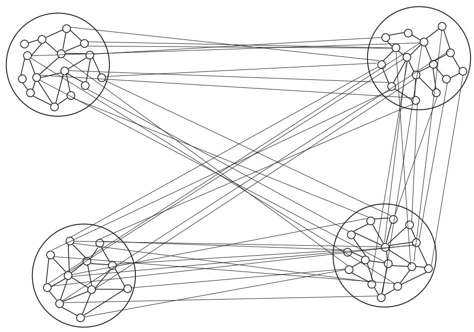

Figure 22: Region Network 2

2.8 Bethe Approximation

We can solve Eq.(17) in a different way. So far, we have used mean field approximation in which the next neighbor degrees are replaced by the mean degree of nearest neighbors, $\langle \xi_{nn} \rangle$. Instead of this method, we will apply a kind of Bethe approximation. Unlike mean field approximation, next neighbor degrees are exact values; however, the next neighbors of next neighbor degrees are replaced by the mean degree of nearest neighbors. As a result, the technology of the agent with degree ξ_j becomes

$$
\begin{aligned}
A_j(\xi_j, t) &\sim \sum_{i\in\partial j} \frac{\xi_i}{\langle \xi_{nn} \rangle} \exp(\rho + \delta_N w \langle \xi_{nn} \rangle) t \\
&\sim \sum_{i\in\partial j} \xi_i \exp(\rho + \delta_N w \langle \xi_{nn} \rangle) t
\end{aligned}
\tag{62}
$$

To show what $\sum_{i\in\partial j} \xi_i$ means clearly, we write it as

$$
\sum_{i\in\partial j} \xi_i = \xi_1 + \xi_2 + \xi_3 + \cdots + \xi_{\xi_j} \tag{63}
$$

$1, 2, 3, \cdots, \xi_j$ are the labels of the vertices connected to agent-j. Eq.(63) is the sum of all the degrees of the vertices connected to agent-j. It is confirmed that Eq.(62) is consistent with our mean field approximation method. (However, bethe approximation is actually a kind of mean field.) Mean field approximation is the way in which all the ξ_i in Eq.(63) are replaced by $\langle \xi_{nn} \rangle$. In this case, Eq.(62) becomes the mean field solution, Eq.(22). The TFP level of agent-j is proportional to $\sum_{i\in\partial j} \xi_i = \xi_1 + \xi_2 + \xi_3 + \cdots + \xi_{\xi_j}$, which is total sum of the degrees of all the agents adjacent to agent-j. In this sense, growth rate is not only dependent on global network structure but also local.

2.9 Weights of the Networks

The weight, w_{ji}, means how strongly firm-i and firm-j are connected. In so far, We have assumed that weights of the networks are the same and are written by w. Bethe approximation is not only better approximation, but also let weights w_{ji} play a role. The TFP level of firm-j is given by

$$
A_j(\xi_j, t) \sim \sum_{i\in\partial j} w_{ji} \xi_i \exp(\rho + \delta_N \langle w \rangle \langle \xi_{nn} \rangle) t \tag{64}
$$

The TFP level is proportional to $\sum_{i\in\partial j} w_{ji}\xi_i$; however, the growth rate is still common across all the agent only if they are connected to the entire network.

3 Comparison of the Growth Rates of Different Network Structures

In this section, we compare growth rates across different kinds of representative networks.

3.1 The Method of Comparing Different Networks

Here, we explain the method to compare a factor across different network structures. Without appropriate method of comparison, complete network explained in Section 1.3.2 must have the largest growth rate, if linking were cost-free because all the vertices have the largest degree among all kinds of networks and each vertex links to all the other vertices, so that knowledge spillover is the greatest, and every vertex can learn from all the other vertices. Thus, we introduce an appropriate way to compare a factor among underlying network structures below.

1. The number of vertices on the entire network, N, is the same.

2. The mean degree of the network is the same.

Under these conditions, we can compare the growth rate among three representative networks: regular, random, and scale-free.

In order to compare growth rate, we use the same three values across three different networks, $\delta_N w = 0.5 \times 10^{-2}, \rho = 0.01, N = 10^9$, these values are network depreciation factor, the self-evolution rate, and the number of firms on the network, respectively. We use $\gamma = 2.9$ for the scale-free network parameter, so that the mean degree becomes $\langle \xi \rangle = 2.1$. Many real scale-free networks fall into this parameter space, $2 < \gamma \leq 3$. The other parameter values such as δ_N, ρ, and N do not have particular meaning; however, we believe that the order of magnitude of growth rates among three representative network is robust, regardless of the values of such parameters. This is because, as we will explain later, growth rates are mainly determined by the mean degree of nearest neighbors $\langle \xi_{nn} \rangle$. When we compare growth rate, as we have already discussed, we must keep the mean degree the same across all the three networks.

First, we need to calculate the mean degree of nearest neighbors, $\langle \xi_{nn} \rangle$.

3.2 Method for calculating $\langle \xi_{nn} \rangle$ for Three types of Networks

We demonstrate how to obtain $\langle \xi_{nn} \rangle$ in three kinds of networks: regular, random, and scale-free, given the that mean degree $\langle \xi \rangle$ is determined already. Here, the mean degree $\langle \xi \rangle$ is the same for all three kinds of networks.

3.2.1 Regular Network

For regular networks, to calculate $\langle \xi_{nn} \rangle$ is straightforward, because every vertex has the same degree. Hence, $\langle \xi_{nn} \rangle = \langle \xi \rangle$.

3.2.2 Random Network

For random networks, degree distribution is $P(\xi) \sim \frac{e^{-\lambda}\lambda^{\xi}}{\xi!}$ which is explained in Section 1.3.3. Thus it is possible for us to make use of the relationship $\langle x^2 \rangle - \langle x \rangle^2 = \langle x \rangle$ for any x that is a stochastic variable following a Poisson distribution.

$$\langle \xi_{nn} \rangle = \frac{\langle \xi^2 \rangle}{\langle \xi \rangle} = \langle \xi \rangle + 1 \tag{65}$$

Finally, we have the mean degree of nearest neighbors $\langle \xi_{nn} \rangle = \langle \xi \rangle + 1$ on the condition that mean degree $\langle \xi \rangle$ is given.

3.2.3 Scale-free Network

Figure 23: Scale-free Degree Distributions

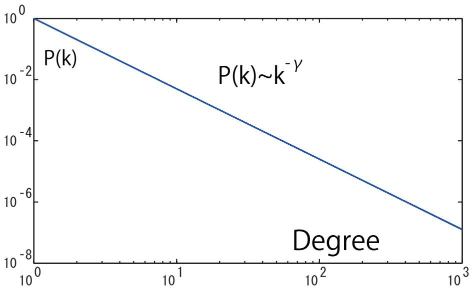

Figure 24: Ideal Degree Distribution

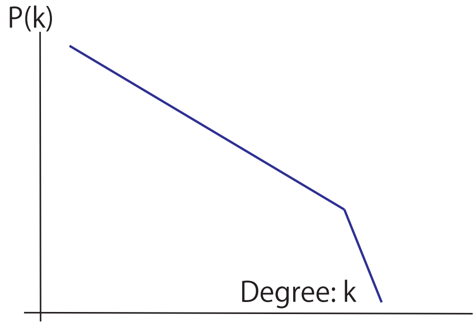

Figure 25: Finite Size

First, let us describe the finite size effect with an example. Thinking of the scale-free network whose parameter is $\gamma = 2$ then $P(\xi) = \xi^{-2}$

So for $\xi = 5$, the probability is $P(5) = 5^{-2} = \frac{1}{25}$. In addition, suppose we have $N = 10,000$ and ignore the normalization factor. N is the size of network. The expected number of firms whose degree is 5 is $N \times P(5) = \frac{10000}{25} = 400$. In this way, the expected number of firms whose degree is $k = 10$ becomes $N \times P(10) = \frac{10000}{10^2} = 100$.

In these two cases, both numbers are larger than 1, so it can be observed in reality. However, for $k = 100$, the number of firms with $\xi = 100$ is $N \times P(100) = 1$. For $\xi = 1,000$, the expected number of firms is $N \times P(\xi) = \frac{10,000}{1000^2} = 0.01 < 1$. This is less than 1. If the expected number of firms whose degree is ξ becomes less than 1, in reality a vertex with such a large degree is hard to observe.

We see the kink in Fig.25, in this example, the kink begins where the degree is 100. At this point, the expected number of firms becomes less than 1. Fig.24 illustrates the degree distribution of a scale-free network where the number of vertices are infinite, $N = \infty$. By contrast, Fig.25 illustrates the degree distribution of a network with finite vertices. There are two main methods for determining the cut off degree. In one method, we regard $\xi = 100$ as a cut-off parameter. We define $\xi_{max} = 100$. The cut-off parameter, ξ_{max}, is obtained by $N \times P(\xi_{max}) = 1$. However, we are going to use the other method, which ill be explained later,to determine the cut off degree.

Note that, we have used $P(\xi) \sim \xi^{-\gamma}$ so far without explicitly considering normalization. However, $P(k) = Z^{-1}\xi^{-\gamma}$ is right rather. Z is defined as $Z = \int P(\xi)d\xi$. Z^{-1} the is normalization factor. Then, we have

$$Z \equiv \int_1^\infty \xi^{-\gamma} d\xi = \frac{1}{\gamma - 1} \tag{66}$$

Together with this normalization factor Z, ξ_{max} is obtained in the following way.

$$\begin{aligned} N(\gamma - 1) \int_{\xi_{max}}^\infty \xi^{-\gamma} d\xi = 1 \\ \Rightarrow \xi_{max} = N^{\frac{1}{\gamma-1}} \end{aligned} \tag{67}$$

In a nutshell, Eq.(67) states that the number of firms that have more than ξ_{max} links is equal to one. In this way, we determine cut off degree, ξ_{max}.

Therefore, $\langle \xi \rangle$ under the finite size effect is represented by

$$\begin{aligned} \langle \xi \rangle &= \int_1^{\xi_{max}} Z^{-1} \xi \xi^{-\gamma} d\xi \\ &= \frac{\gamma - 1}{\gamma - 2} \left[1 - \xi_{max}^{2-\gamma}\right] \end{aligned} \tag{68}$$

$\langle \xi^2 \rangle$ and $\langle \xi_{nn} \rangle$ under the finite size effect are also determined by

$$\begin{aligned} \langle \xi^2 \rangle &= \int_1^{\xi_{max}} \xi^2 \xi^{-\gamma} Z^{-1} d\xi \\ &= \frac{\gamma - 1}{3 - \gamma} \left[\xi_{max}^{3-\gamma} - 1\right] \end{aligned} \tag{69}$$

$$\begin{aligned} \langle \xi_{nn} \rangle &= \frac{\langle \xi^2 \rangle}{\langle \xi \rangle} \\ &= \frac{\gamma - 2}{3 - \gamma} \left[\frac{\xi_{max}^{3-\gamma} - 1}{1 - \xi_{max}^{2-\gamma}}\right] \end{aligned} \tag{70}$$

3.3 Comparison of Growth Rates among Three Representative Networks

The growth rates comparison result is summarized in Table 1.

Table 1: Growth Rate

	Regular Network	Random Network	Scale-free Network
Growth Rate	0.021	0.026	0.059

The growth rates of regular and network do not differ much because the mean degree of nearest neighbors $\langle \xi_{nn} \rangle$ does not differ greatly between them.

The scale-free network illustrates the case where there exists some concentrated research center such as Silicon Valley or big business cities. On the other hand, the regular network is viewed as benefiting each place equally and having no industrially concentrated locations. In conclusion, we state that scale-free network supports a more efficient growth rate than does the regular or random network.

3.4 Network Structure: an Empirical Study

We showed that the scale-free network is the most efficient of three representative networks. The next logical question is what network structure the real economy has. We mention the empirical study by Konno (2009) who studied the inter-firm transaction network among 800,000 Japanese firms.

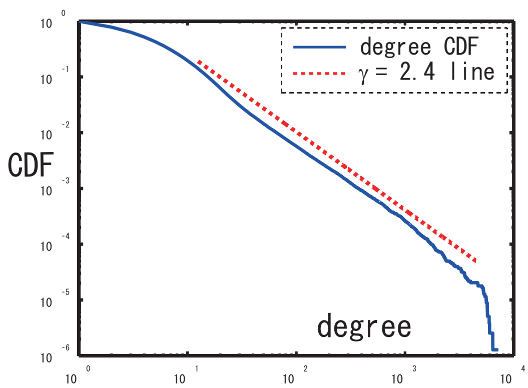

Figure 26: Degree Distribution of the Transaction Network

Fig.26 illustrates the degree distribution of the Japanese inter-firm relationship network. The figure actually shows that, contrary to our naive assumption that the network structure of the firms' transaction is a random network, the Japanese firms' transactions form scale-free network, in which degree distribution follows $P(\xi) \sim \xi^{-2.4}$. Konno (2009) also discovers hierarchical structure by using clustering coefficient, which is a measure for the local density of the network, and the existence of a degree-degree correlation, such as $\xi_{nn} \sim \xi^{\frac{1}{2}}$ exists. Therefore, scale-free network must have something to do with firms' relationship. Konno (2009) did not study knowledge spillover network directly, but the inter-firm transaction network; however, the transaction network is related to the spillover network, because it is also a network of contacts, and contacts are likely result in knowledge spillover.

3.5 Spillover in Star Network

We call the network illustrated in Fig.27 as a star network. The question that which network, scale-free or star network has higher growth rate arises. We will answer the question within the mean field framework.

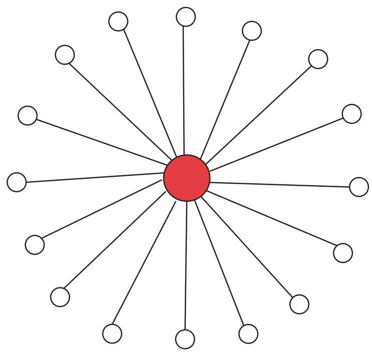

Figure 27: Star Network

The spillover equation for a star network is represented by

$$\begin{aligned} \frac{d}{dt}A(1,t) &= \rho A(1,t) + \delta_N w A(N,t) \\ \frac{d}{dt}A(N,t) &= \rho A(N,t) + \sum_{i=1}^{N} \delta_N A(1,t) = \rho A(N,t) + \delta_N w N A(1,t) \end{aligned} \tag{71}$$

where N denotes the number of vertices in the entire network, and find

$$\begin{aligned} A(1,t) &= A_0 e^{(\rho+\delta_N w\sqrt{N})t} \\ A(N,t) &= A_0\sqrt{N} e^{(\rho+\delta_N w\sqrt{N})t} \end{aligned} \tag{72}$$

The growth rate in a star network proves to be $\rho + \delta_N w\sqrt{N}$. Thus, the question arises whether the growth rate of star network, $\rho+\delta_N w\sqrt{N}$, or that of scale-free, $\rho + \delta_N w\langle\xi_{nn}\rangle$, is larger. The problem becomes whether $\sqrt{N}$ or $\langle\xi_{nn}\rangle$ is larger. The mean degree of star network is given by

$$\begin{aligned}\langle\xi\rangle^{\text{star}} &= 1\frac{N-1}{N} + (N-1)\frac{1}{N} \\ &\sim 2 \end{aligned} \tag{73}$$

That of star and scale-free network, and network sizes are kept fixed for comparison. We study the case where $2 < \gamma$, because if $\gamma \leq 2$ mean degree diverges as N goes infinity. The mean degree of the scale-free network and normalization factor ,Z, are given by

$$\langle\xi\rangle = Z^{-1}\int_1^N \xi\xi^{-\gamma}\,d\xi \tag{74}$$

$$Z = \int_1^N \xi^{-\gamma}\,d\xi \tag{75}$$

Because we must make mean degree of scale-free network equal to that of star network, $\langle\xi\rangle^{\text{star}} = 2$, from the above equations we have $\gamma = 3$ and $Z = 1/2$. The mean degree of nearest neighbors is given by

$$\begin{aligned}\langle\xi_{nn}\rangle &= \frac{\langle\xi^2\rangle}{\langle\xi\rangle} \\ &= \log(N) \end{aligned} \tag{76}$$

Because, $\log(N) < \sqrt{N}$ holds for large enough N, the growth rate in star network is higher than that of scale-free network. However, to keep mean degree and size of the two networks constant we cannot help imposing the special condition that $\gamma = 3$. It is noted that we compared star network with particular scale-free network only.

4 Numerical Simulation

We show that our solution Eq.(22), $A(\xi, t) \sim \xi \exp\left(\rho + \delta_N w\langle\xi_{nn}\rangle\right) t$, is valid by numerical simulations. We fix the mean degree and network size such that $\langle\xi\rangle = 10$ and $N = 1500$ for all the three networks: regular, random, and scale-free. In all the cases, $\delta_N \cdot w = 0.5 \times 10^{-3}$. The scale-free network is set to be $\gamma = 3$. It should be noted that the results are robust to initial values of A, because the they are obtained by largest eigenvalue of $\rho I + \delta_N\langle\xi_{nn}\rangle$, and the corresponding eigenvector.

4.1 Growth Rate

The results are summarized in Table 2. We did three cases: $\rho = 0.01,\ 0.005$, and 0. For each case, the theoretical values, $g_A = \rho + \delta_N w \langle \xi_{nn} \rangle$, written within parenthesis, are in good agreement with simulation values. The growth rate is common across all the agents. This simulation confirms that the growth rate of scale-free network is the highest.

Table 2: Growth Rate: Simulation (Theory)

	Regular Network	Random Network	Scale-free Network
$\rho = 0.01$	0.0150 (0.0150)	0.0156 (0.0155)	0.0295 (0.0297)
$\rho = 0.005$	0.0100 (0.0100)	0.0105 (0.0105)	0.0245 (0.0247)
$\rho = 0$	0.0050 (0.0050)	0.0055 (0.0054)	0.0197 (0.0200)

4.2 TFP Level and Degree

Eq.(22) has the significant property that the TFP level is proportional to the degree. In this section, we demonstrate that this relationship holds true by numerical simulations. The figures show degree and TFP level. All of them are set to be $\rho = 0.01$. The levels are re-scaled so that the slopes are 1. In all the figures, the relationship, TFP level $= 1 \times$ degree $+$ constant, holds true. Thus, the standard error of the slope, R^2, and scatter plots must be checked. We regressed the TFP level on degree. For regular network we cannot draw a scatter plot, because a regular network is defined as having the same degree for all the vertices.

Random Network
In the random network, $N = 1500$, $\langle \xi \rangle = 10$, $\rho = 0.01$, and $\delta_N \cdot w = 0.5 \times 10^{-3}$. As Fig.29 shows, the TFP level is proportional to the degree; the standard error of the slope is 0.0087, R^2 is 0.90.

Scale-free Network
In the scale-free network, $N = 1500$, $\langle \xi \rangle = 10$, $\rho = 0.01$, $\delta_N \cdot w = 0.5 \times 10^{-3}$, and $\gamma = 3$. As Fig.30 shows, the TFP level is proportional to the degree; the standard error of the slope is 0.006, R^2 is 0.94.

These results clearly confirm that the TFP level is proportional to the degree, $A_j \propto \xi_j$, which is expected by Eq.(22).

Figure 28: Degree against TFP Level

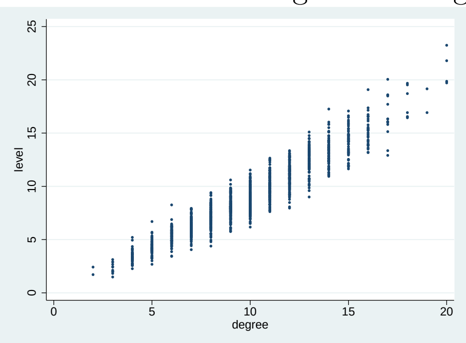

Figure 29: Random Network

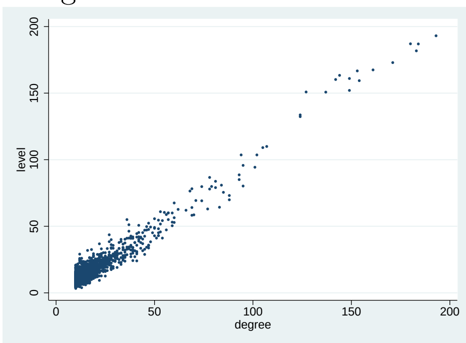

Figure 30: Scale-free Network

4.3 Bethe Approximation

In Section 2.8, we solved the knowledge spillover equation by Bethe approximation to solve Eq.(62). Here, we will confirm the approximation solution, $A(\xi_j, t) \sim (\sum_{i \in \partial j} \xi_i) \exp(\rho + \delta_N w \langle \xi_{nn} \rangle)$, by numerical simulation.

Fig.32 and Fig.33 strongly demonstrate that this approximation method is valid. In both cases, the technology level is re-scaled so that the slope becomes 1; in short, slopes are 1 for both cases. Thus, what should be checked are the R^2 and the standard error for the slopes. For random network, the standard error is only 0.03 and R^2 is 0.99. For scale-free network, the standard error is only 0.01 and $R^2 = 1.00$. Although the solution obtained by Bethe approximation fits better than that of mean field approximation, we believe that mean field solution, $A(\xi, t) \sim \xi \ e^{(\rho + \delta_N w \langle \xi_{nn} \rangle)t}$, is usually more useful. However, Bethe approximation solution, Eq.(62), can be useful for analyzing some kinds of problems like network formation and the like, because the TFP of an agent is determined by the sum of all the degrees of adjacent agents, and the solution fits numerical simulation better than mean field.

Figure 31: Next Neighbor Degrees against TFP Level

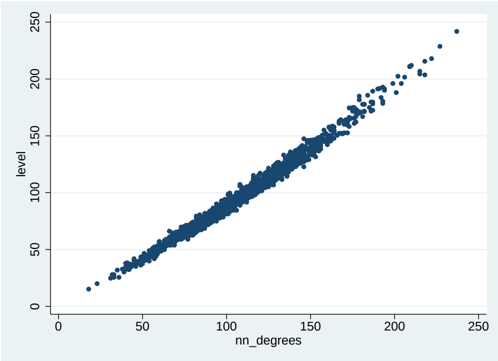

Figure 32: Random Network

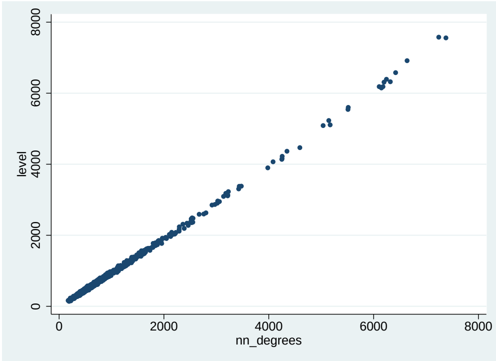

Figure 33: Scale-free Network

5 Spillover, Growth Rate, and Distance

5.1 Basic Model

The purpose of our study is to construct a workable framework for analyzing knowledge spillover through network. In this section, we illustrate the framework with an example. We use knowledge spillover equation to analyze the relationship among growth rate, area, distance, network formation, output, degree, and so on. First, the utility function of consumer is given by

$$U = \left(\int_1^V X_j(\xi_j)^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}} \tag{77}$$

The consumer maximizes this under budget constraint $\int_1^V p_j X_j = I$. Then the demand function for goods X_j becomes

$$X_j(\xi_j) = \frac{p_j(\xi_j)^{-\sigma}}{\int_1^V p_j(\xi_j)^{1-\sigma} dj} I \tag{78}$$

$$= p_j(\xi_j)^{\sigma} \left(\frac{I}{P} \right) \tag{79}$$

where $P \equiv \int_1^V p_j(\xi_j)^{1-\sigma} dj$.

The technology of firms are to change one unit of labor, $l_j(\xi_j)$, to $A_j(\xi_j) unit$ final goods, $X_j(\xi_j) = A_j(\xi_j) l_j(\xi_j)$. Then the firms' maximization problem is

$$\max_{p_j(\xi_j)} \pi_j(\xi_j) = p_j(\xi_j) X_j(\xi_j) - w l_j(\xi_j) \tag{80}$$

$$= p_j(\xi_j) A_j(\xi_j) l_j(\xi_j) - l_j(\xi_j) \tag{81}$$

Substituting Eq.(78), we have the solution of firms' profit maximization problem Eq.(81) as

$$p_j(\xi_j) = \frac{\sigma}{\sigma - 1} w A_j(\xi_j)^{-1} \tag{82}$$

Thus, substituting Eq.(82), we also have

$$X_j(\xi_j) = \left(\frac{\sigma}{\sigma - 1}\right)^{-\sigma} A_j(\xi_j)^{\sigma} \frac{I}{\int_1^V p_j^{1-\sigma} dj} \tag{83}$$

$$l_j(\xi_j) = \left(\frac{\sigma}{\sigma - 1}\right)^{-\sigma} A_j(\xi_j)^{\sigma-1} \frac{I}{\int_1^V p_j^{1-\sigma} dj} \tag{84}$$

From Eqs.(77), (81), (83), and (84), we find

$$X_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} \frac{I}{w} \frac{\sigma - 1}{\sigma} \tag{85}$$

$$l_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} \frac{I}{w} \frac{\sigma - 1}{\sigma} \tag{86}$$

$$\pi_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} I \frac{1}{\sigma} \tag{87}$$

$$\begin{aligned} U &= \left(\int_1^V X_j(\xi_j)^{\frac{\sigma-1}{\sigma}} dj\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\int_1^V A_j(\xi_j)^{\sigma-1} dj\right)^{\frac{1}{\sigma-1}} \frac{I}{w} \frac{\sigma - 1}{\sigma} \end{aligned} \tag{88}$$

From Eqs.(85)-(88), we have the growth rate of each value as

$$X_j(\xi_j, t + \triangle t) = \frac{A_j(\xi_j, t + \triangle t)^{\sigma}}{\int_1^V A_i(\xi_i, t + \triangle t)^{\sigma-1} di} \frac{I}{w} \frac{\sigma - 1}{\sigma} = g_A X_j(t) \tag{89}$$

$$l_j(\xi_j, t + \triangle) = \frac{A_j(\xi_j, t + \triangle t)^{\sigma-1}}{\int_1^V A_i(\xi_i, t + \triangle t)^{\sigma-1} di} \frac{I}{w} \frac{\sigma - 1}{\sigma} = l_j(t) \tag{90}$$

$$\pi_j(\xi_j, t + \triangle t) = \frac{A_j(\xi_j, t + \triangle t)^{\sigma-1}}{\int_1^V A_i(\xi_i, t + \triangle t)^{\sigma-1} di} \frac{1}{\sigma} I = \pi_j(t) \tag{91}$$

$$U(t + \triangle t) = \frac{I}{w} \frac{\sigma - 1}{\sigma} \left(\int_1^V A_j(\xi_j, t + \triangle t)^{\sigma-1} dj\right)^{\frac{1}{\sigma-1}} = g_A U(t) \tag{92}$$

where,

$$g_A \equiv \frac{A_j(t + \triangle t)}{A_j(t)} \tag{93}$$

Remember that g_A is common for all the firms and is independent of j and ξ_j, because they are connected to the entire network, $A_j(\xi_j, t) = \xi_j \exp\left[\rho + \delta_N w_N \langle \xi_{nn} \rangle\right] t$. We used w_N as a weight of the network, for letting w denote wage. The above equation demonstrates that final goods, $X_j(\xi_j, t)$, grows at the rate of g_A which is common for all j.

To close the model, we comment on the income of household, I. Suppose that a household supplies inelastically L unit labor,

$$\int_1^V l_j(\xi_j)dj = L \tag{94}$$

Substituting Eq.(86), we have

$$I = \frac{\sigma}{\sigma - 1} wL \tag{95}$$

Then, Eqs.(85)-(88) become

$$X_j(\xi_j) = \frac{A_j(\xi_j)^\sigma}{\int_1^V A_i(\xi_i)^{\sigma-1} di} L \tag{96}$$

$$l_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} L \tag{97}$$

$$\pi_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} \frac{wL}{\sigma - 1} \tag{98}$$

$$U = \left(\int_1^V A_j(\xi_j)^{\sigma-1} dj \right)^{\frac{1}{\sigma-1}} L \tag{99}$$

Endogenous Network Formation We described the fundamental setting of the model so far. Now, we introduce "distance" and "endogenous network formation" into the model and see what happens in the economy.

5.2 Endogenous Network Formation: Spill over Depreciation with Increased Distance

First, we introduce distance into the knowledge spillover equation. As illustrated in Fig.34, as the distance between two vertices increases, then knowledge spillover between them decreases. This expresses the very simple observation in our lives that we can learn something much easier from a close neighbor than from a distant person.

In Fig.34, two agents receive knowledge spillover from each other. In the upper figure, the distance between them is d, thus the spillover is 100. On the

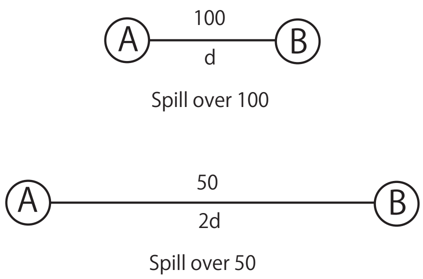

Figure 34: Distance and Spillover

other hand in the lower figure, the distance is 2d, then the spillover becomes 50. In this example, spillover dies out as $1/d$. Generally, we assume that spillover decreases as $d^{-\nu_n}$. In the figure, $\nu_n = 1$. The knowledge spillover equation with the distance depreciation effect is described as follows:

$$A_j(\xi_j, t + \triangle t) = (1 + \rho \triangle t) A_j(\xi_j, t) + \sum_{i \in \partial j} w_{ji} \delta_N d_{ij}^{-\nu_n} A_i(\xi_i, t) \triangle t \tag{100}$$

An example explains the meaning of Eq.(100). The firm-j receives spillover from other firms: firm-1, firm-2, and so on. The spillover terms can be written in the form

$$A_1/d_{1j} + A_2/d_{2j} + A_3/d_{3j} + \cdots \tag{101}$$

where, $\nu_n = 1$. d_{1j} is the distance between firm-1 and firm-j.

We also solve Eq.(100) by mean field approximation to find

$$A(\xi_j, t) = \left(A_{nn}(0)/\langle \xi_{nn} \rangle \langle d_{nn}^{-\nu_n} \rangle\right) \left(d_1^{-\nu_n} + d_2^{-\nu_n} + \cdots + d_{\xi_j}^{-\nu_n}\right) \exp\left(\rho + w_N \delta_N \langle \xi_{nn} \rangle \langle d_{nn}^{-\nu_n} \rangle\right) t \tag{102}$$

where we assumed that the weight of the network is $w_{ij} = w_N$.

We study the symmetric case where the network is regular, so that the distances between any connected firms are the same, d, termed as typical distance, as illustrated in Fig.35.

Then the above equation (102) becomes

$$A(\xi_j, t) = A_{nn}(0) \xi_j \exp\left(\rho + w_N \delta_N \langle \xi_{nn} \rangle d^{-\nu_n}\right) t \tag{103}$$

The growth rate is given by $g_A = \rho + w_N \delta_N \langle \xi_{nn} \rangle d^{-\nu_n}$

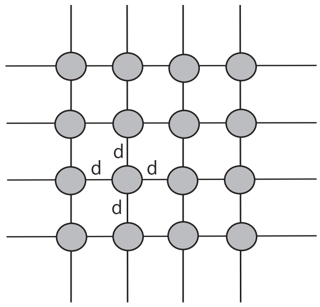

Figure 35: Distance between Adjacent Firms

5.2.1 Network Formation: Linking Cost

We introduce a linking mechanism by imposing a linking cost on each. The firm's problem consists of two stages. In the first stage, the firm chooses how many links they make; at the second stage, they maximize their profit represented by Eq.(87). We assume that linking ξ_j edges with d distant firms requires $C(\xi_j; d)$ unit labor force, given by

$$C(\xi_j; d) = C_N \xi_j^{\mu_w} d^{\mu_d} \tag{104}$$

Thus, the cost is $wC(\xi_j; d)$.

Then, we return to the firm's problem. It follows from the arguments so far that the firm solves the following problem

$$\begin{aligned} &\max_{\xi_j} \ \pi_j(\xi_j) - wC(\xi_j; d) \\ &\Rightarrow \frac{\partial \pi_j(\xi_j)}{\partial \xi_j} = \frac{\partial wC(\xi_j; d)}{\partial \xi_j} \end{aligned} \tag{105}$$

We solve the symmetric solution in which every firm has the same degree. In the symmetric case, every firm is the same; thus, they pay the same amount of cost and receive the same amount of knowledge spillover. To solve Eq.(105) with substituting Eq.(103), we have

$$\xi = \left(\frac{1}{C_N \mu_w V} \frac{\sigma - 1}{\sigma} \frac{I}{w} \right)^{\frac{1}{\mu_w}} d^{-\frac{\mu_d}{\mu_w}} \tag{106}$$

Because this is the symmetric case, then $\xi_j = \xi = \langle \xi \rangle = \langle \xi_{nn} \rangle \ \forall j$ holds true. Remember that the symmetric case is the regular network. We use the market clearing condition for labor, with the household supplying L labor force inelastically, thus;

$$L = \int_1^V l_j(\xi_j)dj + \int_1^V C(\xi_j; d)dj \tag{107}$$

Substituting Eq.(106) into $C(\xi_j; d)$, then above Eq.(107) transforms to

$$L = \frac{\sigma - 1}{\sigma}\frac{I}{w}\frac{1 + \mu_w}{\mu_w} \tag{108}$$

By Eq.(108), the degree becomes

$$\xi = \left(\frac{1}{(1 + \mu_w)C_N}\frac{L}{V}\right)^{\frac{1}{\mu_w}} d^{-\frac{\mu_d}{\mu_w}} \tag{109}$$

From now on, we will normalize $A_{nn}(0) = 1$. Then the TFP $A(\xi, t; d)$ evolves as

$$A(\xi, t; d) = \exp\left[\rho + w_N\delta_N\left(\frac{1}{(1 + \mu_w)C_N}\frac{L}{V}\right)^{\frac{1}{\mu_w}} d^{-\frac{\mu_d}{\mu_w} - \nu_n}\right] t \tag{110}$$

The growth rate is given by

$$g_A \equiv \frac{\dot{A}(\xi, t; d)}{A(\xi, t; d)} = \rho + w_N\delta_N\left(\frac{1}{(1 + \mu_w)C_N}\frac{L}{V}\right)^{\frac{1}{\mu_w}} d^{-\frac{\mu_d}{\mu_w} - \nu_n} \tag{111}$$

We also have

$$X_j(\xi_j) = \frac{A_j(\xi_j)^\sigma}{\int_1^V A_i(\xi_i)^{\sigma - 1}di}L\frac{\mu_w}{1 + \mu_w} \tag{112}$$

$$l_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma - 1}}{\int_1^V A_i(\xi_i)^{\sigma - 1}di}L\frac{\mu_w}{1 + \mu_w} \tag{113}$$

$$\pi_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma - 1}}{\int_1^V A_i(\xi_i)^{\sigma - 1}di}\frac{wL}{\sigma - 1}\frac{\mu_w}{1 + \mu_w} \tag{114}$$

$$U = \left(\int_1^V A_j(\xi_j)^{\sigma - 1}dj\right)^{\frac{1}{\sigma - 1}} L\frac{\mu_w}{1 + \mu_w} \tag{115}$$

We could continue the argument with general parameters, but instead we use specific values to discuss the implication of the model. To do so, we substitute $\nu_n = 1$, $\mu_d = 2$, and $\mu_w = 2$. The spillover into firm-j from adjacent firms with $\nu_n = 1$ is $A_1/d + A_2/d + A_3/d + \cdots$. Thus, $\nu_n = 1$ means that spillover dies out as d^{-1}. The cost function with $\mu_d = 2$ and $\mu_w = 2$

becomes $C(\xi; d) = C_N \xi^2 d^2$. This means that the cost function and marginal cost increase both in degree and distance. For simplicity, we assume $\rho = 0$. We have the degree of the firms in the economy as

$$\begin{aligned}\xi &= \left(\frac{1}{3C_N} \frac{L}{V} \right)^{\frac{1}{2}} d^{-1} \\ &\sim d^{-1} \end{aligned} \tag{116}$$

We also have the growth rate of the economy as

$$\begin{aligned} g_A &= w_N \delta_N \left(\frac{1}{3C_N} \frac{L}{V} \right)^{\frac{1}{2}} d^{-2} \\ &\sim d^{-2} \end{aligned} \tag{117}$$

Proposition 5 (Growth Rate and Distance)**.**

- *The degree of the firm is inversely proportional to distance as* d^{-1}.
- *The growth rate is quadratic inversely proportional to distance as* d^{-2}.

By applying knowledge spillover equation, Eq.(16), we obtained some interesting results as to growth rate and distance.

5.3 Degree Dependent Network Depreciation Rate

We have assumed so far that the depreciation rate is constant even when an agent has very great degree. Instead of this assumption, we now analyze the solution with the degree dependent network depreciation rate $\delta_N(\xi) = \delta_N \xi^{-\eta}$. Because in reality the more links a firm has, the less average spillover from one adjacent firm the firm receives, we study degree dependent spillover model. As discussed in Section 2.2, $0 \leq \eta < 1$. In this case, as is discussed in the same section, the knowledge spillover equation becomes

$$A_j(\xi_j, t + \triangle t) = (1 + \rho \triangle t) A_j(\xi_j, t) + \delta_N \xi^{-\eta} \sum_{i \in \partial j} w_{ji} d_{ij}^{-\nu_n} A_i(\xi_i, t) \triangle t \tag{118}$$

$\delta_N(\xi) = \delta_N \xi^{\eta}$ with $0 \leq \eta < 1$ means that as degree increases, the total spillover from all the adjacent agents also increases; however, the spillover from one adjacent agent on average decreases. Then, we also solve the symmetric case which is a regular network. The degree ξ is

$$\xi = \left(\frac{1 - \eta}{\mu_w + 1 - \eta} \frac{L}{V} \frac{1}{C_N} \right)^{\frac{1}{\mu_w}} d^{-\frac{\mu_d}{\mu_w}} \tag{119}$$

TFP $A(\xi, t)$ evolves as

$$A(\xi, t) = \exp\left[\rho + w_N \delta_N \left(\frac{1-\eta}{\mu_w + 1 - \eta} \frac{L}{V} \frac{1}{C_N}\right)^{\frac{1-\eta}{\mu_w}} d^{-\frac{\mu_d}{\mu_w}(1-\eta) - \nu_n}\right] t \tag{120}$$

The growth rate, g_A, is given by

$$g_A(\xi) \equiv \frac{\dot{A}(\xi, t)}{A(\xi, t)} = \rho + w_N \delta_N \left(\frac{1-\eta}{\mu_w + 1 - \eta} \frac{L}{V} \frac{1}{C_N}\right)^{\frac{1-\eta}{\mu_w}} d^{-\frac{\mu_d}{\mu_w}(1-\eta) - \nu_n} \tag{121}$$

We also have

$$X_j(\xi_j) = \frac{A_j(\xi_j)^\sigma}{\int_1^V A_i(\xi_i)^{\sigma-1} di} L \frac{\mu_w}{\mu_w + 1 - \eta} \tag{122}$$

$$l_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} L \frac{\mu_w}{\mu_w + 1 - \eta} \tag{123}$$

$$\pi_j(\xi_j) = \frac{A_j(\xi_j)^{\sigma-1}}{\int_1^V A_i(\xi_i)^{\sigma-1} di} \frac{wL}{\sigma - 1} \frac{\mu_w}{\mu_w + 1 - \eta} \tag{124}$$

$$U = \left(\int_1^V A_j(\xi_j)^{\sigma-1} dj\right)^{\frac{1}{\sigma-1}} L \frac{\mu_w}{\mu_w + 1 - \eta} \tag{125}$$

Now, we stop the analysis by using general exponent parameters, and we substitute specific values into them. We employ the same values as before, $\nu_n = 1, \mu_w = 2, \mu_d = 2$. Then, the degree and growth rate are

$$\begin{aligned} \xi &= \left(\frac{1-\eta}{3-\eta} \frac{L}{V} \frac{1}{C_N}\right)^{\frac{1}{2}} d^{-1} \\ &\sim d^{-1} \end{aligned} \tag{126}$$

$$\begin{aligned} g_A &= \left(\frac{1-\eta}{3-\eta} \frac{L}{V} \frac{1}{C_N}\right)^{\frac{1-\eta}{2}} d^{-2+\eta} \\ &\sim d^{-2+\eta} \end{aligned} \tag{127}$$

where we assumed $\rho = 0$.

5.4 Endogenously Determined Distance

In the models so far, the distance between connected firms are exogenously given. In the following, we will study the model in which the distance between firms is also determined endogenously.

The cost function for a firm to link to one d distant firm in terms of labor is given by

$$c(d) = c_N d^{\mu_d} \tag{128}$$

Spillover does not die out as distance increases. The effect of distance appears in the linking cost function, Eq.(128), only. The firm's decision rule for network formation is as follows; if a firm is less distant than or equal to critical distance d_c then the firm links to that firm; on the other hand, if a firm is more distant than critical distance, then the firm does not link to that firm. Therefore, all the firms less distant than or equal to d_c link to the firm. Our problem is to find critical distance d_c. Let x denote the density of the firms in the region. The total linking cost for the firm, $TC(d)$, is given by

$$TC(d) = \int_0^{d_c} dr c(r) 2\pi r x = \frac{2\pi x c_N}{\mu_d + 2} d^{\mu_d+2} \tag{129}$$

This equation clearly explains what $c(d)$ means. Because the firm connects to all the firms that are less distant than or equal to d_c, the degree of the firms are

$$\xi = \pi d_c^2 x \tag{130}$$

The network is regular. The same as before, the firm's problem is two stage game; in the first stage, the firm solves network formation problem; in the second stage, the firm maximizes profit. The second stage problem is already solved in Section 5.1 and the profit is given by $\pi(\xi)$. The firms' network formation problem in the first stage is

$$\max_{d_c} \pi(\xi(d_c)) - wTC(d_c) \tag{131}$$

Solving Eq.(131) for d_c, we have

$$d_c = \left[\frac{\sigma - 1}{\sigma}\frac{I}{w}\frac{1}{c_N \pi x V}\right]^{\frac{1}{\mu_d+2}} \tag{132}$$

By labor market clearing condition, we have

$$L = \int_1^V l(\xi_j)dj + \int_1^V TC(d_c)dj \tag{133}$$

The above equation becomes

$$L = \frac{\sigma - 1}{\sigma}\frac{I}{w}\frac{\mu_d + 4}{\mu_d + 2} \tag{134}$$

Substituting Eq.(134) into Eq.(132), we have

$$d_c = \left(\frac{\mu_d + 2}{\mu_d + 4}\frac{L}{c_N \pi V}\right)^{\frac{1}{\mu_d+2}} x^{-\frac{1}{\mu_d+2}} \tag{135}$$

Then, the degree is

$$\xi = \pi d_c^2 x = \pi \left(\frac{\mu_d + 2}{\mu_d + 4}\frac{L}{c_N \pi V}\right)^{\frac{2}{\mu_d+2}} x^{\frac{\mu_d}{\mu_d+2}} \tag{136}$$

From the above discussions we have the following inequalities:

$$\frac{\partial d_c}{\partial x} \propto -\frac{1}{\mu_d + 2} x^{-\frac{\mu_d+3}{\mu_d+2}} < 0 \qquad \frac{\partial^2 d_c}{\partial x^2} \propto \frac{\mu_d + 3}{(\mu_d + 2)^2} x^{-\frac{2\mu_d+5}{\mu_d+2}} > 0 \tag{137}$$

$$\frac{\partial \xi}{\partial x} \propto \frac{\mu_d}{\mu_d + 2} x^{-\frac{2}{\mu_d+2}} > 0 \qquad \frac{\partial^2 \xi}{\partial x^2} \propto -\frac{2\mu_d}{(\mu_d + 2)^2} x^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \tag{138}$$

$$\frac{\partial d_c}{\partial c_N} \propto -\frac{1}{\mu_d + 2} c_N^{-\frac{\mu_d+3}{\mu_d+2}} < 0 \qquad \frac{\partial^2 d_c}{\partial c_N^2} \propto \frac{\mu_d + 3}{(\mu_d + 2)^2} x^{-\frac{2\mu_d+5}{\mu_d+2}} > 0 \tag{139}$$

$$\frac{\partial \xi}{\partial c_N} \propto -\frac{2}{\mu_d + 2} x^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \qquad \frac{\partial^2 \xi}{\partial c_N^2} \propto \frac{2(\mu_d + 4)}{(\mu_d + 2)^2} x^{-\frac{2\mu_d+6}{\mu_d+2}} > 0 \tag{140}$$

For simplicity we assume $\rho = 0$, then the growth rate is

$$g_A = \delta_N w_N \pi \left(\frac{\mu_d + 2}{\mu_d + 4}\frac{L}{c_N \pi V}\right)^{\frac{2}{\mu_d+2}} x^{\frac{\mu_d}{\mu_d+2}} \tag{141}$$

and we have the following inequalities:

$$\frac{\partial g_A}{\partial x} \propto \frac{\mu_d}{\mu_d + 2} x^{-\frac{2}{\mu_d+2}} > 0 \qquad \frac{\partial^2 g_A}{\partial x^2} \propto -\frac{2\mu_d}{(\mu_d + 2)^2} x^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \tag{142}$$

$$\frac{\partial g_A}{\partial c_N} \propto -\frac{2}{\mu_d + 2} x^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \qquad \frac{\partial^2 g_A}{\partial c_N^2} \propto \frac{2(\mu_d + 4)}{(\mu_d + 2)^2} x^{-\frac{2\mu_d+6}{\mu_d+2}} > 0 \tag{143}$$

We let S denote the region size. We are going to see how $1/S$ affects, because we want to know how the parameters change as the region size decreases. We keep the number of firms in the region, V, constant. Even if we increase the density of the firm in the region, the number of the firms is the same. We are interested in comparing the economies that are different only in the densities, x.Therefore, increasing the density, x, is equivalent to decreasing the region size with keeping the number of the firms, V, the same. The critical distance d_C, the degree ξ, and the growth rate g_A in terms of the region size are given by

$$d_c = \left(\frac{\mu_d+2}{\mu_d+4}\frac{L}{c_N\pi V}\right)^{\frac{1}{\mu_d+2}}\left(\frac{1}{S}\right)^{-\frac{1}{\mu_d+2}} \tag{144}$$

$$\xi = \pi\left(\frac{\mu_d+2}{\mu_d+4}\frac{L}{c_N\pi V}\right)^{\frac{2}{\mu_d+2}}\left(\frac{1}{S}\right)^{\frac{\mu_d}{\mu_d+2}} \tag{145}$$

$$g_A = \delta_N w_N \pi\left(\frac{\mu_d+2}{\mu_d+4}\frac{L}{c_N\pi V}\right)^{\frac{2}{\mu_d+2}}\left(\frac{1}{S}\right)^{\frac{\mu_d}{\mu_d+2}} \tag{146}$$

Then, we have the following equations:

$$\frac{\partial d_c}{\partial(1/S)} \propto -\frac{1}{\mu_d+2}(1/S)^{-\frac{\mu_d+3}{\mu_d+2}} < 0 \quad \frac{\partial^2 d_c}{\partial(1/S)^2} \propto \frac{\mu_d+3}{(\mu_d+2)^2}(1/S)^{-\frac{2\mu_d+5}{\mu_d+2}} > 0 \tag{147}$$

$$\frac{\partial \xi}{\partial(1/S)} \propto \frac{\mu_d}{\mu_d+2}(1/S))^{-\frac{2}{\mu_d+2}} > 0 \quad \frac{\partial^2 \xi}{\partial(1/S)^2} \propto -\frac{2\mu_d}{(\mu_d+2)^2}(1/S)^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \tag{148}$$

$$\frac{\partial g_A}{\partial(1/S)} \propto \frac{\mu_d}{\mu_d+2}(1/S))^{-\frac{2}{\mu_d+2}} > 0 \quad \frac{\partial^2 g_A}{\partial(1/S)^2} \propto -\frac{2\mu_d}{(\mu_d+2)^2}(1/S)^{-\frac{\mu_d+4}{\mu_d+2}} < 0 \tag{149}$$

From the inequalities (137)-(143), (147)- (149), we have Fig.36-Fig.38.

Proposition 6 (Endogenously determined distance)**.**
In summary we have, the following properties:

- *As the density of the firms in the region, x, increases, the growth rate, g_A, increases.*
- *As the linking cost, C_N, increases, the growth rate decreases.*
- *As the region size decreases, the growth rate increases.*

6 Network Formation

First, we return to the starting point, knowledge spillover equation, Eq.(16). We provide it again below for convenience.

$$A_j(\xi_j, t+\triangle t) = (1+\rho\triangle t)A_j(\xi_j, t) + \sum_{i\in\partial j} w_{ji}\delta_N A_i(\xi_i, t)\triangle t \tag{150}$$

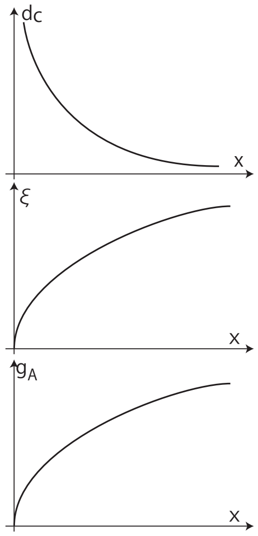

Figure 36: Density

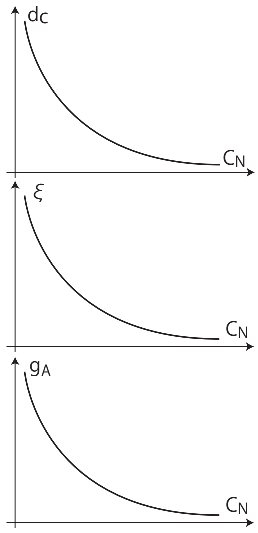

Figure 37: Cost

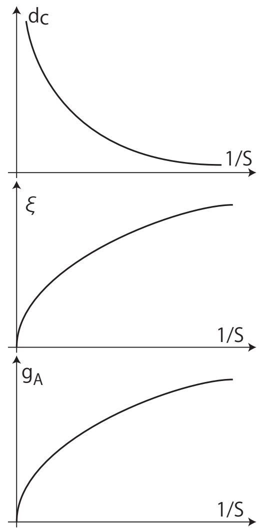

Figure 38: Area

This equation suggests that a rational agent wants to link to high TFP agents to receive large knowledge spillover. A high TFP agent attracts more links than a low TFP agent does. We show that because our knowledge spillover equation has this characteristic, it works well with stochastic network formation mechanisms (Barabasi and Albert (1999); Dorogovtsev et al. (2000); Krapivsky et al. (2000); Krapivsky and Redner (2001)). We combine our framework with the stochastic network formation mechanisms in this order. Although an agent does not know the network structure of knowledge spillover or even the degrees of the other agents, the nature of the solution of our knowledge spillover equation let the problem being that of degree. Note that in stochastic network formation mechanisms that will be discussed, an agent decision problem is not explicitly considered. The decision is only stochastic not deterministic. The agent decision problem is implicitly considered behind the stochastic network formation mechanisms; however, the mechanisms must be related to the deterministic network formation mechanisms in which agent decision problems are explicitly considered.

Network Formation 1 First, we introduce a stochastic mechanism in which the probability that an agent with A_j unit of TFP attracts a new link is given by

$$\Pr(A_j) = \frac{A_j}{\sum_i A_i} \tag{151}$$

At each time step, one new agent–"the firm"–enters the existing network and links to m existing firms with probability $\Pr(A_j)$. This process continues. We let $p(A(\xi), s, t)$ denote the probability that the firm that entered the network at time s has $A(\xi)$ unit TFP at time t. The process can be described by the following master equation:

$$p(A(\xi), s, t+1) = \frac{mA(\xi-1)}{\sum_i A(\xi_i)} p(A(\xi-1), s, t) + \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) p(A(\xi), s, t) \tag{152}$$

This is because the firm of $A(\xi - 1)$ increases the degree by 1 in probability $m \cdot \Pr(A(\xi - 1))$. We investigate asymptotic degree distribution as time t approaches infinity. We let $p(A)$ denote such a distribution presented as

$$p(A) = \lim_{t\to\infty} \sum_s \frac{p(A, s, t)}{t} \tag{153}$$

We take the summation, $\sum_{s=1}^{t+1}$, of both sides of Eq.(152), noting $p(A, t+1, t) = 0$, because at time t, by definition, the firm that enters at time $t+1$ cannot exist. Then the equation becomes

$$\sum_{s=1}^{t+1} p(A(\xi), s, t+1) = \frac{mA(\xi-1)}{\sum_i A(\xi_i)} \sum_{s=1}^{t} p(A(\xi-1), s, t) + \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) \sum_{s=1}^{t} p(A(\xi), s, t) \tag{154}$$

Substituting Eq.(153) into the above Eq.(154) to find asymptotic distribution, we obtain

$$(t+1)p(A(\xi)) = \frac{mA(\xi-1)}{\sum_i A(\xi_i)} tp(A(\xi-1)) + \left(1 - \frac{mA(\xi)}{\sum_i A(\xi_i)}\right) tp(A(\xi)) \tag{155}$$

Remember that the solution of knowledge spillover equation is

$$A(\xi) = \xi \exp(\rho + \delta_N w \langle \xi_{nn} \rangle) t \tag{156}$$

Because at each time step m links are added to the network, if we total the degrees of all the vertices in the network at time t , we obtain $2mt$ edges because every single link is counted twice. Thus, $\sum_i \xi_i \sim 2mt$. We have the following probability using Eq.(156):

$$
\begin{aligned}
m\,\Pr(A(\xi-1)) &= \frac{mA(\xi-1)}{\sum_i A(\xi_i)} \\
&= \frac{m(\xi-1)\exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t}{\sum_i \xi_i \exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t} \\
&= \frac{\xi-1}{2t}
\end{aligned} \tag{157}
$$

Substituting Eq.(157) into Eq.(155), we find

$$
(t+1)p(A(\xi)) = \frac{\xi-1}{t}t\cdot p(A(\xi-1) + \left(1-\frac{\xi}{2t}\right)t\cdot p(A(\xi)) \tag{158}
$$

which becomes

$$
p(A(\xi)) = \frac{\xi-1}{\xi-2}p(A(\xi-1)) \tag{159}
$$

To solve this equation, we have

$$
p(A(\xi)) = \frac{\text{Const}}{\xi(\xi+1)(\xi+3)} \sim \xi^{-3} \tag{160}
$$

which can be read as $p(A(\xi)) = p(\xi) \sim \xi^{-3}$. It suggests that the network generated by the simple stochastic mechanism described by Eq.(151) is a scale-free network with the exponent 3.

Network Formation 2 In stead of Eq.(151), we have the following probability that a vertex with TFP $A(\xi)$ attracts a new link, represented as

$$
\Pr(A_j) = \frac{A_j + A_0}{\sum_i (A_i + A_0)} \tag{161}
$$

Remember that at each time step, a new vertex links to m existing vertices, and the master equation is

$$
p(A(\xi), s, t+1) = m\frac{A(\xi-1)+A_0}{\sum_i (A(\xi_i)+A_0)}p(A(\xi-1), s, t) + \left(1 - m\frac{A(\xi)+A_0}{\sum_i (A(\xi_i)+A_0)}\right)p(A(\xi), s, t) \tag{162}
$$

We assume that the constant A_0 grows as $\xi_0 \exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t$. Note that there is no agent indexed by 0, so A_0 is just a constant. Because $A(\xi, t) \sim \xi \exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t$, so that the probability, Eq.(161), becomes

$$
\frac{(\xi_j+\xi_0)\exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t}{\sum_i (\xi_i+\xi_0)\exp(\rho+\delta_N w\langle\xi_{nn}\rangle)t} \sim \frac{\xi_j+\xi_0}{\sum_i (\xi_i+\xi_0)} \tag{163}
$$

Then the master equation becomes

$$p(A(\xi), s, t+1) = m\frac{\xi_j + \xi_0}{\sum_i(\xi_i + \xi_0)}p(A(\xi-1), s, t) + \left(1 - m\frac{\xi_j + \xi_0}{\sum_i(\xi_i + \xi_0)}\right)p(A(\xi), s, t) \tag{164}$$

We let $\bar{\xi}(s,t)$ denote the mean degree of the vertex entering the network at time s when the time is t. As time t passes, the vertex entering the network at time s has more links. If $t < s$, $\bar{\xi}(s,t) = 0$ because before time s such vertices, by definition, do not exist in the network. Then it follows that

$$\frac{\partial \bar{\xi}(s,t)}{\partial t} = m\frac{\xi_j + \xi_0}{\sum_i(\xi_i + \xi_0)} \tag{165}$$

With the initial condition, $\bar{\xi}(t,t) = m$, we have the solution,

$$\bar{\xi}(s,t) \sim (\frac{t}{s})^{\frac{m}{2m+\xi_0}} \tag{166}$$

With the familiar formula of the delta functions, $\int \delta(g(x))dx = \frac{1}{g(x)'}|_{x:g(x)=0}$, the degree distribution at time t, $P(\xi, t)$, is represented as

$$\begin{aligned} P(\xi,t) &= \frac{1}{t}\int_0^t ds\delta(\xi - \bar{\xi}(s,t)) \\ &= -\frac{1}{t}\left(\frac{\partial \bar{\xi}(s,t)}{\partial s}\right)^{-1} \end{aligned} \tag{167}$$

By these two Eqs.(166) and (167), we have the scale-free degree distribution:

$$\begin{aligned} P(\xi) &= t^{-1}\frac{\partial s(\xi,t)}{\partial \xi} \\ &\sim \xi^{-(3+\xi_0/m)} \end{aligned} \tag{168}$$

where, ξ_0 is the degree of corresponding vertex with A_0 TFP level. $3 + \xi_0/m$ can take $(2, \infty]$.

Network Formation 3 We have another stochastic network formation mechanism, in which a vertex attracts a new vertex edge by the probability represented as

$$\Pr(A_j) = \frac{A_j^x}{\sum_i A_i^x} \tag{169}$$

The master equation is

$$p(A(\xi), s, t+1) = m\frac{A_j^x}{\sum_i A_i^x} p(A(\xi-1), s, t) + \left(1 - m\frac{A_j^x}{\sum_i A_i^x}\right) p(A(\xi), s, t) \tag{170}$$

In a similar argument to those preceding, this process generates degree distribution:

$$P(\xi) \sim \xi^{-x} \exp[-\mu\xi^{1-x}/(1-x)] \tag{171}$$

where $0 < x < 1$ and μ is defined by $\mu = \int_0^1 dz\kappa^x(x)$, and $\kappa(s/t) = \bar{\xi}(s,t)$

As we have seen, the solution of our framework has the property that the TFP level is proportional to its degree at each time. Our framework works well with stochastic network generating mechanisms as shown in this study. The property of our framework would be helpful in understanding the network formation mechanism. We believe that the property of our framework is helpful not only for a stochastic network formation mechanism but also for the deterministic agent decision network formation problem.

7 Conclusion

We introduce knowledge spillover equation, Eq.(16), and solve it. We find some characteristics as follows:

1. Growth rate g_A of each agent does not depend on degree ξ. It is common across all the vertices regardless of their degree ξ. The growth rate is given by $g_A = \rho + w\delta_N\langle\xi_{nn}\rangle$
2. Growth rate is dependent on network structure.
3. The level of productivity A is proportional to degree ξ

We also study knowledge spillover equation for its relationship with existing models, the CES spillover, multi-technology, hierarchical network, bethe approximation, and numerical simulation.

We compare the growth rates among different representative network structures: regular, random, and scale-free networks. We find that the growth rate in scale-free networks is the largest and the smallest in regular networks. In the perspectives of the growth rate, the scale-free network is the optimal of these three representative networks.

We apply knowledge spillover equation to the model with distance and endogenous network formation, and we find the relationship between distance, region size and growth rate. We show that the distance and the region size significantly affect growth rate.

We also apply knowledge spillover equation to network formation mechanism.

To conclude, we construct a workable tool for analyzing knowledge spillover.

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