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Two-Sided Platforms: Pricing and Social Efficiency-Extensions

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Two-Sided Platforms: Pricing and Social Efficiency - Extensions

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Abstract

This paper contains two extensions of the modelling framework proposed by Hagiu (2004a) for studying two-sided market platforms. First, introducing vertical differentiation among both users and developers, we show that the optimal platform pricing structure continues to shift towards making a larger share of profits on developers relative to users when the latter have a stronger preference for product variety. Also, when developers are vertically differentiated, a two-sided proprietary platform may induce socially excessive product variety, a scenario which never occurs in the horizontal differentiation model. Second, we introduce developer investment in product quality and show that a two-sided proprietary platform may be more socially efficient than an open platform in terms of the product quality it induces, even when it is less efficient with respect to the level of product variety. In this context we also determine the profit-maximizing proportional variable fee charged by a proprietary platform to developers and show that it is is increasing in the degree of developer risk-aversion and is used by the platform to trade product variety for product quality when developers' marginal cost of quality provision increases.

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1 Introduction

This paper presents two extensions of the framework developed in Hagiu (2004a), hereafter TSP, for studying two-sided platforms in markets where product variety is important: vertical differentiation (both among users and developers) and developer incentives for investment in product quality. Indeed, while the horizontal differentiation framework developed in TSP is analytically very tractable and yields appealing and intuitive results, it is important to check the robustness of our results to the -realistic- introduction of vertical differentiation. After all, in reality not all users have the same *incremental* valuation for products, be they games, computer software, digital content, etc. Conversely, not all products have the same value to users: some applications are more valuable than others or at least have a higher quality, which allows them to command higher prices in the marketplace. Furthermore, when discussing innovation induced by platforms in TSP, we have restricted ourselves to product variety, by assuming the quality of all applications was fixed. Once again, in reality, innovation in applications comprises both variety and quality, therefore another necessary extension is to endogenize quality choice by developers. It is somewhat remarkable that our main conclusions remain unchanged under these extensions.

First, the intensity of users' preferences for product variety has the same influence on the optimal platform pricing structure when users and developers are vertically differentiated as in the horizontal differentiation version of the model presented in TSP: the more users care about variety, the larger the relative share of platform profits made on developers relative to users. Second, when developers are vertically differentiated, the possibility arises that a two-sided proprietary platform induces socially excessive product variety, which is never the case with horizontal differentiation on both sides. The key reason is that in the horizontal differentiation model the two-sided proprietary platform fully internalizes indirect network effects, therefore the only distortion it introduces comes from deadweight loss through monopoly pricing on both sides of the market, leading -just like in a one-sided market- to insufficient entry. By contrast, with vertical differentiation among developers, the platform does not fully internalize the indirect network externalities and therefore might *overestimate* the marginal value of an additional developer relative to his true marginal social value, resulting in socially *excessive* entry. Third, introducing developer investment in product quality in the horizontal differentiation model we show that even when a proprietary (closed) platform induces socially insufficient product variety and the open platform comes closer to the social optimum, the level of product quality may turn out to be closer to (and even exceed) the socially optimal level, under a proprietary platform. Together, the last two results confirm the insight that despite two-sided deadweight loss due to monopoly pricing, proprietary platforms are in a better position to internalize indirect network externalities and therefore may sometimes be more socially efficient than open platforms. The fourth and final result concerns the role of variable fees (royalties) charged by a proprietary platform to developers when the latter are risk-averse and there is uncertainty with respect to user adoption. In this case, the platform faces the following tradeoff in setting its royalty rate: high royalties allow it to take on some of the risk faced by developers and therefore alleviate the inefficiency associated with risk-aversion, but at the same time they reduce developers' inevestment incentives in product quality. Consequently, we show that there exists a unique profit-maximizing proportional royalty rate strictly between 0 and 1. By contrast, in TSP, since there is no developer risk-aversion and no investment in product quality, variable fees and fixed access fees are perfect pricing substitutes (which is why we have chosen to focus on access fees in TSP), whereas in this paper, when developers invest in quality but are risk-neutral, the profit-maximizing royalty rate is exactly 0. Furthermore we show that the optimal royalty rate is increasing in the degree of developer risk-aversion and in the elasticity of the marginal cost function of quality provision. This last result reveals the following insight: when the marginal cost of providing an additional unit of quality *increases* relative to the marginal cost of providing an additional unit of variety, the proprietary platform finds it profitable to trade quality for variety, i.e. it *increases* the royalty rate, which *decreases* the average quality supplied by developers, which at the same time *increases* the number of developers who enter the market.

The remainder of the paper is organized as follows. The next section develops the vertical differentiation model. Subsection 1 focuses on user vertical differentiation and derives the optimal platform pricing structure while subsection 2 focuses on developer vertical differentiation and provides an example in which a proprietary platform induces socially excessive product diversity. Section 3 reverts to the horizontal model developed in TSP and introduces developer investment in product quality. It first compares product quality under proprietary platforms, open platforms and a social planner and then determines the optimal royalty rate charged by a proprietary platform to developers when the latter are risk-averse. Section 4 concludes.

2 Vertical differentiation

In this section we show that the results regarding the optimal platform pricing structure derived in TSP with horizontal differentiation on both sides of the market extend to the case when users and developers are vertically differentiated.

We assume users have unitary demand for each application, i.e. buy either 0 or 1 units (cf. example 2 in TSP).

In the vertical differentiation model, the net utility of user θ from buying the platform and consuming a subset $I \subset \{1, .., n\}$ of applications is:

$$u(\theta, I) = \theta V\left(\sum_{i \in I} q_i\right) - \sum_{i \in I} p_i - P^U$$

where q_i is the "quality" of application i. By contrast, in the horizontal differentiation model we had:

$$u(\theta, I) = V\left(\sum_{i \in I} q_i\right) - \sum_{i \in I} p_i - P^U - \theta$$

The differentiation parameter θ is distributed on an interval $[\theta_L, \theta_H]$, $\theta_H > \theta_L \geq 0$, with c.d.f. $F(.)$ and density $f(.)$. $V(.)$ is a strictly increasing and concave function, whose elasticity $\varepsilon_V = \frac{QV'(Q)}{V(Q)}$ measures the intensity of users' preferences for variety. Finally, P^U is the access price for users charged by the platform.

It is convenient to define:

$$Q_I = \sum_{i \in I} q_i$$

and:

$$\theta_P = \arg\max_{\theta} \theta \left(1 - F(\theta)\right)$$

The timing of the pricing/adoption game is exactly like in TSP:

- Stage 1) The platform sets prices P^U and P^D for consumers and developers simultaneously
- Stage 2) Users and developers make their adoption decision simultaneously
- Stage 3) Developers set prices for consumers and those consumers who have acquired the platform in the second stage decide which applications to buy.

The first task will be to determine the price equilibrium between developers, when n of them support the platform. Things are slightly more complicated here than in the horizontal differentiation model since users no longer agree on the marginal valuation of additional applications. The following lemma characterizes the price equilibrium when users and developers are vertically differentiated.

Lemma *Assume the platform has been adopted by developers $i \in \{1, .., n\}$ and all users with $\theta \geq \theta_m$.*

If $\theta_m \geq \theta_P$ then there exists a unique pure-strategy price equilibrium given by:

$$p_i = \theta_m \left[V\left(Q_{\{1,..,n\}}\right) - V\left(Q_{\{1,..,n\}} - q_i\right)\right] \tag{1}$$

In this equilibrium all users buy all applications.

Conversely, if this price equilibrium exists, then necessarily $\theta_m \geq \theta_P$.

Proof See appendix.■

The interpretation of this price equilibrium is quite simple: each complementor is able to extract his marginal contribution to the surplus of the consumer with the lowest valuation, θ_m, present in the market. The condition $\theta_m \geq \theta_P$ has an intuitive interpretation: if θ_m is too low, each individual complementor could profitably increase his price and exclude consumers with low valuations. In fact, when all q_i's are equal and $\theta_m < \theta_P$, no symmetric price equilibrium exists.

Like in TSP, we treat developers as a continuum, i.e. we assume the quality q is distributed on an interval $[q_L, q_H]$ with c.d.f. $H(.)$ and continuously differentiable density $h(.)$. All developers are however assumed to have the same fixed development cost $f > 0$. Lemma 1 is easily extended to this case; the price equilibrium when users $\theta \geq \theta_m \geq \theta_P$[1] and developers $q \geq q_m$ have adopted the platform is[2]:

$$p(q) = \theta_m q V'(Q)$$

where:

$$Q = \int_{q_m}^{q_H} q h(q)\, dq$$

In this case, user $\theta \geq \theta_m$ derives net utility:

$$\theta V(Q) - \theta_m Q V'(Q) - P^U$$

from joining the platform, whereas developer $q \geq q_m$ obtains net profits:

$$\theta_m q V'(Q)(1 - F(\theta_m)) - P^D - f$$

Working backwards to the adoption stage, given the platform's price P^U and P^D, it is indeed an equilibrium for users $\theta \geq \theta_m$ and developers $q \geq q_m$ to adopt the platform if and only if the following two conditions hold:

$$\theta_m q_m V'(Q)(1 - F(\theta_m)) - P^D - f = 0 \qquad (2)$$

$$\theta_m V(Q) - \theta_m Q V'(Q) - P^U = 0 \qquad (3)$$

The expression of platform profits is then:

$$\Pi^P = P^U (1 - F(\theta_m)) + P^D (1 - H(q_m))$$

$$= (V(Q) - QV'(Q) + (1 - H(q_m)) q_m V'(Q)) \theta_m (1 - F(\theta_m)) - (1 - H(q_m)) f$$

$$= (V(Q) - E(q_m)) \theta_m (1 - F(\theta_m)) - (1 - H(q_m)) f \qquad (4)$$

[1] Below we show that at the optimum we indeed have $\theta_m \geq \theta_P$.

[2] As it will become clear below, this is the only possibility in equilibrium.

where:

$$E(q_m) = V'(Q)(Q - (1 - H(q_m))q_m) > 0$$

is the difference between total developer surplus gross of fixed costs and the portion thereof which is extracted by the platform. In other words, it is the portion of developer gross surplus uninternalized by the platform. Note that when all developers have the same quality, $E(Q) = 0$, which brings us back to the case studied in TSP.

Thus, just like in TSP, it turns out that platform profits depend only on (θ_m, q_m)[3]. Assuming sufficient second-order conditions, the profit-maximizing (θ_m, q_m) is defined by the following two first-order conditions:

$$\theta_m = \theta_P \tag{5}$$

$$q_m (V'(Q) - E'(Q))\,\theta_P (1 - F(\theta_P)) = f \tag{6}$$

where we have used:

$$\frac{dQ}{dq_m} = -h(q_m)\,q_m$$

In particular, the assumption we have made that $\theta_m \geq \theta_P$ is satisfied at the optimum[4].

2.1 Optimal platform pricing structure

In order to avoid cumbersome calculations which yield no additional insights, in this subsection we will focus on the case in which only users are vertically differentiated and assume developers are horizontally differentiated, i.e. all applications have quality $q = 1$ and $H(.)$, $h(.)$ are the c.d.f., respectively density of the distribution of f on an interval $[0, f_H]$.

The expression of plaform profits (4) becomes:

$$\Pi^P = V(n)\,\theta_m (1 - F(\theta_m)) - nH^{-1}(n)$$

[3]The same discussion as in TSP applies: given a set of prices (P^U, P^D), there might be multiple adoption equilibria (θ_m, q_m), however we will assume the platform can coordinate both sides on its most preferred stable equilibrium. Just like in TSP, one can then impose sufficient conditions ensuring stability and concavity of the maximization problems considered, without unduly restricting the validity of the results.

[4]In fact, in order for this to be an equilibrium, we need to make some further assumptions about the developer pricing game ensuring that when the platform's prices yield $\theta_m < \theta_P$, the platform makes strictly lower profits than when $\theta_m = \theta_P$.

and the first-order conditions are:

$$\theta_m = \theta_P \tag{7}$$

$$V'(n)\,\theta_P\,(1 - F(\theta_P)) = nH^{-1\prime}(n) + H^{-1}(n) \tag{8}$$

Using (2) and (3), we obtain the following pricing structure[5]:

$$\Pi^{PU} = (V(n) - nV'(n))\,\theta_P\,(1 - F(\theta_P))$$

$$\Pi^{PD} = n^2 H^{-1\prime}(n)$$

or, in relative terms:

$$\frac{\Pi^{PD}}{\Pi^{PU}} = \frac{n^2 H^{-1\prime}(n)}{(V(n) - nV'(n))\,\theta_P\,(1 - F(\theta_P))} = \frac{1}{\left(1 + \frac{H^{-1}(n)}{nH^{-1\prime}(n)}\right)\left(\frac{V(n)}{nV'(n)} - 1\right)}$$

Denoting by $\varepsilon_H = \frac{H^{-1}(n)}{nH^{-1\prime}(n)} > 0$ the elasticity of H, we have proven the following result:

We have thus proven the following result:

Proposition 1 *When developer demand is elastic, the optimal platform pricing structure is given by:*

$$\frac{\Pi^{PD}}{\Pi^{PU}} = \frac{\varepsilon_V}{(1 + \varepsilon_H)(1 - \varepsilon_V)}$$

It is such that the relative share of profits made on the developer side is decreasing in the elasticity of developer demand ε_H for the platform and increasing in the intensity of users' preference for diversity ε_V.

■

This result is very similar to the one contained in Proposition 1 in TSP, most importantly with respect to the influence of the strength of users' tastes for variety.

It is then easily seen how one can introduce uncertainty, user risk-aversion and limited supply of applications in the vertical differentiation model, just like we have done in the horizontal differentiation model (sections 3.1 and 3.2 in TSP): we obtain similar expressions and the exact same comparative statics.

[5] Π^{PU} is the portion of total platform profits made on users and Π^{PD} the portion made on developers.

2.2 Product variety

In section 4 of TSP we have shown that when both users and developers are solely horizontally differentiated, a two-sided proprietary platform always induces insufficient product variety and user adoption. This is because with horizontal differentiation a two-sided platform fully internalizes the indirect network effects between users and developers. This is apparent in the expression of platform profits derived in TSP, which we reproduce here for convenience:

$$\Pi^P = \left(P^U + n\pi\left(n\right)\right) F\left(u\left(n\right) - P^U\right) - nH^{-1}\left(n\right)$$

$$= \left(V\left(n\right) - \theta_m\right) F\left(\theta_m\right) - nH^{-1}\left(n\right)$$

Therefore there will be no bias due to business stealing or product diversity. The only distortion is deadweight loss due to monopoly pricing on both sides of the market and leads to insufficient entry of both users and developers.

Intuitively however, it should be clear that this feature cannot be robust to more general formulations of user and developer demand. Even though a two-sided platform extracts only a part of *total* user and developer surplus, there is no reason why the *marginal* contribution of an additional developer to platform profits should necessarily be lower than the marginal contribution of that developer to total social surplus. In particular, if developers are sufficiently vertically differentiated by the benefits they offer users (as opposed to being simply heterogeneous in their fixed costs) and if the platform is unable to perfectly price discriminate, then it might overestimate the value of the positive indirect network effects with respect to the value of negative direct network effects and therefore induce socially *excessive* entry.

Developers are vertically differentiated as in the beginning of the present section but in order to make the comparison with the horizontal differentiation model more clear, we assume users are horizontally differentiated as in TSP. The analysis above carries over however (it is in fact easier in this case) and we obtain a slightly different version of the expression of platform profits (4):

$$\Pi^P = [V(Q) - E(q_m) - \theta_m] F(\theta_m) - (1 - H(q_m)) f$$

The first-order conditions are[6]:

$$(V(Q) - E(q_m) - \theta_m) f(\theta_m) = F(\theta_m) \tag{9}$$

$$\left(\frac{dQ}{dq_m} V'(Q) - E'(q_m)\right) F(\theta_m) + h(q_m) f = 0 \tag{10}$$

Under an open platform, the marginal developer and the marginal user respectively are given by[7]:

$$V(Q) - QV'(Q) - \theta_m = 0 \tag{11}$$

$$q_m V'(Q) F(\theta_m) = f$$

Given that $\frac{dQ}{dq_m} = -h(q_m) q_m$, the second condition is equivalent to:

$$\frac{dQ}{dq_m} V'(Q) F(\theta_m) + h(q_m) f = 0 \tag{12}$$

Social welfare has the following expression:

$$W = V(Q) F(\theta_m) - \int_0^{\theta_m} \theta f(\theta) d\theta - (1 - H(q_m)) f$$

so that the socially optimal level of product variety and user entry are defined by:

$$V(Q) - \theta_m = 0 \tag{13}$$

$$\frac{dQ}{dq_m} V'(Q) F(\theta_m) + h(q_m) f = 0 \tag{14}$$

Comparing (27) and (14) to (24) and (12), it is clear that the levels of product variety and user adoption under an open platform are always insufficient here: $\theta_m^{fe} < \theta_m^{so}$ and $q_m^{fe} > q_m^{so} \Longleftrightarrow n^{fe} < n^{so}$.

However, comparing (27) and (14) to (9) and (10), it is no longer obvious whether the two-sided proprietary platform will induce too little or too much

[6] We assume second-order conditions are satisfied.

[7] Note that $V(Q) - QV'(Q)$ is increasing in Q, therefore decreasing in q_m and increasing in the number of developers who enter $(1 - H(q_m))$. Also, $q_m V'(Q)$ is increasing in q_m.

variety and user adoption, i.e. whether $q_m^{2sp} > q_m^{so}$ or $q_m^{2sp} < q_m^{so}$. Indeed, while the monopoly pricing distortion on the user side still tends to render user adoption sub-optimal[8], on the developer side it all depends on the sign of $E'(q_m)$. Specifically, if $E'(q_m) > 0$, then the left hand side of (10) is lower than the left-hand side of (14) and consequently, since both expressions are decreasing in q_m (required by our assumption that the maximization problem is well-defined), it might turn out that $q_m^{2sp} < q_m^{so}$. This means that the bias towards excessive entry on the developer side exceeds the bias towards insufficient user adoption on the user side.

Let us provide an example in which this happens. $E'(q_m^{so}) > 0$ if and only if:

$$\frac{dQ}{dq_m} V''(Q)(Q - (1 - H(q_m)) q_m) + V'(Q)\left(\frac{dQ}{dq_m} - (1 - H(q_m)) + h(q_m) q_m\right) > 0$$

Several lines of calculation show that this is also equivalent to:

$$\frac{q_m h(q_m) \varepsilon_{V'}(Q) - (1 - H(q_m))}{1 - H(q_m)} Q > \varepsilon_{V'}(Q) q_m^2 h(q_m) \qquad (15)$$

where:

$$\varepsilon_{V'}(Q) = -\frac{QV"(Q)}{V'(Q)} > 0$$

Assume H is uniform on $[q_L, q_H]$, so that $h(q_m) = \frac{1}{\Delta q}$, $Q = \frac{q_H^2 - q_m^2}{2\Delta q}$ and let:

$$V(Q) = \frac{1}{Q_0^\beta} - \frac{1}{(Q_0 + Q)^\beta}$$

which implies $\varepsilon_{V'}(Q) = \beta + 1$.

Then condition (15) becomes:

$$(q_m(\beta + 2) - q_H)(q_H - q_m) > 2(\beta + 1) q_m^2$$

which is equivalent to:

$$q_m > \frac{q_H}{\beta}$$

Also, it can be verified that for this example, the left-hand side of (10) and (14) are both decreasing in q_m which ensures that the respective maximization problems of the two-sided platform and the social planner are well-defined.

[8] Indeed, note that given the same q_m, (9) yields a lower θ_m than (27).

Thus, if $\beta > 1$ and $q_L > \frac{q_H}{\beta}$ then $E'(q_m) > 0$ for all $q_m \in [q_L, q_H[$. If in addition user demand is inelastic so that the platform entirely extracts user surplus and therefore there is no deadweight loss on the user side, then we have:

$$q_m^{2sp} < q_m^{so} \Longleftrightarrow n^{2sp} > n^{so}$$

Proposition 2 *Assume developers are vertically differentiated by their quality q, distributed uniformly on $[q_L, q_H]$, users are horizontally differentiated, user demand for the platform is inelastic and the user marginal surplus function has constant elasticity: $\frac{-QV''(Q)}{V'(Q)} = \beta + 1$. Then, when $q_L > \frac{q_H}{\beta}$ the two-sided platform induces excessive variety from a social standpoint, i.e. $q_m^{2sp} < q_m^{so}$.*

■

We have thus produced an example in which the level of product variety chosen by a two-sided profit-maximizing platform can be socially excessive. It remains to compare it with the open platform: clearly, in the example above, since user demand is inelastic, the open platform will exactly choose the socially optimal level of product variety ((12) and (14) are identical and all users enter). However, when user demand becomes slightly elastic, the monopoly pricing distortion on the user side contained in (9) tends to lower n^{2sp} towards n^{so}, while n^{fe} is lowered *away* from n^{so}. Therefore, for some positive elasticity of user demand ε_F, the variety chosen by the two-sided platform will be exactly socially optimal, whereas the open platform results in insufficient product variety. And similarly for user adoption. Also, just like in TSP, it should be clear that social welfare can be either higher or lower with a proprietary platform relative to an open platform.

3 Developer investment in product quality

Up to here we have assumed that the value or quality of each application was cast in stone, i.e. did not depend on the amount of investment or effort its developer expended. Developers faced a simple binary decision - support the platform or not. This may be realistic in cases in which the applications have already been produced (for example for another platform) and developers face a fixed cost of "porting" them, independent of application quality.

In most cases however, applications are produced as a consequence of deciding whether or not to support a given platform, therefore platform pricing also affects the amount of effort developers invest in producing their applications. For example, the higher the installed user base of a videogame console and the *lower* the royalty rate charged by the console vendor, the higher the marginal revenues per additional unit of investment in game quality for game developers, and thus the higer the "quality" of the games which will be produced.

Consequently, it is important to investigate how the level of investment by developers induced by the two-sided platform compares with the socially optimal level of investment and with the one induced by an open platform.

In order to do so, we extend the framework developed in TSP in the following way: we assume all developers are identical in the marginal productivity of their investment in product quality, but they differ in the fixed, quality-independent portion of their costs. We also assume product quality is deterministic and depends positively on the level of investment. Specifically, the fixed cost for developer f of producing an application of quality q is:

$$f + c(q)$$

where $c(0) = 0$, $c'(q) > 0$, $c''(q) > 0$. As before, f is distributed according to c.d.f. $H(.)$ and density $h(.)$ on $[0, f_H]$, which we assume wide enough so that all solutions are interior in what follows.

The timing is modified by adding an investment stage for developers:

- Stage 1) The platform sets prices for consumers and developers simultaneously
- Stage 2) Users and developers make their adoption decision simultaneously
- Stage 3) Developers which have entered choose their respective levels of investment in product quality, or equivalently their respective levels of product quality.
- Stage 4) Given the product qualities chosen in stage 3), developers set prices for consumers and those consumers who have acquired the

platform in the second stage decide which applications to buy.

User θ purchasing the platform and n applications of quality $q(i)$, $i \in [0, n]$, priced at $p(i)$, derives net surplus:

$$V\left(\int_{i=0}^{n} q(i)\, di\right) - \int_{i=0}^{n} p(i)\, di - P^{U} - \theta$$

The equilibrium price in the developer pricing subgame (stage 4)) is:

$$p(i) = q(i)\, V'\left(\int_{i=0}^{n} q(i)\, di\right)$$

In this section we also allow the platform to charge developers not only fixed access fees P^{D} but also variable fees (royalties) λ proportional to the price they charge. These proportional royalties do not have any impact on the equilibrium price above, therefore net profits for developer f choosing quality q, when users $\theta \geq \theta_m$ and n developers enter, are:

$$(1-\lambda)\, qV'\left(\int_{i=0}^{n} q(i)\, di\right) F(\theta_m) - P^{D} - f - c(q)$$

His quality choice q in stage 3 is therefore given by:

$$(1-\lambda)\, V'\left(\int_{i=0}^{n} q(i)\, di\right) F(\theta_m) = c'(q)$$

Thus, the royalty λ is essential in determining the quality chosen by developers. In the absence of investment by developers (the TSP framework) nothing changes with the introduction of royalties as will become clear below.

Under standard regularity assumptions, the best response functions over i determine a unique symmetric equilibrium, in which all developers who have entered choose the same quality q defined as an implicit function of (θ_m, n, λ)[9]:

$$(1-\lambda)\, V'(nq(n, \theta_m, \lambda))\, F(\theta_m) = c'(q(n, \theta_m, \lambda)) \tag{16}$$

Working our way backwards, the adoption equilibrium (θ_m, n) in stage 2 is defined by:

$$qV'(nq(n, \theta_m, \lambda))\, F(\theta_m) - P^{D} - H^{-1}(n) - c(q(n, \theta_m, \lambda)) = 0$$

[9]Note that since V is concave and c is convex, $\frac{\partial q}{\partial \lambda} < 0$.

and:

$$V(nq) - nqV'(nq) - P^U - \theta_m = 0$$

yielding the following expression of platform profits:

$$\Pi^P = P^U F(\theta_m) + P^D n + \lambda q V'(nq)\, nF(\theta_m)$$

$$= (V(nq(n, \theta_m, \lambda)) - \theta_m) F(\theta_m) - nH^{-1}(n) - nc(q(n, \theta_m, \lambda))$$

Thus, Π^P depends on λ only through its influence on developers' choice of product quality q. Assume for a moment that the platform can *choose* q directly. Then its optimal choice given (θ_m, n) is defined by the following first-order condition:

$$nV'(nq) F(\theta_m) - nc'(q) = 0 \tag{17}$$

But this is exactly equivalent to (16) when $\lambda = 0$. Thus, for *any* (θ_m, n) the platform can obtain its most preferred (i.e. profit-maximizing) level of product quality q by setting $\lambda = 0$. Therefore the optimal (θ_m, n, q) are given by (33) and the following two first order conditions with respect to θ_m and n[10]:

$$(V(nq) - \theta_m) f(\theta_m) - F(\theta_m) = 0 \tag{18}$$

$$qV'(nq) F(\theta_m) - H^{-1}(n) - nH^{-1\prime}(n) - c(q) = 0 \tag{19}$$

Combining (33), (18) and (19), we obtain the following three equations determining $\left(q^{2sp}, \theta^{2sp}, n^{2sp}\right)$:

$$qc'(q) - c(q) = H^{-1}(n) + nH^{-1\prime}(n) \tag{20}$$

$$\frac{V(nq) - \theta_m}{\theta_m} = \frac{1}{\varepsilon_F} \tag{21}$$

$$V'(nq) F\left(\frac{\varepsilon_F V(nq)}{1 + \varepsilon_F}\right) = c'(q) \tag{22}$$

In the case of an open platform, the equilibrium product quality given θ_m and n is also defined by (33), whereas θ_m and n are given by:

$$V(nq) - nqV'(nq) = \theta_m$$

[10] In other words, everything is as if the platform could ignore the influence of θ_m and n on the choice of q: setting $\lambda = 0$ is sufficient.

$$qV'(nq)F(\theta_m) - H^{-1}(n) - c(q) = 0$$

Therefore application quality, variety and user adoption induced by an open platform are defined by the 3 following equations:

$$qc'(q) - c(q) = H^{-1}(n) \tag{23}$$

$$\theta_m = V(nq) - nqV'(nq) \tag{24}$$

$$V'(nq)F(V(nq) - nqV'(nq)) = c'(q) \tag{25}$$

Finally, the expression of social welfare is:

$$W = V(nq)F(\theta_m) - \int_0^{\theta_m} \theta f(\theta)\,d\theta - \int_0^{H^{-1}(n)} fh(f)\,df - nc(q)$$

The three corresponding first-order conditions are:

$$nV'(nq)F(\theta_m) = nc'(q)$$

$$qV'(nq)F(\theta_m) - H^{-1}(n) - c(q) = 0$$

$$V(nq) - \theta_m = 0$$

or, equivalently:

$$qc'(q) - c(q) = H^{-1}(n) \tag{26}$$

$$\theta_m = V(nq) \tag{27}$$

$$V'(nq)F(V(nq)) = c'(q) \tag{28}$$

We obtain the following proposition:

Proposition 3 *If all developers have the same marginal productivity of investment in application quality, then:*

a) The level of product variety induced by an open platform is higher than that induced by a two-sided proprietary platform and lower than the socially optimal level: $n^{2sp} < n^{fe} < n^{so}$.

b) The level of product quality (or equivalently developer investment) induced by an open platform is lower than the socially optimal level, whereas the level of product quality induced by a two-sided proprietary platform can be either higher or lower than both the socially optimal level and the level induced by the open platform: $q^{2sp} \gtrless q^{fe}$, $q^{2sp} \gtrless q^{so}$

c) If in addition all developers have the same quality-independent fixed cost, f_0, then the three levels of product quality (developer investment) are identical and equal to the unique solution of:

$$qc'(q) = f_0 + c(q)$$

Proof Under sufficient regularity conditions[11], the left-hand sides of (22), (25) and (28) are all decreasing in nq. Therefore, (22), (25) and (28) can be equivalently written as $n = N_{2sp}(q)$, $n = N_{fe}(q)$ and $n = N_{so}(q)$ respectively, where N_{2sp}, N_{fe}, N_{so} are strictly decreasing functions of q, verifying:

$$N_{2sp}(q) < N_{fe}(q) < N_{so}(q)$$

for all q.

On the other hand:

$$\frac{d\left(qc'(q) - c(q)\right)}{dq} = qc''(q) > 0$$

Therefore, assuming both $H^{-1}(n)$ and $H^{-1}(n) + nH^{-1\prime}(n)$ are increasing in n[12], (20), (23) and (26) can be equivalently written as $q = Q_{2sp}(n)$, $q = Q_{fe}(n)$ and $q = Q_{so}(n)$ respectively, where Q_{2sp}, Q_{fe}, Q_{so} are strictly increasing functions of q, verifying:

$$Q_{2sp}(q) > Q_{fe}(q) = Q_{so}(q)$$

Part a) of the proposition is then obvious from figure (1).

From figure (**??**) it is also apparent that $q_{fe} < q_{so}$. However, q_{2sp} can be either lower or higher than q_{so}. For example, if users are undifferentiated (F inelastic) then:

$$N_{2sp}(q) = N_{fe}(q) = N_{so}(q) = \frac{V'^{-1}\left(c'(q)\right)}{q}$$

which implies:

$$q_{2sp} > q_{so} = q_{fe}$$

It is then clear that if F is slightly elastic, i.e. the curves $N_{2sp}(q)$, $N_{fe}(q)$ and $N_{so}(q)$ are close enough to one another, then $q_{2sp} > q_{so} > q_{fe}$. When

[11] Tthe 3 conditions provided in appendix A1 of TSP are an example.

[12] This is the case for example when $H(.)$ has constant elasticity.

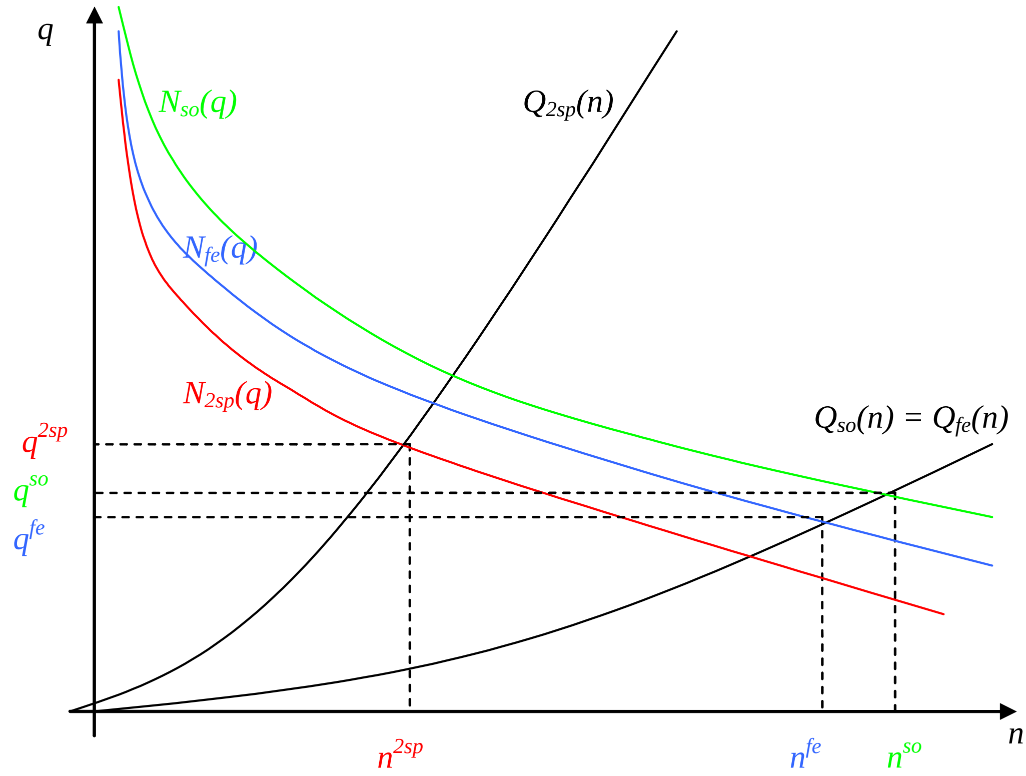

Figure 1:

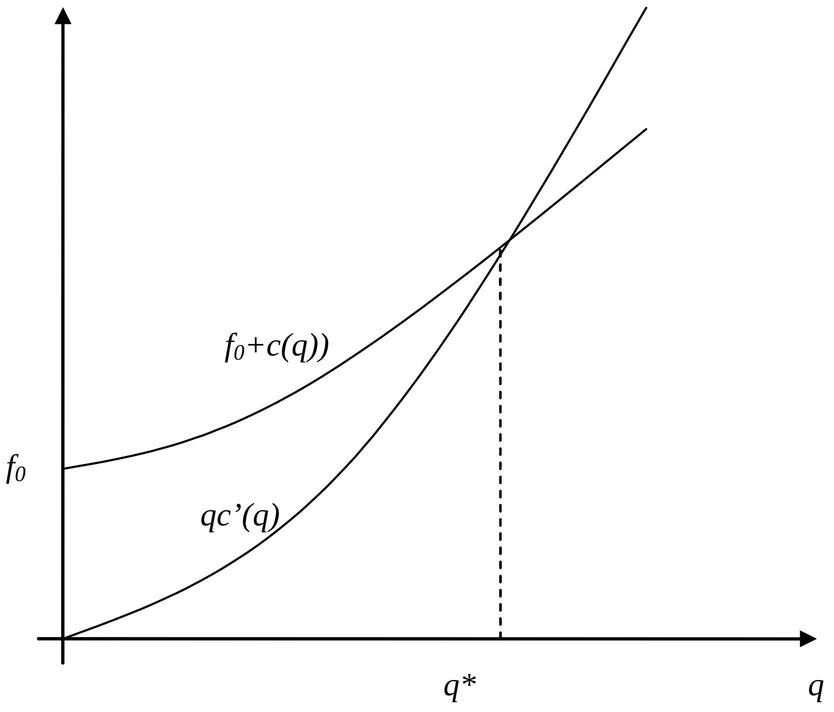

Figure 2:

the distance between the three curves increases, we have $q_{so} > q_{2sp} > q_{fe}$ and finally, when the three curves are sufficiently far apart $q_{so} > q_{fe} > q_{2sp}$.

Finally, for part c), if h is concentrated at one point f_0, (22), (25) and (28) are all equivalent to:

$$qc'(q) = f_0 + c(q)$$

which has a unique solution, illustrated by figure (**??**).

■

Proposition 5 shows that the product quality or investment bias is very different from the product variety bias. In particular, although the open platform always entails a level of product variety closer to the social optimum than the two-sided proprietary platform in this particular framework with horizontal differentiation on both sides, the level of product quality (or equivalently developer investment) induced by the latter can in turn be closer to the socially optimum than the one induced by an open platform. And when developers are identical, the quality levels induced by both types of platform are identical and equal to the socially optimal quality level. In this case, the market inefficiency is entirely concentrated in the level of product diversity.

One could also study the case in which developers are vertically differentiated by their marginal productivity of investment, by assuming for example that developers' cost of quality provision is $f_0 + \gamma c(q)$, with γ distributed on an interval $[\gamma_L, \gamma_H]$ with c.d.f. $H(.)$ and density $h(.)$. However, it should be clear that this case would not entail any new insights: we would simply obtain that both the quality and product diversity levels induced by the proprietary platform can be either insufficient or excessive.

3.1 Developer investment, risk-aversion and variable fees

As we have shown above, variable fees charged by a proprietary platform to developers play no role when there is no investment in quality by developers (then P^D and λ are perfect substitutes as pricing instruments) and when there is investment but no uncertainty, $\lambda = 0$ ensures the first-best choice of product quality from the point of the platform. This means that the platform allows developers to earn all the revenues from their investment in order to obtain the optimal level of quality, but extracts some of these revenues *ex-ante* through the fixed access fee P^D.

However, some two-sided platform in the real world, videogame consoles in particular, do charge positive royalties to developers. According to some industry analysts, the reason is uncertainty with respect to the size of the future user installed base of the platform (no developer is ready to pay access fees for porting his games to a platform that will attract no users) combined with game publishers' risk-aversion. In this case, it is optimal for the platform to offer a contract specifying a positive royalty rate whose role is to take some of the risk associated with this uncertainty. However, even when the platform were risk-neutral, it would not be optimal to charge the maximum royalty rate (i.e. extract all revenues) because then there will be no incentive for developers to invest in product quality. Given these two opposite forces, the optimal royalty rate will lie somewhere strictly between 0 and 1 (in proportional terms). In what follows we formalize this idea.

We introduce uncertainty in a slightly different way in this section than in TSP. Consistent with the discussion above, we will assume that in stage 3, when $F(\theta_m)$ users *should* have entered in the market given the platform's prices and developer demand, in reality $\gamma F(\theta_m)$ have entered, where $E\gamma = 1$. Moreoevr, developers are risk-averse so that:

$$E_D(\gamma) = 1 - r < E\gamma = 1$$

where $E_D(\gamma)$ is the expectation operator from the point of view of developers.

The uncertainty is resolved between stages 3 and 4, i.e. after developers have chosen their investment or quality levels q. Given that all developers who have entered have the same revenue function and the same marginal cost of investment in quality $c'(q)$, in equilibrium all developers having entered choose the same quality q. From the perspective of the beginning of stage 3, expected profits for developer f (net of the access fee P^D) are:

$$(1-\lambda)\, qV'(nq)\, E_D(\gamma)\, F(\theta_m) - c(q) - f$$

so that, given λ, θ_m and n, the equilibrium quality q is given by:

$$(1-r)(1-\lambda)\, V'(nq)\, F(\theta_m) = c'(q) \tag{29}$$

In the adoption stage 2, developer demand n and user demand $F(\theta_m)$ are defined by:

$$(1-r)(1-\lambda)\, qV'(nq)\, F(\theta_m) - P^D - c(q) - H^{-1}(n) = 0 \tag{30}$$

$$V(nq) - nqV'(nq) - P^U - \theta_m = 0 \tag{31}$$

where q is given by (29).

The expression of platform profits is then[13]:

$$\Pi^P\left(P^U, P^D, \lambda\right) = P^U F(\theta_m) + P^D n + n\lambda qV'(nq)\, F(\theta_m)$$

[13]The platform is assumed to be risk-neutral.

or, using the expressions determining user and developer demand:

$$\Pi^P(\theta_m, n, \lambda) = (V(nq) - (1-\lambda) rnqV'(nq) - \theta_m) F(\theta_m) - nc(q) - nH^{-1}(n)$$

Using (29) we can rewrite this expression as a function of θ_m, n and q only:

$$\Pi^P(\theta_m, n, q) = (V(nq) - \theta_m) F(\theta_m) - nq\frac{rc'(q)}{1-r} - nc(q) - nH^{-1}(n)$$

and λ is determined by (29)[14].

The first-order condition with respect to θ_m is:

$$\theta_m = \frac{\varepsilon_F V(nq)}{1+\varepsilon_F}$$

With respect to n:

$$qV'(nq) F(\theta_m) - \frac{r}{1-r} qc'(q) - c(q) - nH^{-1\prime}(n) - H^{-1}(n) = 0 \quad (32)$$

With respect to q (after simplifying by n):

$$V'(nq) F(\theta_m) - \frac{r}{1-r}\left(c'(q) + qc''(q)\right) - c'(q) = 0 \quad (33)$$

Combining (32) and (33) to eliminate the term $V'(nq) F(\theta_m)$, we obtain:

$$\frac{r}{1-r} q^2 c''(q) + qc'(q) = c(q) + nH^{-1\prime}(n) + H^{-1}(n) \quad (34)$$

Similarly, combining (33) and (29), we also have:

$$c'(q) = (1-\lambda)\left(r\left(c'(q) + qc''(q)\right) + (1-r) c'(q)\right)$$

implying:

$$\lambda = \frac{1}{1+\frac{c'(q)}{rqc''(q)}} \in (0,1)$$

Assume the cost of investment in quality has the following form:

$$c(q) = cq^{m+1}$$

[14] Note indeed that given (θ_m, n) (29) determines a strictly decreasing and continous relation between λ and q on the relevant domains.

where convexity requires $m > 0$. Then:

$$\lambda = \frac{1}{1 + \frac{1}{rm}}$$

Thus, the optimal royalty rate is strictly between 0 and 1 and is *increasing* in the degree of developer risk-aversion r[15] and in the "elasticity" m of the marginal cost function, which measures the convexity of $c(.)$: $m = \frac{qc'(q)}{c(q)} - 1$. These two results are quite intuitive. When developers are more risk-averse, the platform will find it optimal to take on more of the risk by charging higher variable fees; however it will never take on the entire risk because otherwise there will be no incentive left for developers to invest in quality.

When $c(.)$ is more convex, quality provision is increasingly costly, therefore the platform finds it profitable to trade quality for variety, in the sense that it will charge higher variable fees which will induce lower quality but at the same time more developers will enter (because $V'(nq)$ is decreasing in q) so that variety is increased. Indeed, since quality and variety are perfect substitutes for users, i.e. they only care about the product nq, when the marginal cost of an extra unit of quality $c'(q)$ increases relative to the marginal cost of an extra application f, it is more profitable to induce more variety and lower quality.

In order to see this clearly, assume all developers have the same fixed development cost f_0, $c(q) = cq^{m+1}$, $F(\theta_m) = \theta_m$ and $V(n) = n^{\beta}$, with $\beta < \frac{1}{2}$[16]. Then, solving (34) for q and (32) for n, we obtain:

$$q = \left[\frac{f_0(1-r)}{cm(1+rm)}\right]^{\frac{1}{m+1}}$$

and:

$$n = \left[\frac{\beta(1-r)}{2(m+1)c(1+rm)q^{m+1-2\beta}}\right]^{\frac{1}{1-2\beta}}$$

Clearly, q is decreasing in m. n is *increasing* in m if and only if:

$$\frac{(m+1)(1+rm)}{m^{\frac{m+1-2\beta}{m+1}}(1+rm)^{\frac{m+1-2\beta}{m+1}}}$$

[15]Note that if developers are risk-neutral ($r = 0$) then $\lambda = 0$, the result we have obtained at the beginning of this section when there was no uncertainty.

[16]$\beta < \frac{1}{2}$ is necessary for the solutions to be well-defined.

is *decreasing* in m, i.e. if and only if:

$$\frac{m+1}{m} m^{\frac{2\beta}{m+1}} (1+rm)^{\frac{2\beta}{m+1}}$$

is *decreasing* in m, which is true for all m larger than 3.

For the same reason, q is decreasing in c, whereas n is increasing in c and, conversely, q is increasing in f_0, whereas n is decreasing in f_0. However, neither c nor f_0 have any influence on the royalty rate λ.

These results are summarized in the following proposition:

Proposition 4 *Assume there is uncertainty with respect to the extent of user platform adoption, developers are risk-averse with a coefficient of relative risk-aversion* $(1-r)$ *and have the same marginal cost function of quality provision* $c'(.)$. *Then the profit-maximizing proportional royalty rate charged by the platform is* $\lambda = \frac{1}{1+\frac{c'(q)}{rqc''(q)}}$, *increasing in* r *and in the elasticity of* $c'(.)$.

■

4 Conclusion

In this paper we have explored two extensions of the model of two-sided market platforms developed in TSP. First, we have introduced vertical differentiation on both sides of the market and have shown that the result regarding the influence of users' preference for product variety on the optimal platform pricing structure derived in TSP generalizes to this setting: the intensity of users' preference for variety shifts prices in their favor. Also, developer vertical differentiation allowed us to produce an example of socially excessive product diversity under a proprietary platform, a scenario which was impossible in the horizontal differentiation framework. Second, we have introduced investment in product quality by developers and have shown that just like with user adoption and product diversity, by internalizing indirect network effects a proprietary platform may induce a level of quality closer to the social optimum than an open platform. Furthermore, we have demonstrated that the profit-maximizing royalty rate charged by a proprietary platform to developers is increasing in the degree of developer risk-aversion and in the elasticity of the marginal cost of quality provision, meaning that the platform trades quality for variety when the marginal cost of quality increases.

Clearly, there are still many important issues which we have not yet covered, such as platform competition (Hagiu (2004b) contains an initial exploration of this topic), developer exclusivity and multihoming, dynamic platform pricing, vertical integration decisions, etc. We hope the framework developed here and in TSP will provide a formal basis for future study of these themes.

References

[1] Hagiu, A. (2004a). "Two-Sided Platforms: Pricing and Social Efficiency," mimeo Princeton University and Research Institute for the Economy, Trade and Industry.

[2] Hagiu, A. (2004b). "Merchants of Two-Sided Platforms?" mimeo Princeton University and Research Institute for the Economy, Trade and Industry.

5 Appendix

Proof of Lemma 1 Let us first prove the existence of the equilibrium. Assume all developers charge the prices defined by (1) above. Then, for all $\theta \geq \theta_m$, any $I \subsetneq \{1, .., n\}$ and $i \in \{1, .., n\} \backslash I$:

$$\begin{aligned}\theta \left[V\left(Q_I + q_i\right) - V\left(Q_I\right)\right] &\geq \theta_m \left[V\left(Q_I + q_i\right) - V\left(Q_I\right)\right] \\ &\geq \theta_m \left[V\left(Q_{\{1,..,n\}}\right) - V\left(Q_{\{1,..,n\}} - q_i\right)\right] \\ &= p_i\end{aligned}$$

where we use the concavity of $V(.)$ in the second inequality.

This implies that at these prices, all users who have adopted the platform will buy all complements available.

Let us now verify that there is no profitable deviation for any complementor i. Clearly, $p_i' < p_i$ cannot be profitable because all available consumers are served at p_i.

For $p_i' > p_i$, users with valuations close to θ_m will stop purchasing complement i, hence profits are:

$$p_i'\left(1-F\left(\frac{p_i'}{V\left(Q_{\{1,..,n\}}\right)-V\left(Q_{\{1,..,n\}}-q_i\right)}\right)\right)$$

and the derivative with respect to p_i' at $p_i'=p_i$ is:

$$1-F\left(\theta_m\right)-\theta_m f\left(\theta_m\right)\leq 0$$

when $\theta_m \geq \theta_P$, so that no deviation is profitable.

Also, it is clear that existence of this equilinrium implies $\theta_m \geq \theta_P$.

Let us now turn to uniqueness. Start with any equilibrium p_i, $i=1,..n$. If in equilibrium all platform users buy all complements, then necessarily:

$$p_i \leq \theta_m\left[V\left(Q_{\{1,..,n\}}\right)-V\left(Q_{\{1,..,n\}}-q_i\right)\right]$$

and we have seen above that none of these inequalities can be strict in equilibrium. We find the same equilibrium above. In order for this equilibrium to have a chance to be different, it must be that some users do not purchase all complements. which we assume.

Start with the highest valuation user, θ_H. Denote by $I_1, I_2, .., I_p$, $p\geq 1$, his most preferred subsets of complements[17], i.e.:

$$\theta_H V\left(Q_{I_1}\right)-\sum_{i\in I_1}p_i = ... = \theta_H V\left(Q_{I_p}\right)-\sum_{i\in I_p}p_i > \theta_H V\left(Q_I\right)-\sum_{i\in I}p_i$$

for all $I \notin \{I_1,..,I_p\}$ and let the I_i's be arranged so that:

$$V\left(Q_{I_1}\right)\leq V\left(Q_{I_2}\right)\leq ... \leq V\left(Q_{I_p}\right)$$

Suppose $p>1$ and $V\left(Q_{I_1}\right)=V\left(Q_{I_2}\right)$. This implies $\sum_{i\in I_1}p_i = \sum_{i\in I_2}p_i$ so that all users are indifferent between I_1 and I_2. Then there exists $\varepsilon>0$ such that *all* users with $\theta\in[\theta_H-\varepsilon,\theta_H]$ rank I_1 and I_2 among their most preferred subsets of complements. Indeed:

$$\theta V\left(Q_{I_1}\right)-\sum_{i\in I_1}p_i > \theta V\left(Q_I\right)-\sum_{i\in I}p_i$$

[17] Naturally, we assume all I_i's are distinct.

for all $I \notin \{I_1, .., I_p\}$ and θ close enough to θ_H and:

$$0 \geq \theta \left(V\left(Q_{I_1}\right) - V\left(Q_{I_k}\right)\right) \geq \theta_H \left(V\left(Q_{I_1}\right) - V\left(Q_{I_k}\right)\right) = \sum_{i \in I_1} p_i - \sum_{i \in I_k} p_i$$

for all $\theta \leq \theta_H$, $k \in \{2, .., p\}$

Then consumers with $\theta \in [\theta_H - \varepsilon, \theta_H]$ choose randomly between I_1, I_2 and possibly (a finite number of) other subsets, each being equally likely to be chosen. Consider then any complementor $i \in I_1 \backslash I_2$: by slightly reducing its price, it ensures that all consumers with $\theta \in [\theta_H - \varepsilon, \theta_H]$ now strictly prefer I_1 to I_2 (not necessarily to all other previously tied subsets), so that they randomize with equal probabilities among a strictly smaller number of options. Thus, the slightest price cut induces a discrete jump in demand and hence in profits for complementor i, so that his initial price could not have been an equilibrium.

Therefore $p = 1$ or $V\left(Q_{I_1}\right) < V\left(Q_{I_2}\right)$. In any event, this immediately implies that there exists θ_1, $\theta_m \leq \theta_1 < \theta_H$, such that all consumers with $\theta \in]\theta_1, \theta_H[$ *strictly* prefer I_1 to any other subset, whereas θ_1 is indifferent between I_1 and possibly some other subsets $J_1, J_2, .., J_r$, where $V\left(Q_{J_1}\right) \leq V\left(Q_{J_2}\right) \leq ... \leq V\left(Q_{J_r}\right)$, $r \geq 1$. Indeed, this is clearly true for any $I \notin \{I_1, .., I_p\}$ and for $k \in \{2, .., p\}$:

$$0 > \theta \left(V\left(Q_{I_1}\right) - V\left(Q_{I_k}\right)\right) > \theta_H \left(V\left(Q_{I_1}\right) - V\left(Q_{I_k}\right)\right) = \sum_{i \in I_1} p_i - \sum_{i \in I_k} p_i$$

In fact, by the same argument, all consumers $\theta < \theta_H$ strictly prefer I_1 over any I such that $V\left(Q_I\right) > V\left(Q_{I_1}\right)$, therefore it must be that:

$$V\left(Q_{J_1}\right) \leq V\left(Q_{J_2}\right) \leq ... \leq V\left(Q_{J_r}\right) < V\left(Q_{I_1}\right)$$

If $\theta_1 = \theta_m$ then *all* available consumers (except perhaps those with $\theta = \theta_H$ and $\theta = \theta_m$, a set of measure 0) strictly prefer subset I_1 to all other subsets, which means that all developers $j \in \{1, .., n\} \backslash I_1$ have 0 demand and make 0 profits, which is impossible in equilibrium since they could guarantee themselves strictly positive profits by charging:

$$p_j = \theta_m \left(V\left(Q_{I_1} + q_j\right) - V\left(Q_{I_1}\right)\right)$$

Thus, since $I_1 \subsetneq \{1, .., n\}$ by assumption, we must have $\theta_1 > \theta_m$.

Then, for all $\theta < \theta_1$, $k \in \{1, .., r\}$:

$$0 > \theta \left(V\left(Q_{J_k}\right) - V\left(Q_{I_1}\right)\right) > \theta_1 \left(V\left(Q_{J_k}\right) - V\left(Q_{I_1}\right)\right) = \sum_{i \in J_k} p_i - \sum_{i \in I_1} p_i$$

so I_1 cannot be among the most preferred subsets by user θ.

We can then repeat the same reasoning starting with θ_1 rather than θ_H, obtaining a finite sequence $\theta_H > \theta_1 > ... > \theta_p > \theta_{p+1} = \theta_m$ and $I_1, .., I_{p+1}$ (where I_{p+1} can possibly be the empty set), such that $V\left(Q_{I_1}\right) > V\left(Q_{I_2}\right) > ... > V\left(Q_{I_{p+1}}\right)$ and, for all $k = 1, .., p$, all users $\theta \in \left]\theta_{k+1}, \theta_k\right[$ *strictly* prefer I_{k+1} to all other subsets and user θ_k is indifferent between I_k and I_{k+1}.

Since $V\left(Q_{I_p}\right) > V\left(Q_{I_{p+1}}\right)$, there exists $i \in I_p \backslash I_{p+1}$. Then, by definition of i and continuity on both sides of θ_p:

$$\theta_p \left(V\left(Q_{I_p}\right) - V\left(Q_{I_p} - q_i\right)\right) \geq p_i \geq \theta_p \left(V\left(Q_{I_{p+1}} + q_i\right) - V\left(Q_{I_{p+1}}\right)\right)$$

Concavity of V then requires:

$$Q_{I_{p+1}} < Q_{I_p} \leq Q_{I_{p+1}} + q_i$$

which combined with the above inequalities implies:

$$\frac{p_i}{V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)} \geq \theta_p \tag{35}$$

Consider now complementor i. Its demand $d\left(p_j\right)$ is lower than or equal to $\left(1 - F\left(\theta_p\right)\right)$. If it slightly lowers its price by $\varepsilon > 0$, it obtains a variation in profits of:

$$\begin{aligned} d\pi_i^D &= -\varepsilon d\left(p_i\right) + p_i d\left(p_i - \varepsilon\right) - p_i d\left(p_i\right) + O\left(\varepsilon^2\right) \\ &\geq -\varepsilon \left(1 - F\left(\theta_p\right)\right) + p_i d\left(p_i - \varepsilon\right) - p_i d\left(p_i\right) + O\left(\varepsilon^2\right) \end{aligned}$$

Let:

$$\theta_\varepsilon = \theta_p - \frac{\varepsilon}{V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)}$$

Then user θ strictly prefers I_p to I_{p+1} if and only if:

$$\theta \left(V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)\right) > \sum_{i \in I_p} p_i - \varepsilon - \sum_{i \in I_{p+1}} p_i = \theta_p \left(V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)\right) - \varepsilon$$

which is equivalent to:

$$\theta > \theta_\varepsilon$$

Let ε be small enough and consider any subset I. If $i \notin I$ then all users $\theta \in]\theta_\varepsilon, \theta_p]$ still prefer I_{p+1} to I and therefore I_p is strictly preferred to both. Thus, for all users $\theta \in]\theta_\varepsilon, \theta_p]$ the most preferred subset necessarily includes i. Therefore:

$$d\left(p_i - \varepsilon\right) \geq d\left(p_i\right) + F\left(\theta_p\right) - F\left(\theta_\varepsilon\right) = d\left(p_i\right) + f\left(\theta_p\right) \frac{\varepsilon}{V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)} + O\left(\varepsilon^2\right)$$

We obtain:

$$\begin{aligned}
d\pi_i^D &\geq -\varepsilon\left(1 - F\left(\theta_p\right)\right) + f\left(\theta_p\right) \frac{\varepsilon p_i}{V\left(Q_{I_p}\right) - V\left(Q_{I_{p+1}}\right)} + O\left(\varepsilon^2\right) \\
&\geq -\varepsilon\left(1 - F\left(\theta_p\right) - \theta_p f\left(\theta_p\right)\right) + O\left(\varepsilon^2\right) \\
&> 0
\end{aligned}$$

for ε small enough since $\theta_p > \theta_m \geq \theta_P$.

Therefore complementor i has a profitable deviation, so the initial prices could not have constituted an equilibrium.

This completes the proof of the uniqueness of the equilibrium.

■